Chapter 10 Local Subcell Models of Fine Geometrical Features

10.1 Introduction

Problem - many problems have key features that vary greatly in size

options - use spatial cells scaled to fit smallest features of interest (very computationally expensive)

- use variable spatial cells (both size + shape) to model problem (extensive preprocessing to properly grid problem)

- use uniform mesh w/ local subcell modeling to handle features that are smaller than the spatial cell sizes.

10.2 Basis of Contour-Path FDTD Modeling

Originally Yee used the differential forms of Faraday's Law $\frac{\partial \bar{\mathcal{B}}}{\partial t} = -\bar{\nabla} \times \bar{\mathcal{E}} - \bar{\mathcal{M}}$

and

Ampere's Law $\frac{\partial \bar{\mathcal{D}}}{\partial t} = \bar{\nabla} \times \bar{\mathcal{H}} - \bar{\mathcal{J}}$

to derive the second-order accurate, central-difference update equations.

Drawback – no way to account for variations in materials/geometry smaller than a spatial step

In subcell modeling, we apply the <u>integral</u> forms of

Faraday's Law $\frac{\partial}{\partial t} \iint_S \bar{\mathcal{B}} \cdot d\bar{s} = -\oint_C \bar{\mathcal{E}} \cdot d\bar{\ell} - \iint_S \bar{\mathcal{M}} \cdot d\bar{s}$

and

Ampere's Law $\frac{\partial}{\partial t} \iint_S \bar{\mathcal{D}} \cdot d\bar{s} = \oint_C \bar{\mathcal{H}} \cdot d\bar{\ell} - \iint_S \bar{\mathcal{J}} \cdot d\bar{s}$

over the small (but macroscopic) areas defined by the spatial cells in the FDTD grid.

This allows one to account for variations in materials, geometry, or the <u>fields</u> w/in the spatial cell (if underlying physics understood).

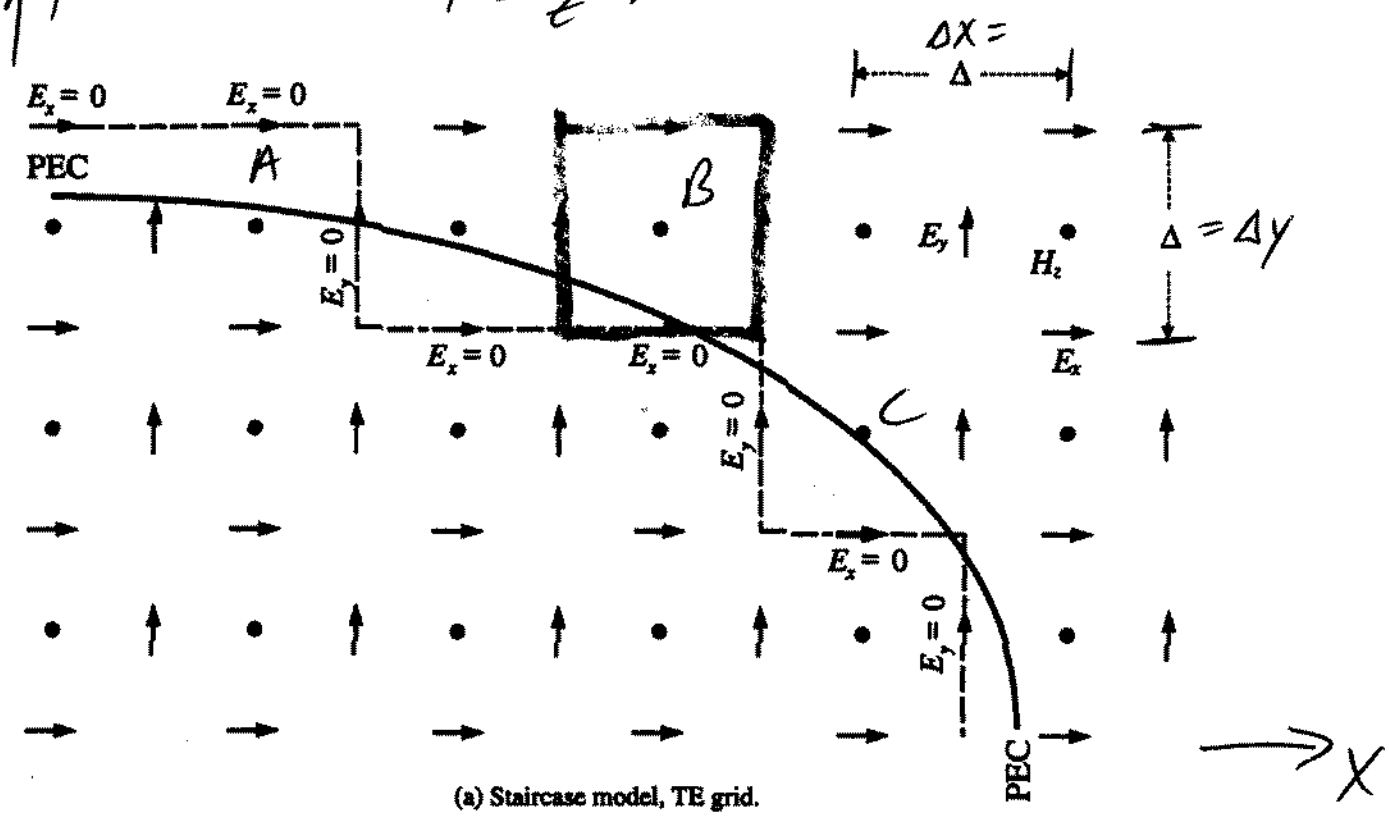
10.3 The Simplest Contour-Path Subcell Models

10.3.1 Diagonal Split-Cell Model for PEC Surfaces

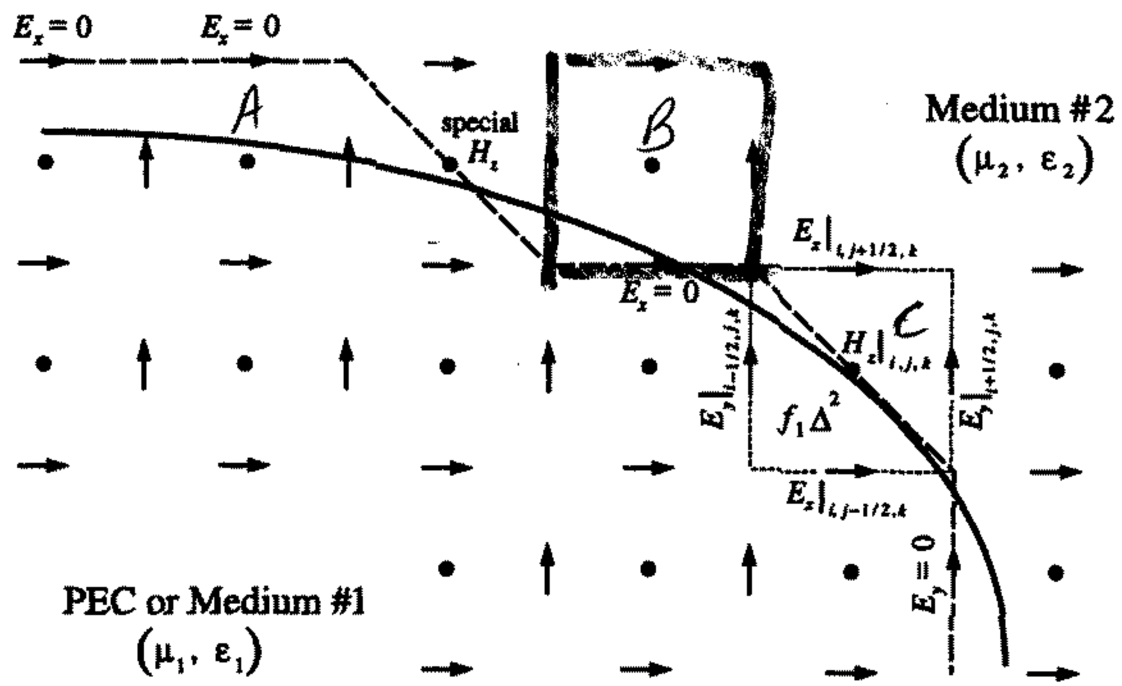
Consider the PEC boundary w/ some material shown in Fig 10.1. As shown in Fig 10.1a), in normal FDTD modeling we get a <u>staircase approximation</u>, i.e. if an E_x, E_y, or H_z node is w/in the PEC, the whole Δx, Δy, or $\Delta x \Delta y$ assoc. w/ that field quantity is assumed to have E_x, E_y, +/or $H_y = 0$

10.3.1 cont.

TE_z Mode Grid (part of 3D)



(a) Staircase model, TE grid.



(b) Diagonal split-cell model for PEC surface, average-properties model for material interface.

Fig. 10.1 The simplest subcell models for PEC surfaces and material interfaces.

With diagonal split-cell modeling, we do a best fit to the PEC surface.

For example:

At location A, over half of the area $\Delta x \Delta y$ associated w/ that H_z component (including H_z itself) is w/in the PEC

→ set H_z and all E_x + E_y on perimeter to zero. Remember $E_{tan} = 0$ near PEC when setting top $E_x = 0$, even though it's outside PEC.

At location B, only a small portion of the area $\Delta x \Delta y$ associated w/ that H_z component is w/in the PEC

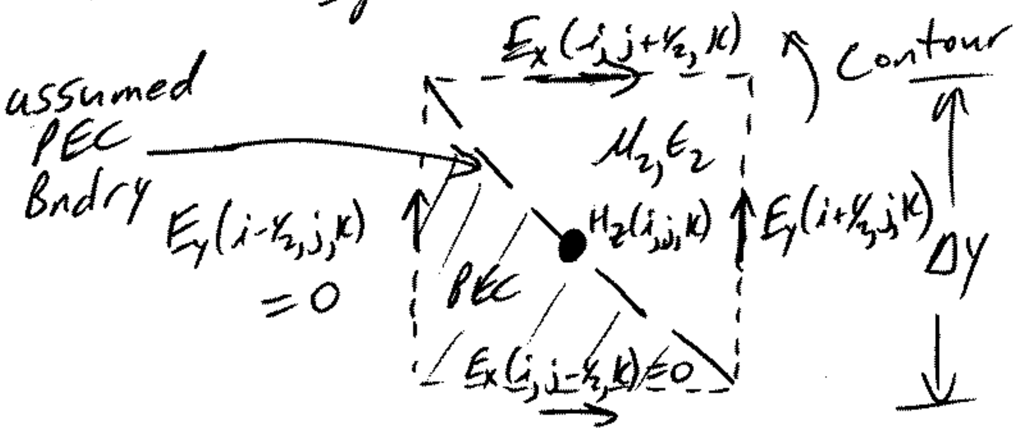
→ only set the single E_x component on the bottom to zero, leave other components alone + assume that they are uniform over Δx, Δy, +/or $\Delta x \Delta y$.

<u>At location C</u>, slightly less than half of the area associated w/ $H_z(i,j,k)$ is w/in the PEC. $H_z(i,j,k)$, $E_x(i,j+\frac{1}{2},k)$, + $E_y(i+\frac{1}{2},j,k)$ are outside the PEC while $E_x(i,j-\frac{1}{2},k)$ + $E_y(i-\frac{1}{2},j,k)$ are inside the PEC.

→ $E_{tan}=0$ does <u>NOT</u> necessarily mean that $E_x(i,j+\frac{1}{2},k)$ + $E_y(i+\frac{1}{2},j,k)$ must be zero as they are partially normal + partially tangential to the surface

→ Since $|H_{tan}|=|J_s|$, there is no issue w/ $H_z(i,j,k)$

→ Use split cell model!



Apply the integral form of Faraday's Law (assume $\overline{M}=0$) to the split cell model

$$\frac{\partial}{\partial t}\iint \overline{B}\cdot d\overline{s} = -\oint_c \overline{\mathcal{E}}\cdot d\overline{l}$$

→ assume $H_z(i,j,k)$ is uniform in upper right half of split cell

→ assume $E_x(i,j+\frac{1}{2},k)$ + $E_y(i+\frac{1}{2},j,k)$ are uniform along their respective sides

Then, the integrals become (at $t=n\Delta t$)

$$\frac{d}{dt}\mu_z H_z(i,j,k)\underbrace{\iint_{\text{Triangle}} ds_z}_{\frac{1}{2}\Delta x\Delta y} = -E_y(i+\tfrac{1}{2},j,k)\Delta y + E_x(i,j+\tfrac{1}{2},k)\Delta x + O(\Delta y) - O(\Delta x)$$

$$\mu_z\frac{d H_z^n(i,j,k)}{dt}\left(\frac{\Delta x\Delta y}{2}\right) = E_x(i,j+\tfrac{1}{2},k)\Delta x - E_y(i+\tfrac{1}{2},j,k)\Delta y$$

Using the standard central-difference approximation for the time-derivative

10.3.1 cont.

yields

$$\frac{\Delta x \Delta y \mu_z}{2}\left(\frac{H_z^{n+1/2}(i,j,k) - H_z^{n-1/2}(i,j,k)}{\Delta t}\right) = E_x^n(i,j+1/2,k)\Delta x - E_y^n(i+1/2,j,k)\Delta y$$

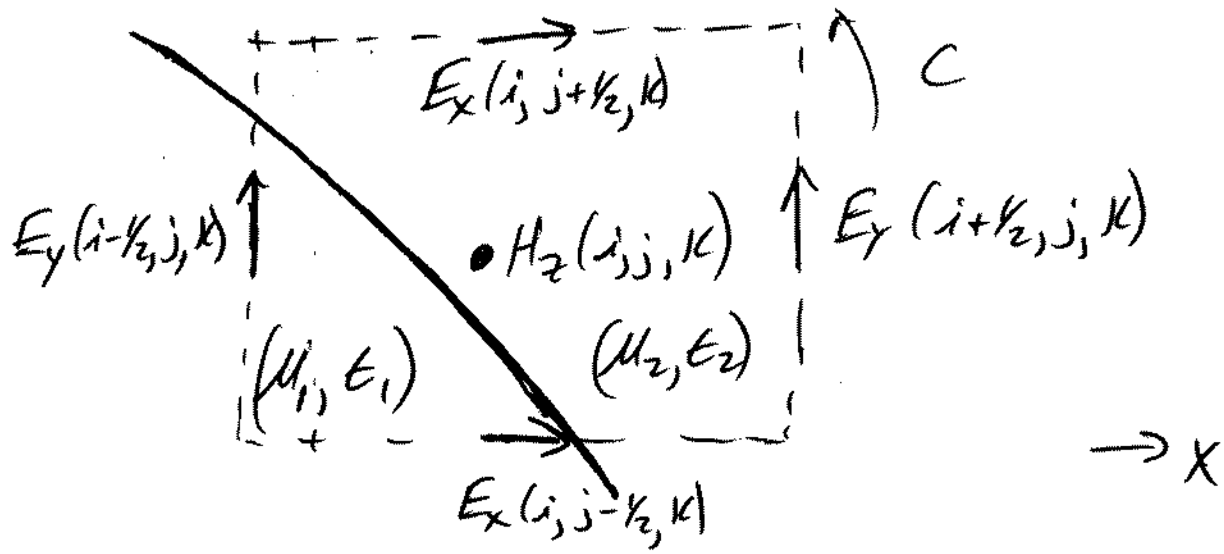
$$\boxed{H_z^{n+1/2}(i,j,k) = H_z^{n-1/2}(i,j,k) + \frac{2\Delta t}{\mu_z \Delta y} E_x^n(i,j+1/2,k) - \frac{2\Delta t}{\mu_z \Delta x} E_y^n(i+1/2,j,k)}$$

→ Similar process in 3D grid for H_x & H_y

→ No change for $E_x(i,j+1/2,k)$ & $E_y(i+1/2,j,k)$ updates

10.3.2 Average Properties Model for Material Interfaces

Now consider the boundary in Fig 10.1b to be the interface between two materials — Medium #1 (μ_1, ϵ_1) and Medium #2 (μ_2, ϵ_2), and consider the cell around $H_z(i, j, k)$



→ assume all field quantities are <u>uniform</u> & <u>continuous</u>. Not strictly accurate wrt

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \quad (\text{if } J_s = 0),$$

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n} \quad (\text{if } \rho_s = 0)$$

but it works!

10.3.2 cont.

Again, apply the integral form of Faraday's Law @ $t = n\Delta t$

$$\frac{d}{dt}\iint_S \bar{B}\cdot d\bar{s} = -\oint_c \bar{\mathcal{E}}\cdot d\bar{\ell}$$

$$\frac{d(H_z^n(i,j,k))}{dt}\underbrace{\iint \mu\, ds_z}_{\text{area of cell}} = -E_y(i+\tfrac{1}{2},j,k)\Delta y + E_x(i,j+\tfrac{1}{2},k)\Delta x + E_y(i-\tfrac{1}{2},j,k)\Delta y - E_x(i,j-\tfrac{1}{2},k)\Delta x$$

Time-invariant

$$\left(\frac{H_z^{n+1/2}(i,j,k) - H_z^{n-1/2}(i,j,k)}{\Delta t}\right)\left[\underbrace{\iint \mu_1\,dx\,dy}_{\text{fraction in med \#1}} + \underbrace{\iint \mu_2\,dx\,dy}_{\text{fraction in med \#2}}\right]$$

$$= \left[E_x^n(i,j+\tfrac{1}{2},k) - E_x^n(i,j-\tfrac{1}{2},k)\right]\Delta x - \left[E_y^n(i+\tfrac{1}{2},j,k) - E_y^n(i-\tfrac{1}{2},j,k)\right]\Delta y$$

→ defining $f_1 = \dfrac{\text{Area in region 1}}{\Delta x\,\Delta y}$ $\quad f_2 = 1 - f_1$

$$\left(\frac{H_z^{n+1/2}(i,j,k) - H_z^{n-1/2}(i,j,k)}{\Delta t}\right)\left[f_1\,\Delta x\Delta y\,\mu_1 + (1-f_1)\Delta x\Delta y\,\mu_2\right] =$$

$$\left[E_x^n(i,j+\tfrac{1}{2},k) - E_x^n(i,j-\tfrac{1}{2},k)\right]\Delta x - \left[E_y^n(i+\tfrac{1}{2},j,k) - E_y^n(i-\tfrac{1}{2},j,k)\right]\Delta y$$

10.3.2 cont.

$$\boxed{\begin{aligned} H_z^{n+1/2}(i,j,k) = H_z^{n-1/2}(i,j,k) &+ \frac{\Delta t}{\left(f_1\mu_1 + (1-f_1)\mu_2\right)\Delta y}\left[E_x^n(i,j+1/2,k) - E_x^n(i,j-1/2,k)\right] \\ &- \frac{\Delta t}{\left(f_1\mu_1 + (1-f_1)\mu_2\right)\Delta x}\left[E_y^n(i+1/2,j,k) - E_y^n(i-1/2,j,k)\right] \end{aligned}}$$

Note: $\mu_{AVE} = f_1\mu_1 + (1-f_1)\mu_2$

→ Similar developement for E-components cells cut by dielectric boundary (section 10.6.3)

→ σ^* + σ can also be taken into account.

10.4 The Contour-Path Model of the Narrow Slot

Fig 10.2 shows a thin PEC sheet (in some x-y plane) which has a slot of width $g < \Delta x$ cut in it. Here, we'll consider 3 possibilities

Contour 1 - near edge of PEC sheet (no slot)

Key Assumptions

* E_x on PEC sheet is zero as are E_y + H_z on PEC sheet
* E_x, E_y, + H_z NOT on PEC sheet are assumed to be uniform for portions NOT on PEC sheet (approx)

Contour 2 - at slot opening into free space

* portions of E_x + E_y in sheet are zero
* E_x, E_y, + H_z Not in sheet assumed uniform along respective paths/area

10.4 cont.

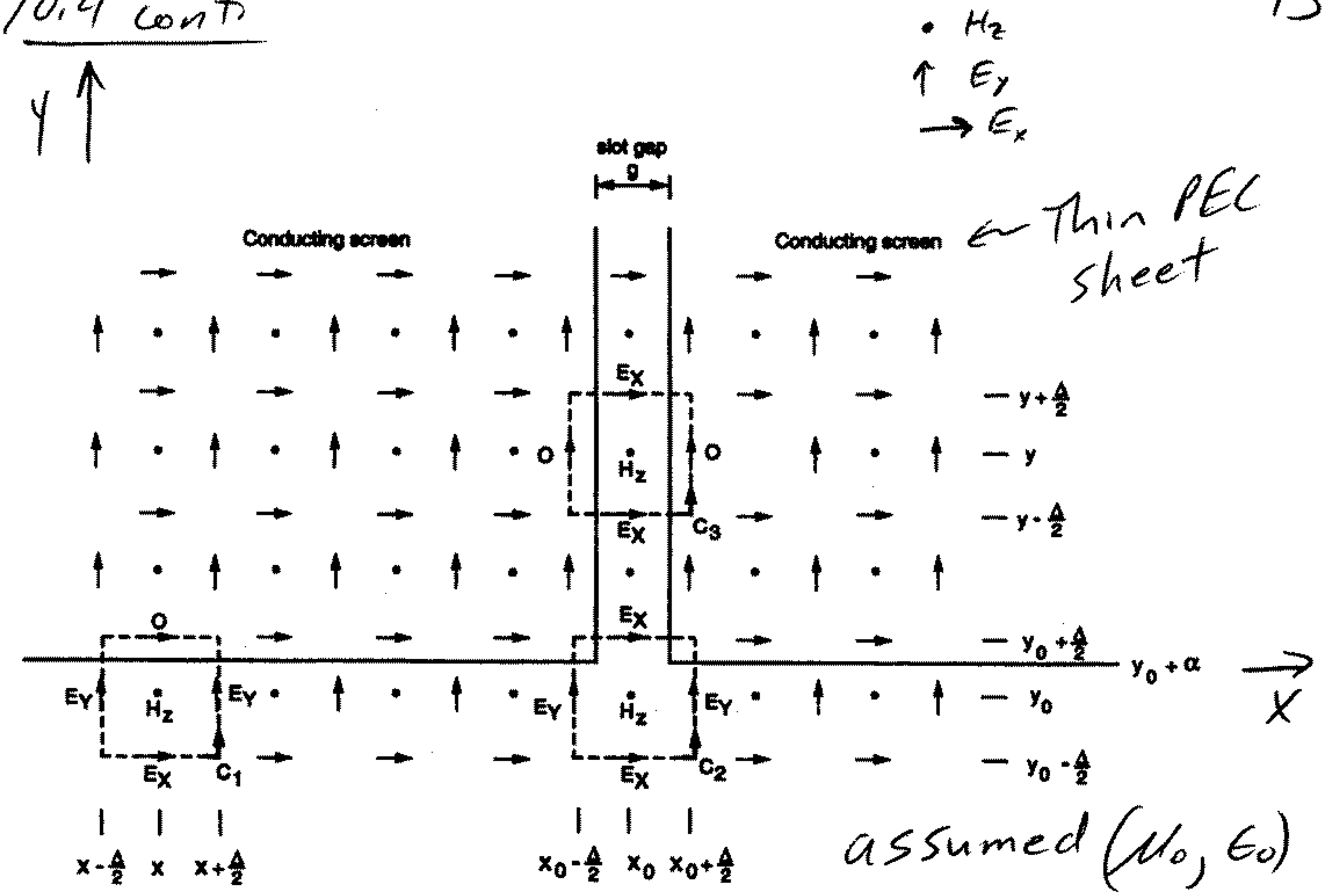


Fig. 10.2 Faraday's law contour paths for the narrow slot in a PEC screen. *Source:* Taflove et al., *IEEE Trans. Antennas and Propagation*, 1988, pp. 247–257, © 1988 IEEE.

Contour 3 – in the slot

* H_z uniform in slot (ave. value)
* E_y on left + right are zero
* E_x on top + bottom uniform w/in slot and zero for portions in/on PEC sheet.

For convenience, I'll assume that $H_z(i,j,k)$ at the center of each contour. Apply Faraday's Law (assume $\bar{M}=0$) in the integral form. @ $t = n\Delta t$

Contour 1

$$\frac{d}{dt}\iint_S \bar{B}\cdot d\bar{s} = -\oint_c \bar{E}\cdot d\bar{l}$$

$$\frac{d(H_z^n(i,j,k))}{dt}\underset{\substack{\text{Non-PEC}\\ \text{area}}}{\iint}\mu_0\, dx\, dy = -E_y^n(i+\tfrac{1}{2},j,k)\left(\frac{\Delta y}{2}+\alpha\right) + 0$$
$$+E_y^n(i-\tfrac{1}{2},j,k)\left(\frac{\Delta y}{2}+\alpha\right) - E_x^n(i,j-\tfrac{1}{2},k)\Delta x$$

$$\left(\frac{H_z^{n+1/2}(i,j,k) - H_z^{n-1/2}(i,j,k)}{\Delta t}\right)\left[\mu_0\Delta x\left(\frac{\Delta y}{2}+\alpha\right)\right] = -E_x^n(i,j-\tfrac{1}{2},k)\,\Delta x$$
$$-\left[E_y^n(i+\tfrac{1}{2},j,k) - E_y^n(i-\tfrac{1}{2},j,k)\right]\left(\frac{\Delta y}{2}+\alpha\right)$$

$$\boxed{\begin{aligned} H_z^{n+1/2}(i,j,k) = H_z^{n-1/2}(i,j,k) &- \frac{\Delta t}{\mu_0\left(\frac{\Delta y}{2}+\alpha\right)}E_x^n(i,j-\tfrac{1}{2},k) \\ &- \frac{\Delta t}{\mu_0\Delta x}\left[E_y^n(i+\tfrac{1}{2},j,k) - E_y^n(i-\tfrac{1}{2},j,k)\right]\end{aligned}}$$

10.4 cont.

Contour 2

$$\frac{d\left(H_z^n(i,j,k)\right)}{dt}\iint\limits_{\substack{\text{Non-PEC}\\ \text{Area}}}\mu_0\,dx\,dy = -E_y^n(i+\tfrac{1}{2},j,k)\left(\tfrac{\Delta y}{2}+\alpha\right)+E_x^n(i,j+\tfrac{1}{2},k)\,g$$
$$+E_y^n(i-\tfrac{1}{2},j,k)\left(\tfrac{\Delta y}{2}+\alpha\right)-E_x^n(i,j-\tfrac{1}{2},k)\,\Delta x$$

$$\frac{H_z^{n+1}(i,j,k)-H_z^n(i,j,k)}{\Delta t}\,\mu_0\left[\Delta x\left(\tfrac{\Delta y}{2}+\alpha\right)+g\left(\tfrac{\Delta y}{2}-\alpha\right)\right] = \text{ditto}$$

$$\boxed{\begin{aligned}H_z^{n+1}(i,j,k) = H_z^n(i,j,k) &+ \left(\frac{E_x^n(i,j+\tfrac{1}{2},k)\,g - E_x^n(i,j-\tfrac{1}{2},k)\,\Delta x}{\mu_0\left[\Delta x\left(\tfrac{\Delta y}{2}+\alpha\right)+g\left(\tfrac{\Delta y}{2}-\alpha\right)\right]}\right)\\ &- \left(\frac{E_y^n(i+\tfrac{1}{2},j,k)\left(\tfrac{\Delta y}{2}+\alpha\right) - E_y^n(i-\tfrac{1}{2},j,k)\left(\tfrac{\Delta y}{2}+\alpha\right)}{\mu_0\left[\Delta x\left(\tfrac{\Delta y}{2}+\alpha\right)+g\left(\tfrac{\Delta y}{2}-\alpha\right)\right]}\right)\end{aligned}}$$

Contour 3

$$\left(\frac{H_z^{n+1}(i,j,k)-H_z^n(i,j,k)}{\Delta t}\right)\left(\mu_0\,g\,\Delta y\right) = 0 + E_x^n(i,j+\tfrac{1}{2},k)\,g + 0$$
$$-E_x^n(i,j-\tfrac{1}{2},k)\,g$$

← same as free space!

$$\boxed{H_z^{n+1}(i,j,k) = H_z^n(i,j,k) + \left(\frac{\Delta t}{\mu_0\,\Delta y}\right)\left[E_x^n(i,j+\tfrac{1}{2},k) - E_x^n(i,j-\tfrac{1}{2},k)\right]}$$

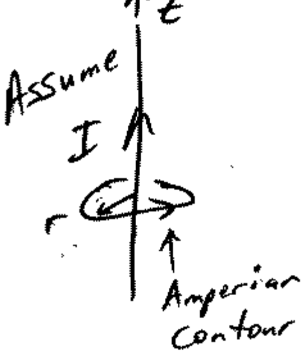
Original Taflove approach [2]

→ assumes wire radius $r_0 \le \Delta x/2$ (see Fig 10.4) on next page

Quasi-Static Field Distribution Assumptions

* Looping magnetic fields fall off as $1/r$

Why/where? Ampere's Law $\oint \bar{H} \cdot d\bar{l} = I_{enc}$



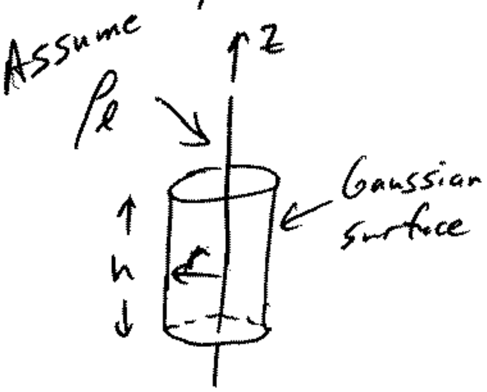
leads to $\bar{H} = \hat{a}_\phi \dfrac{I}{2\pi r}$

Therefore, $H_y \propto \dfrac{1}{x}$ and $H_x \propto \dfrac{1}{y}$

near thin wire w/ Cartesian coord.

* Radial electric fields fall off as $1/r$

Why/where? Gauss' Law $\oint \bar{D} \cdot d\bar{s} = Q_{enc}$



leads to $\bar{D} = \hat{a}_r \dfrac{\rho_\ell h}{2\pi r h} = \hat{a}_r \dfrac{\rho_\ell}{2\pi r}$

↳ $E_x \propto \dfrac{1}{x}$ and $E_y \propto \dfrac{1}{y}$

* Assumed field distributions for Fig 10.4

$$H_y(x, y_0, z) = H_y\left(\tfrac{\Delta x}{2}, y_0, z_0\right) \frac{\Delta x/2}{x}\left[1 + C_1(z - z_0)\right]$$

$$E_x(x, y_0, z_0 \pm \Delta z/2) = E_x\left(\tfrac{\Delta x}{2}, y_0, z_0 \pm \Delta z/2\right) \frac{(\Delta x/2)}{x}$$

$$E_z(\Delta x, y_0, z) = E_z(\Delta x, y_0, z_0)\left[1 + C_2(z - z_0)\right]$$

* $\frac{1}{r}$ wrt x
* linear wrt z

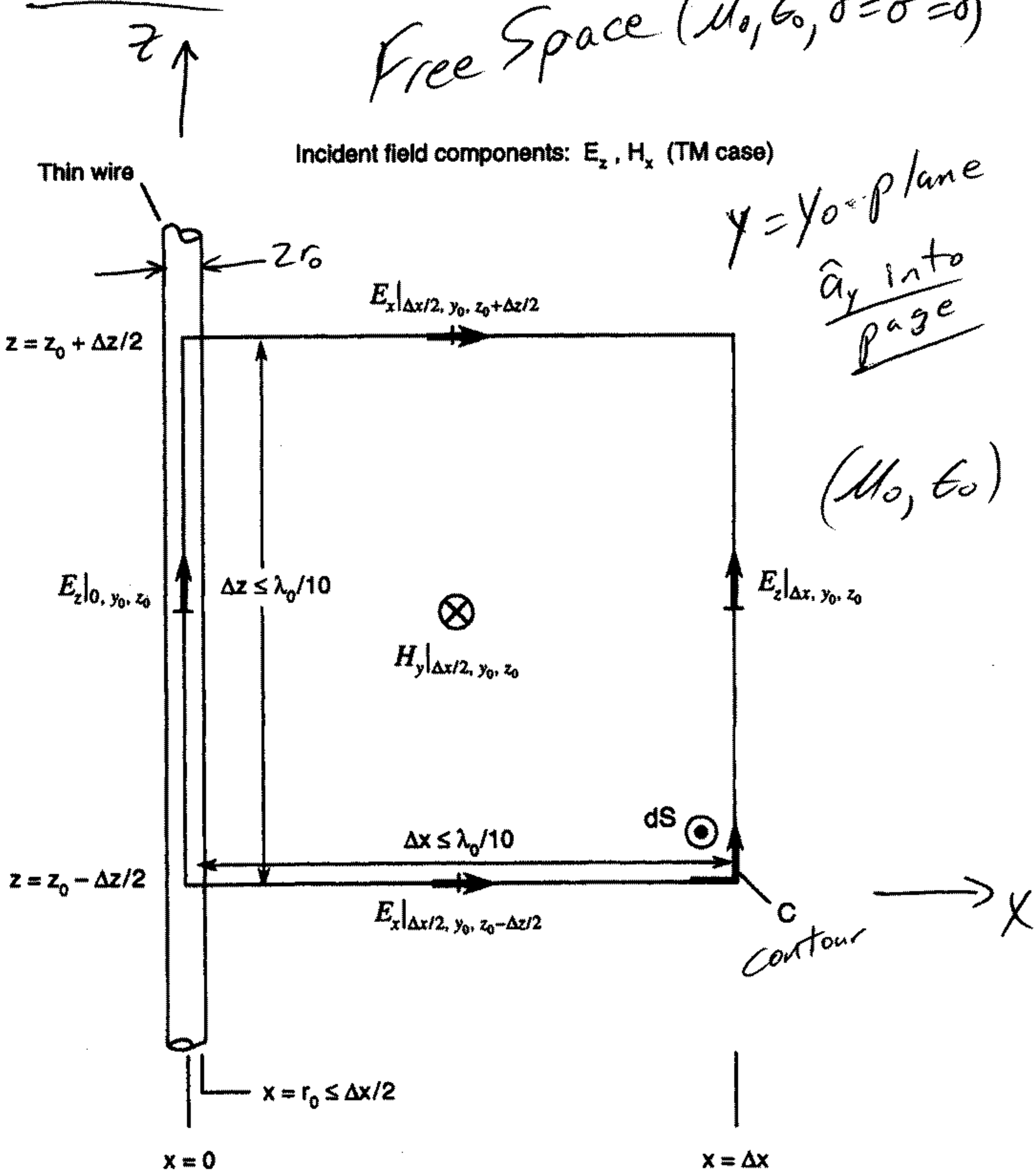


Fig. 10.4 Faraday's law contour path for the thin wire. *Source:* Umashankar et al., *IEEE Trans. Antennas and Propagation*, 1987, pp. 1248–1257, © 1987 IEEE.

10.5 cont.

Apply integral form of Faraday's Law (assume $\bar{M}=0$) at time $t = n\Delta t$

$$\frac{\partial}{\partial t}\iint_S \bar{B}\cdot d\bar{s} = -\oint_C \bar{\mathcal{E}}\cdot d\bar{l}$$

RHR w/contour

$$\simeq \frac{\partial}{\partial t}\iint_S \hat{a}_y \mu_0 H_y^n(x,y_0,z)\cdot -\hat{a}_y\,dx\,dz = -\frac{d}{dt}\iint \mu_0 H_y^n\left(\frac{\Delta x}{2},y_0,z_0\right)\frac{\Delta x/2}{x}[\;]\,dx\,dz$$

$$= -\frac{\mu_0 \Delta x}{2}\,\frac{d H_y^n\left(\frac{\Delta x}{2},y_0,z_0\right)}{dt}\int_{z_0-\frac{\Delta z}{2}}^{z_0+\frac{\Delta z}{2}}\left[1+C_1(z-z_0)\right]dz\int_{x=r_0}^{\Delta x}\frac{dx}{x}$$

$$= -\frac{\mu_0 \Delta x}{2}\left[\frac{H_y^{n+0.5}\left(\frac{\Delta x}{2},y_0,z_0\right)-H_y^{n-0.5}\left(\frac{\Delta x}{2},y_0,z_0\right)}{\Delta t}\right]\left[z+\frac{C_1}{2}z^2-C_1 z_0 z\right]\Bigg|_{z_0-\Delta z/2}^{z_0+\Delta z/2}\ln(x)\Bigg|_{r_0}^{\Delta x}$$

$$= -\frac{\mu_0 \Delta x}{2}\left[\frac{H_y^{n+0.5}\left(\frac{\Delta x}{2},y_0,z_0\right)-H_y^{n-0.5}\left(\frac{\Delta x}{2},y_0,z_0\right)}{\Delta t}\right](\Delta z)\ln\left(\frac{\Delta x}{r_0}\right)$$

$$\frac{\partial}{\partial t}\iint_S \bar{B}\cdot d\bar{s} \simeq -\frac{\mu_0\,\Delta x\,\Delta z\ln\left(\frac{\Delta x}{r_0}\right)}{2\Delta t}\left[H_y^{n+0.5}\left(\frac{\Delta x}{2},y_0,z_0\right)-H_y^{n-0.5}\left(\frac{\Delta x}{2},y_0,z_0\right)\right]$$

Now, for the RHS of Faraday's Law, follow the contour C counter-clockwise (as indicated on Fig 10.4)

On the RHS,

$$-\oint \bar{\mathcal{E}}\cdot d\bar{l} = -\int_{z_0-\Delta z/2}^{z_0+\frac{\Delta z}{2}} E_z^n(\Delta x, y_0, z_0)\left[1+C_2(z-z_0)\right]dz$$

in wire

$$-\int_{\Delta x}^{r_0} E_x^n\left(\frac{\Delta x}{2}, y_0, z_0+\frac{\Delta z}{2}\right)\frac{\Delta x/2}{x}\,dx + \int 0\,dz$$

$$-\int_{r_0}^{\Delta x} E_x^n\left(\frac{\Delta x}{2}, y_0, z_0-\frac{\Delta z}{2}\right)\frac{\Delta x/2}{x}\,dx$$

$$= -E_z^n(\Delta x, y_0, z_0)\left[z+\frac{C_2 z^2}{2} - C_2 z_0 z\right]\Bigg|_{z_0-\Delta z/2}^{z_0+\frac{\Delta z}{2}}$$

$$-\frac{\Delta x}{2}E_x^n\left(\frac{\Delta x}{2}, y_0, z_0+\frac{\Delta z}{2}\right)\ln x\Big|_{\Delta x}^{r_0}$$

$$-\frac{\Delta x}{2}E_x^n\left(\frac{\Delta x}{2}, y_0, z_0-\frac{\Delta z}{2}\right)\ln x\Big|_{r_0}^{\Delta x}$$

$$= -\Delta z\, E_z^n(\Delta x, y_0, z_0) + \frac{\Delta x}{2}\ln\left(\frac{\Delta x}{r_0}\right)E_x^n\left(\frac{\Delta x}{2}, y_0, z_0+\frac{\Delta z}{2}\right)$$

$$-\frac{\Delta x}{2}\ln\left(\frac{\Delta x}{r_0}\right)E_x^n\left(\frac{\Delta x}{2}, y_0, z_0-\frac{\Delta z}{2}\right)$$

$$= \frac{\Delta x}{2}\ln\left(\frac{\Delta x}{r_0}\right)\left[E_x^n\left(\frac{\Delta x}{2}, y_0, z_0+\frac{\Delta z}{2}\right) - E_x^n\left(\frac{\Delta x}{2}, y_0, z_0-\frac{\Delta z}{2}\right)\right]$$

$$-\Delta z\, E_z^n(\Delta x, y_0, z_0)$$

10.5 cont.

$$-\frac{\mu_0 \Delta x \Delta z \ln\left(\frac{\Delta x}{r_0}\right)}{2\Delta t}\left[H_y^{n+0.5}\left(\frac{\Delta x}{2}, y_0, z_0\right) - H_y^{n-0.5}\left(\Delta x/2, y_0, z_0\right)\right]$$

$$= \frac{\Delta x \ln\left(\frac{\Delta x}{r_0}\right)}{2}\left[E_x^n\left(\frac{\Delta x}{2}, y_0, z_0+\frac{\Delta z}{2}\right) - E_x^n\left(\frac{\Delta x}{2}, y_0, z_0-\Delta z/2\right)\right]$$

$$- \Delta z\, E_z^n(\Delta x, y_0, z_0)$$

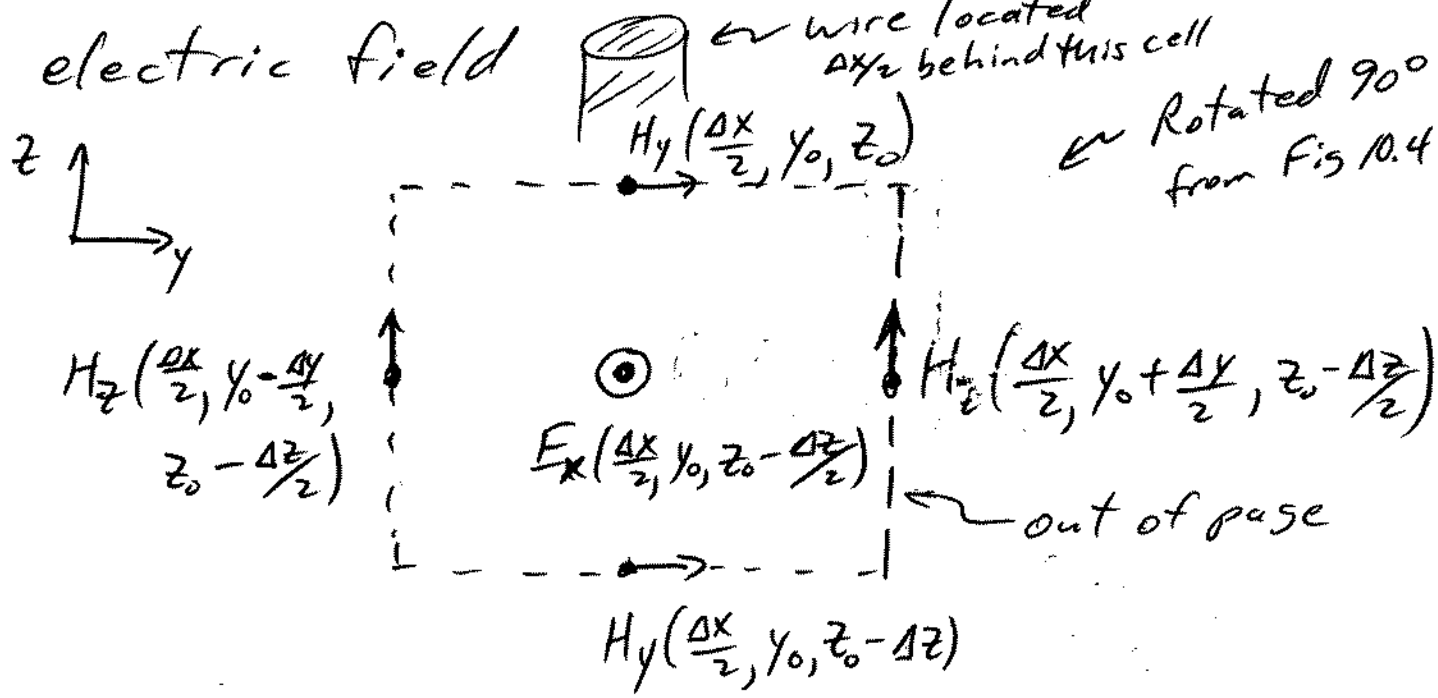
$$\boxed{H_y^{n+0.5}\left(\frac{\Delta x}{2}, y_0, z_0\right) = H_y^{n-0.5}\left(\frac{\Delta x}{2}, y_0, z_0\right) - \frac{\Delta t}{\mu_0 \Delta z}\left[E_x^n\left(\frac{\Delta x}{2}, y_0, z_0+\frac{\Delta z}{2}\right) - E_x^n\left(\frac{\Delta x}{2}, y_0, z_0-\frac{\Delta z}{2}\right)\right] + \frac{\Delta t}{\mu_0 \Delta x}\left(\frac{2}{\ln(\Delta x/r_0)}\right) E_z^n(\Delta x, y_0, z_0)}$$

→ Similar derivations for remaining H_x + H_y looping around wire (only fields immediately adjacent to wire)

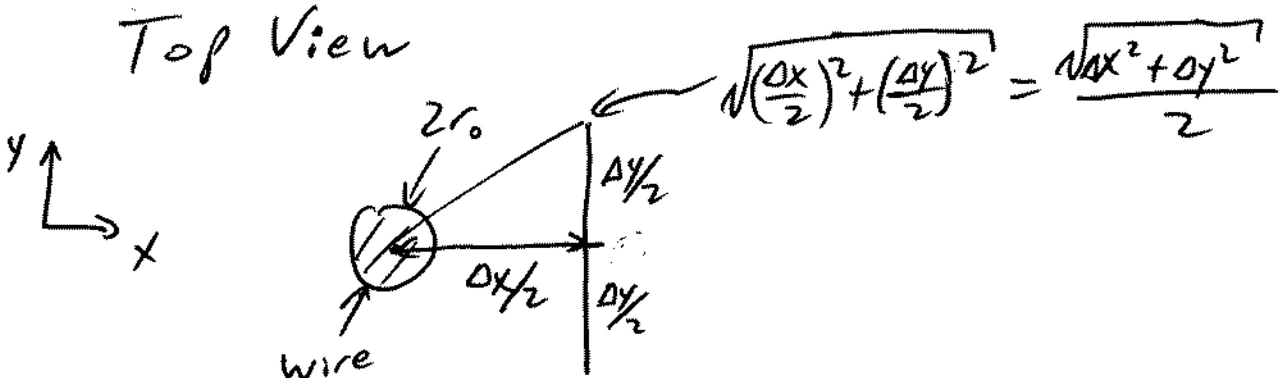
→ Update equations for the radial electric field equations were left unchanged (not strictly accurate) by Umashankar + Taflove + Beker

10.5 cont.

Contour Path for Ampere's Law for radial electric field



Top View



Problems:

→ over the surface E_x varies in distance from wire

→ over Δy, H_y varies in distance from wire (on top and bottom)

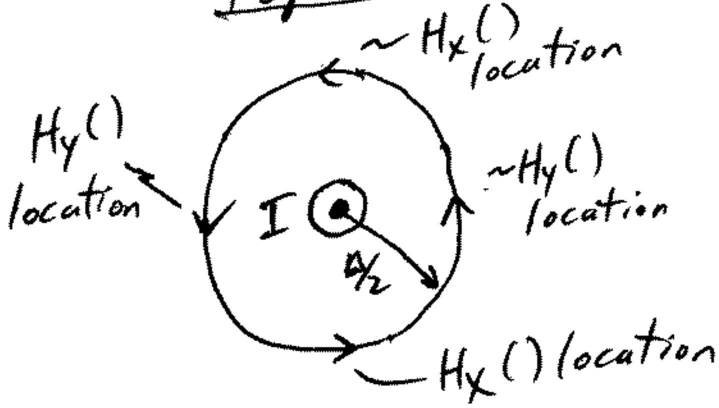
→ H_z is at a constant distance from wire (on left + right)

Mäkinen, Juntunen, + Kivikoski [5] improved the subcell model for the looping magnetic field and radial electric field components near/adjacent to a thin wire by:

* Keeping $\frac{1}{r}$ variation for these components e.g. $H_x \propto \frac{1}{y}$, $H_y \propto \frac{1}{x}$, $E_x \propto \frac{1}{x}$, + $E_y \propto \frac{1}{y}$

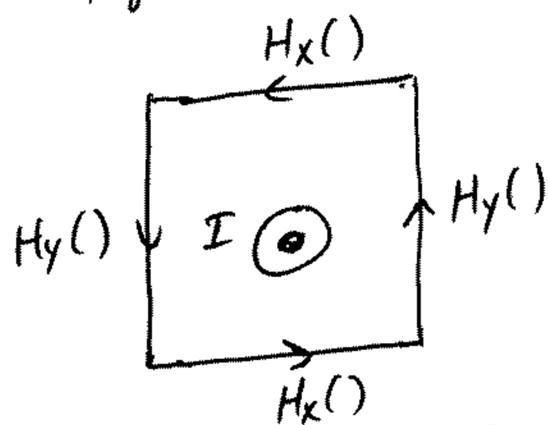
* Projecting looping magnetic field components (e.g. H_x + H_y) onto Cartesian cell edges for radial electric field update equations ($E_x + E_y$)
  $\Rightarrow$ take into account varying distance from wire + path length for Cartesian versus circular looping path

Top View



Magnetic field
$\bar{H} = \hat{a}_\phi \frac{I}{2\pi \Delta/2}$ (Ampere's Law)
↳ circular path ($\Delta/2$ radius)

Top View / FDTD grid



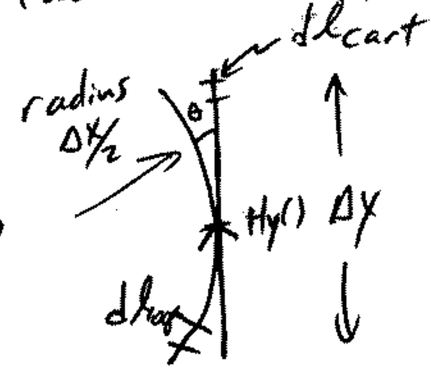
perimeter $= 4\Delta$
circumference $= \pi\Delta$

obviously different path lengths!

10.5 cont.

To account for the fact that the $H_y()$ components (for the radial $E_x()$ components) are NOT truly constant on the Cartesian FDTD cell edges, a scaling factor is introduced

$$K_{H_y} = \frac{\int_{C_{loop}} d\ell_{loop}}{\int_{C_{cart}} d\ell_{cart}} \Rightarrow$$



$$= \frac{\int_0^{\Delta Y/2} \cos\theta \left(\frac{\Delta X/2}{\sqrt{(\Delta X/2)^2 + y^2}} \right) dy}{\Delta Y/2}$$

Note: Symmetric about middle

$\cos\theta$ projects curved path onto straight path; term inside ( ) accounts for $1/r$ field strength variation

$$\underline{K_{H_y} = \frac{\Delta X}{\Delta Y} \tan^{-1}\left(\Delta Y / \Delta X \right)}$$

Similarly,

$$\underline{K_{H_x} = \frac{\Delta Y}{\Delta X} \tan^{-1}\left(\Delta X / \Delta Y \right)}$$

In the standard Yee update eqns for the radial $E_x()$ + $E_y()$ components, replace $H_y()$ w/ $K_{H_y} H_y()$ and $H_x()$ w/ $K_{H_x} H_x()$. The $H_z()$ terms are NOT changed.

10.5 cont.

To right of wire in Fig 10.4

$$E_x^{n+1}\left(\frac{\Delta x}{2}, y_0, z_0-\frac{\Delta z}{2}\right) = E_x^{n}\left(\frac{\Delta x}{2}, y_0, z-\frac{\Delta z}{2}\right)$$
$$-\frac{\Delta t}{\epsilon_0 \Delta z}\left[K_{H_y} H_y^{n+0.5}\left(\frac{\Delta x}{2}, y_0, z_0\right) - K_{H_y} H_y^{n+0.5}\left(\frac{\Delta x}{2}, y_0, z_0-\Delta z\right)\right]$$
$$+\frac{\Delta t}{\epsilon_0 \Delta y}\left[H_z^{n+0.5}\left(\frac{\Delta x}{2}, y_0+\frac{\Delta y}{2}, z_0-\frac{\Delta z}{2}\right) - H_z^{n+0.5}\left(\frac{\Delta x}{2}, y_0-\frac{\Delta y}{2}, z_0-\frac{\Delta z}{2}\right)\right]$$

where $K_{H_y} = \frac{\Delta x}{\Delta y}\tan^{-1}(\Delta y/\Delta x)$

Similarly

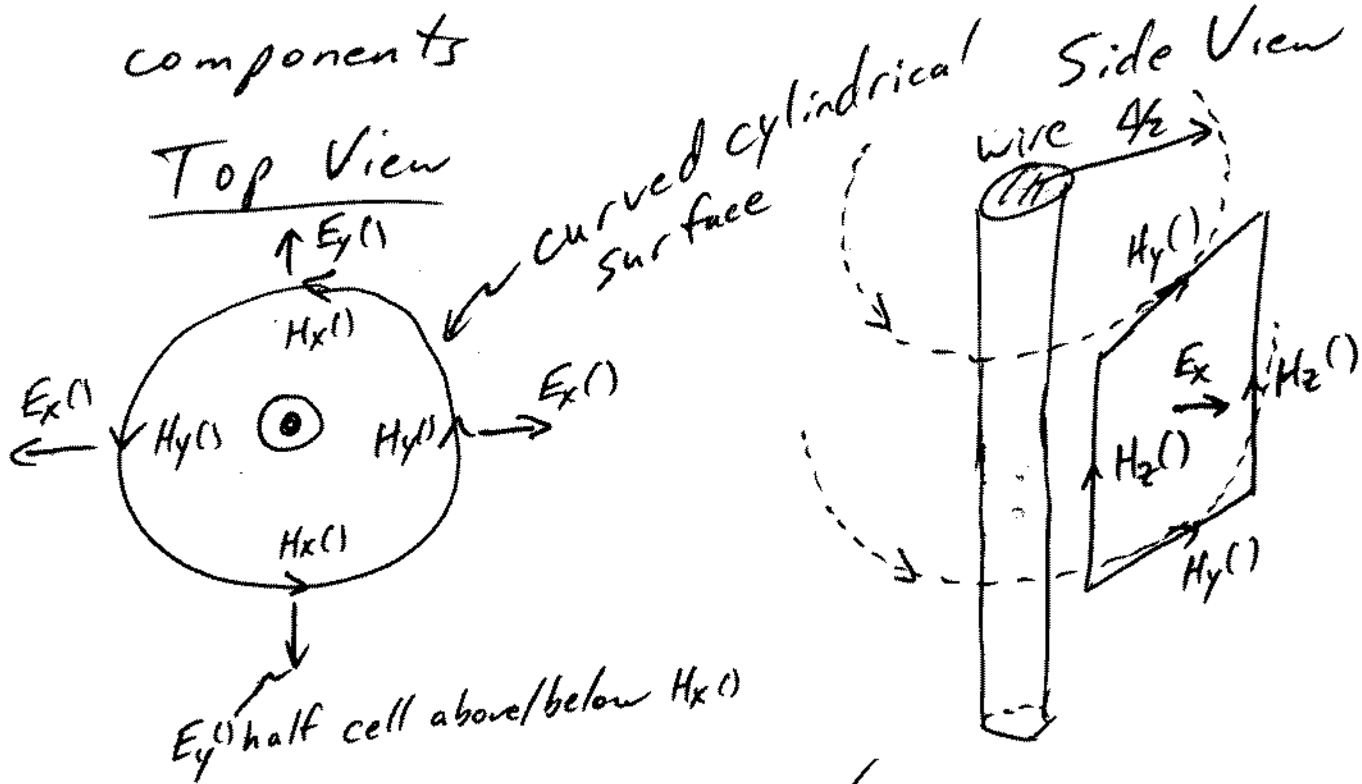
in front of wire in Fig 10.4

$$E_y^{n+1}\left(0, \frac{\Delta y}{2}, z_0-\frac{\Delta z}{2}\right) = E_y^{n}\left(0, \frac{\Delta y}{2}, z_0-\frac{\Delta z}{2}\right)$$
$$+\frac{\Delta t}{\epsilon_0 \Delta z}\left[K_{H_x} H_x^{n+0.5}\left(0, \frac{\Delta y}{2}, z_0\right) - K_{H_x} H_x^{n+0.5}\left(0, \frac{\Delta y}{2}, z_0-\Delta z\right)\right]$$
$$-\frac{\Delta t}{\epsilon_0 \Delta x}\left[H_z^{n+0.5}\left(\frac{\Delta x}{2}, \frac{\Delta y}{2}, z_0-\frac{\Delta z}{2}\right) - H_z^{n+0.5}\left(-\frac{\Delta x}{2}, \frac{\Delta y}{2}, z_0-\frac{\Delta z}{2}\right)\right]$$

where $K_{H_x} = \frac{\Delta y}{\Delta x}\tan^{-1}\left(\frac{\Delta x}{\Delta y}\right)$

10.5 cont.

* Radial electric field components are projected onto the cylindrical surface occupied by looping magnetic field components



↳ $H_z()$ components @ constant radius from wire on Cartesian path

↳ $H_y()$ components @ varying radius from wire on Cartesian FDTD update path

10.5 cont.

Here, similar scaling factors ($\mathcal{K}_{E_x}$ + $\mathcal{K}_{E_y}$) are used to replace $E_x()$ w/ $\mathcal{K}_{E_x} E_x()$ and $E_y()$ w/ $\mathcal{K}_{E_y} E_y()$ in the update equations for $H_x()$ and $H_y()$ looping about the wire

$$\mathcal{K}_E = \frac{\int_{S_{cart}} dS_{cart}}{\int_{S_{cyl}} dS_{cyl}} = \frac{\Delta z \int_{L_{cart,\perp}} d\ell_{cart,\perp}}{\Delta z \int_{L_{cyl,\perp}} d\ell_{cyl}} = \frac{1}{\mathcal{K}_H}$$

related to E in question

$$\mathcal{K}_{E_x} = \frac{1}{\frac{\Delta x}{\Delta y} \tan^{-1}(\Delta y/\Delta x)} = \frac{1}{\mathcal{K}_{H_y}}$$

$$\mathcal{K}_{E_y} = \frac{1}{\frac{\Delta y}{\Delta x} \tan^{-1}(\Delta x/\Delta y)} = \frac{1}{\mathcal{K}_{H_x}}$$

For example

$$\boxed{\begin{aligned} H_y^{n+0.5}\left(\tfrac{\Delta x}{2}, y_0, z_0\right) &= H_y^{n-0.5}\left(\Delta x/2, y_0, z_0\right) \\ &- \frac{\Delta t}{\mu_0 \Delta z}\left[\mathcal{K}_{E_x} E_x^n\left(\tfrac{\Delta x}{2}, y_0, z_0 + \tfrac{\Delta z}{2}\right) - \mathcal{K}_{E_x} E_x^n\left(\tfrac{\Delta x}{2}, y_0, z_0 - \tfrac{\Delta z}{2}\right)\right] \\ &+ \left(\frac{\Delta t}{\mu_0 \Delta x}\right)\left(\frac{2}{\ln(\Delta x/r_0)}\right) E_z^n(\Delta x, y_0, z_0) \end{aligned}}$$

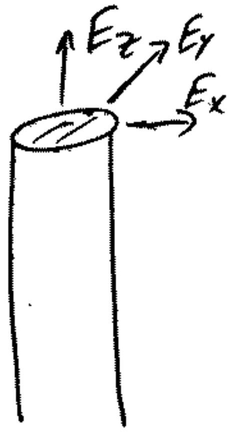
10.5 cont.

Note: If $\Delta x = \Delta y = \Delta$

ratio of perimeter of sqr to circum. of circle!

$$K_E = \frac{1}{(1)\tan^{-1}(1)} = 4/\pi = 1.273$$

$$K_H = (1)\tan^{-1}(1) = \pi/4 = 0.785$$



What about axial (E_z) and radial ($E_x + E_y$) electric fields near open ends of wire?

Mäkinen et. al dealt w/ this situation as well. See eq'ns (10.10) – (10.15) in text or [5]. → quite complicated underlying physics for wires of non-negligible radius due to charge accumulation on ends

10.7 Maloney-Smith Technique for Thin Material Sheets

→ deal w/ thin ($d < \Delta/2$) planar material ($\epsilon_s, \sigma_s, \mu_s = \mu_0$) sheets where sheet is perpend/parallel to a Cartesian coordinate axes.

10.7.1 Basis

→ consider fig 10.12 where we see a thin material sheet on a plane of constant x between $x = i\Delta x$ + $x = (i + 1/2)\Delta x$

→ In the x-direction for the H_y cells containing the material sheet, E_x is split into the parts w/in the sheet $E_{x,in}$ and outside the sheet $E_{x,out}$

New aux. variable

→ Must satisfy regular boundary (normal) condition for electric fields

→ No problems w/ magnetic fields since $\mu_{sht} = \mu_0$ (assume constant)

10.7.1 Cont.

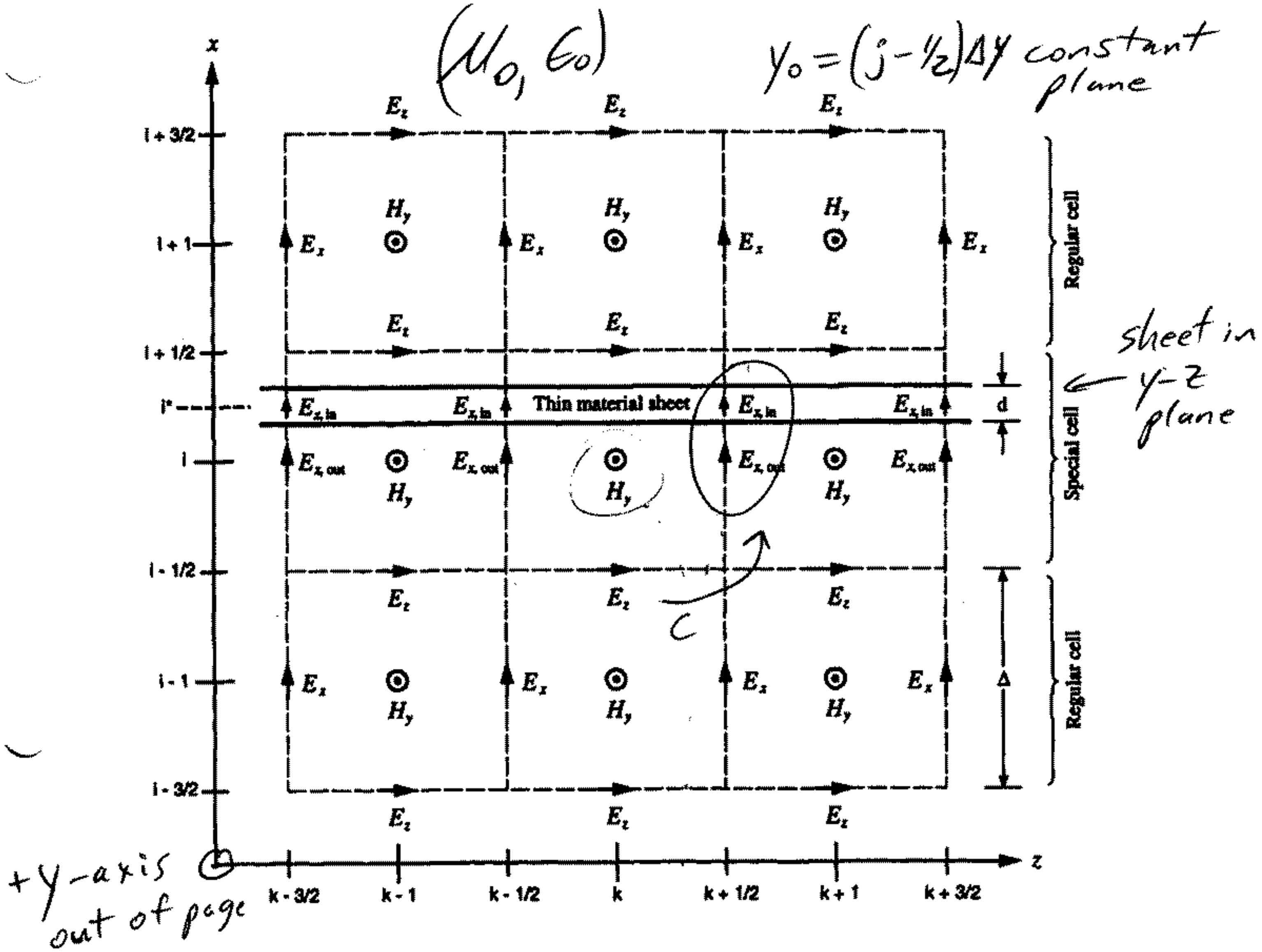


Fig. 10.12 A slice of the three-dimensional FDTD lattice at a j = constant plane, showing the locations of the field components used in the thin-sheet model. *Source:* Maloney and Smith, *IEEE Trans. Antennas and Propagation*, 1992, pp. 323–330, © 1992 IEEE.

Electric Field Updates

The Amperian contour for $E_{x,out}(i, j-1/2, k+1/2)$ is on a x-y plane parallel to (but outside of) the sheet. → Regular update eqn

10.7.1 cont.

$$E_{x,out}^{n+1}(i, j-1/2, k+1/2) = E_{x,out}^{n}(i, j-1/2, k+1/2) - \frac{\Delta t}{\epsilon_0 \Delta z}\left[H_y^{n+1/2}(i, j-1/2, k+1) - H_y^{n+1/2}(i, j-1/2, k)\right] + \frac{\Delta t}{\epsilon_0 \Delta y}\left[H_z^{n+1/2}(i, j, k+1/2) - H_z^{n+1/2}(i, j-1, k+1/2)\right]$$

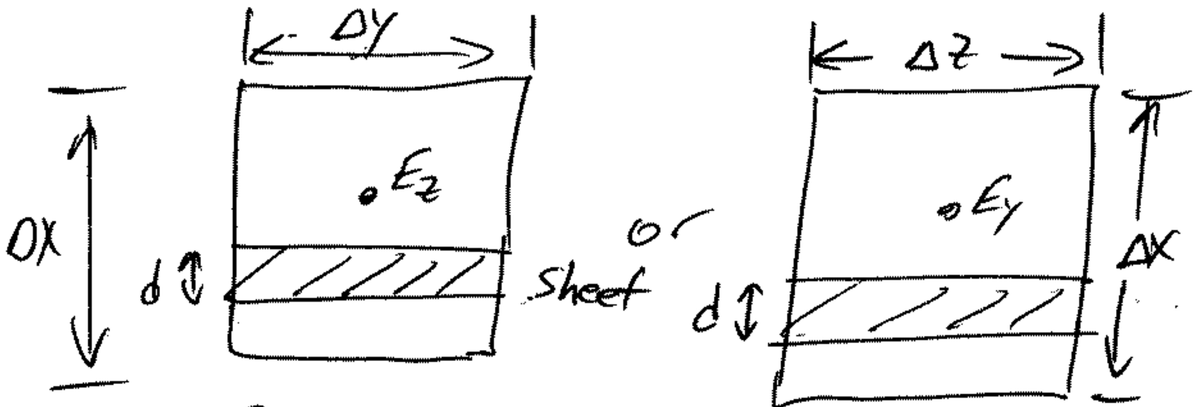
Normal free-space update eqn

Later, (for the H_y update) we'll need $E_{x,in}$. So, using the assumption that H_y is <u>constant</u> w/in the cell, the update eqn for $E_{x,in}$ is the standard one for a material where the electrical properties are $(\sigma_s, \epsilon_s, \mu_0)$

$$E_{x,in}^{n+1}(i^*, j-1/2, k+1/2) = \left(\frac{1 - \frac{\sigma_s \Delta t}{2\epsilon_s}}{1 + \frac{\sigma_s \Delta t}{2\epsilon_s}}\right) E_{x,in}^{n}(i^*, j-1/2, k+1/2) - \frac{\Delta t}{\epsilon_s \Delta z\left(1 + \frac{\sigma_s \Delta t}{2\epsilon_s}\right)}\left[H_y^{n+1/2}(i, j-1/2, k+1) - H_y^{n+1/2}(i, j-1/2, k)\right] + \frac{\Delta t}{\epsilon_s \Delta y\left(1 + \frac{\sigma_s \Delta t}{2\epsilon_s}\right)}\left[H_z^{n+1/2}(i, j, k+1/2) - H_z^{n+1/2}(i, j-1, k+1/2)\right]$$

10.7.1 cont.

What about E_z + E_y adjacent to sheet?



→ E_y + E_z are tangential to sheet ($E_{t_1} = E_{t_2}$)
→ just use average conductivity + permittivity in regular update eqns

$$\epsilon_{ave} = \left(1 - \frac{d}{\Delta x}\right)\epsilon_0 + \frac{d}{\Delta x}\epsilon_s$$

$$\sigma_{ave} = \left(1 - \frac{d}{\Delta x}\right)0 + \frac{d}{\Delta x}\sigma_s = \frac{d}{\Delta x}\sigma_s$$

Magnetic Field updates

H_x → Faraday contour outside of sheet, No change

10.7.1 cont.

H_y :→ Apply integral form of Faraday's Law $(t = n\Delta t)$

$$\frac{\partial}{\partial t}\iint \bar{B}\cdot d\bar{s} = -\oint_c \bar{\mathcal{E}}\cdot d\bar{\ell}$$

$$\frac{d}{dt}\left(\mu_0 H_y^n(i, j-\tfrac{1}{2}, k)\right)\Delta x \Delta z = -E_z^n(i-\tfrac{1}{2}, j-\tfrac{1}{2}, k)\Delta z$$

$$- E^n_{x,out}(i, j-\tfrac{1}{2}, k+\tfrac{1}{2})(\Delta x - d) - E^n_{x,in}(i^*, j-\tfrac{1}{2}, k+\tfrac{1}{2})\, d$$

$$+ E_z^n(i+\tfrac{1}{2}, j-\tfrac{1}{2}, k)\Delta z + E^n_{x,out}(i, j-\tfrac{1}{2}, k-\tfrac{1}{2})(\Delta x - d)$$

$$+ E^n_{x,in}(i^*, j-\tfrac{1}{2}, k-\tfrac{1}{2})\, d$$

$$\mu_0\left[\frac{H_y^{n+0.5}(i, j-\tfrac{1}{2}, k) - H_y^{n-0.5}(i, j-\tfrac{1}{2}, k)}{\Delta t}\right]\Delta x \Delta z$$

$$= -(\Delta x - d)\left[E^n_{x,out}(i, j-\tfrac{1}{2}, k+\tfrac{1}{2}) - E^n_{x,out}(i, j-\tfrac{1}{2}, k-\tfrac{1}{2})\right]$$

$$- d\left[E^n_{x,in}(i^*, j-\tfrac{1}{2}, k+\tfrac{1}{2}) - E^n_{x,in}(i^*, j-\tfrac{1}{2}, k-\tfrac{1}{2})\right]$$

$$+ \Delta z\left[E_z^n(i+\tfrac{1}{2}, j-\tfrac{1}{2}, k) - E_z^n(i-\tfrac{1}{2}, j-\tfrac{1}{2}, k)\right]$$

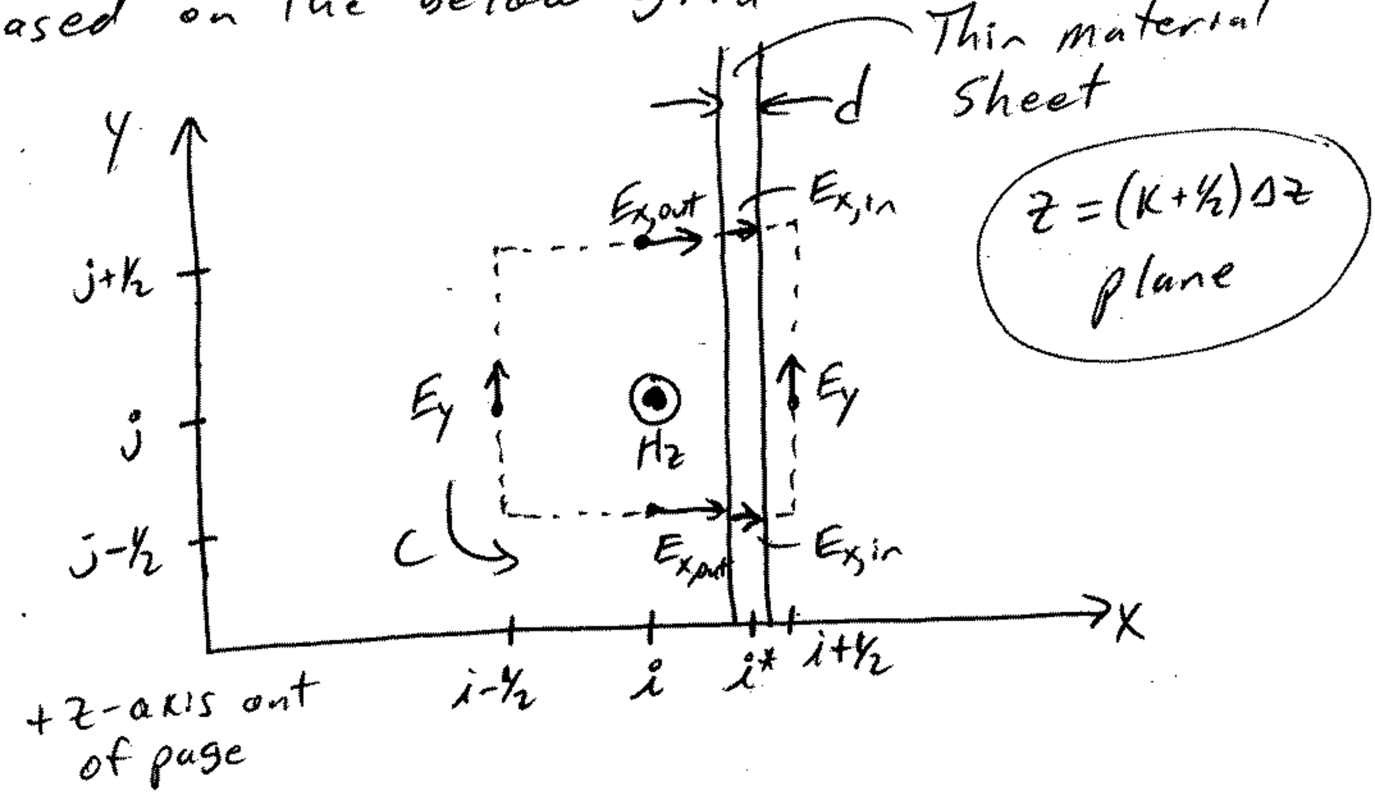
10.71 cont

Solving for the update equation yields;

$$\boxed{\begin{aligned} H_y^{n+0.5}(i, j-\tfrac{1}{2}, k) &= H_y^{n-0.5}(i, j-\tfrac{1}{2}, k) \\ &- \frac{\Delta t}{\mu_0 \Delta x \Delta z}\Big[(\Delta x - d)\left[E^n_{x,out}(i, j-\tfrac{1}{2}, k+\tfrac{1}{2}) - E^n_{x,out}(i, j-\tfrac{1}{2}, k-\tfrac{1}{2})\right] \\ &\quad + d\left[E^n_{x,in}(i^*, j-\tfrac{1}{2}, k+\tfrac{1}{2}) - E^n_{x,in}(i^*, j-\tfrac{1}{2}, k-\tfrac{1}{2})\right] \\ &\quad - \Delta z\left[E^n_z(i+\tfrac{1}{2}, j-\tfrac{1}{2}, k) - E^n_z(i-\tfrac{1}{2}, j-\tfrac{1}{2}, k)\right]\Big] \end{aligned}}$$

* I believe (10.23) in text has sign errors.

Similarly, the update equation for H_z near the thin material sheet can be found based on the below grid



$$\frac{\partial}{\partial t}\iint_S \bar{B}\cdot d\bar{s} = \frac{\mu_0\,\Delta x\,\Delta y}{\Delta t}\left[H_z^{n+0.5}(i,j,k+1/2) - H_z^{n-0.5}(i,j,k+1/2)\right]$$

$$= -\oint_C \bar{\mathcal{E}}\cdot d\bar{l} = -E_{x,out}^{n}(i,j-1/2,k+1/2)(\Delta x - d) - E_{x,in}^{n}(i^*,j-1/2,k+1/2)\,d$$

$$- E_y^{n}(i+1/2,j,k+1/2)\,\Delta y + E_{x,in}^{n}(i^*,j+1/2,k+1/2)\,d$$

$$+ E_{x,out}^{n}(i,j+1/2,k+1/2)(\Delta x - d) + E_y^{n}(i-1/2,j,k+1/2)\,\Delta y$$

$$\boxed{\begin{aligned} H_z^{n+0.5}(i,j,k+1/2) &= H_z^{n-0.5}(i,j,k+1/2) \\ &+ \frac{\Delta t}{\mu_0\,\Delta x\,\Delta y}\left[\begin{aligned} &(\Delta x - d)\left[E_{x,out}^{n}(i,j+1/2,k+1/2) - E_{x,out}^{n}(i,j-1/2,k+1/2)\right] \\ &+ d\left[E_{x,in}^{n}(i^*,j+1/2,k+1/2) - E_{x,in}^{*}(i^*,j-1/2,k+1/2)\right] \\ &- \Delta y\left[E_y^{n}(i+1/2,j,k+1/2) - E_y^{n}(i-1/2,j,k+1/2)\right] \end{aligned}\right] \end{aligned}}$$