Chapter 9 Dispersive, Nonlinear, and Gain Materials

9.1 Introduction

→ Some applications require the modeling of media which are dispersive, frequency dependent dielectric properties that can be nonlinear.

→ Good example is soil which is a lossy dispersive dielectric material.

4 Key EM characteristics

* Linear Dispersion – ϵ or μ varies w/ f at low field intensities

Nonlinearity – ϵ or μ vary w/ $|E|$ or $|H|$, usually at a high values

Nonlinear Dispersion – strength of nonlinearity varies w/ f

Gain – model a power transfer into an EM wave, causing it to gain strength as it propagates. Can be dispersive + nonlinear

9.2 Generic Isotropic Material Dispersions

→ list three important classes of linear isotropic material dispersions: Debye relaxation, Lorentzian resonance, + Drude model of metals

9.2.1 Debye Media

– characterized by a susceptibility function $\chi(\omega)$

e.g. Single pole

$$\chi_p(\omega) = \frac{\epsilon_{s,p} - \epsilon_{\infty,p}}{1 + j\omega\tau_p} = \frac{\Delta\epsilon_p}{1 + j\omega\tau_p} \qquad (9.1)$$

where

ϵ_{sp} is the static/DC relative permittivity,

$\epsilon_{\infty,p}$ is the high frequency relative permittivity,

$\Delta\epsilon_p$ is the change in ϵ_r due to the Debye pole,

and τ_p is the pole relaxation time.

In the time domain

$$\chi_p(t) = \frac{\Delta\epsilon_p}{\tau_p} e^{-t/\tau_p} U(t)$$

For multiple (P) poles, the relative permittivity becomes

$$\epsilon_r(\omega) = \epsilon_{r,\infty} + \sum_{p=1}^{P} \frac{\Delta\epsilon_p}{1 + j\omega\tau_p}$$

9.2.1 cont.

As a little review, remember

$$\underset{\uparrow \text{complex permittivity}}{\epsilon_c} = \epsilon' - j\epsilon'' = \epsilon_0(\epsilon_r' - j\epsilon_r'') = \underbrace{\epsilon\left[1 - j\frac{\sigma'}{\omega\epsilon}\right]}_{\Downarrow}$$

from $\bar{\nabla}\times\bar{H} = \bar{J} + j\omega\bar{D} = \underset{\uparrow \text{static conductivity}}{\sigma_0}\bar{E} + j\omega\epsilon\left[1 - j\frac{\sigma'}{\omega\epsilon}\right]\bar{E}$

$$= \sigma_0\bar{E} + (j\omega\epsilon + \sigma')\bar{E}$$

$$= (\sigma_0 + \sigma')\bar{E} + j\omega\epsilon\bar{E}$$

where $\bar{D} = \epsilon_0\bar{E} + \underset{\uparrow \text{polarization vector}}{\bar{P}} = \epsilon_0\bar{E} + \chi\epsilon_0\bar{E}$

for a single pole

$$\chi_p = \frac{\epsilon_{sp} - \epsilon_{\infty p}}{1 + j\omega\tau_p}\left(\frac{1 - j\omega\tau_p}{1 - j\omega\tau_p}\right) = \chi_p' - j\chi_p''$$

$$= \frac{\epsilon_{sp} - \epsilon_{\infty p}}{1 + \omega^2\tau_p^2} - j\frac{(\epsilon_{sp} - \epsilon_{\infty p})\omega\tau_p}{1 + \omega^2\tau_p^2}$$

Now $\chi = \underset{\substack{\uparrow \\ \text{Debye component}}}{\chi_p} + \underset{\substack{\uparrow \\ \text{static component (all freqs)}}}{\chi_t}$ ← overall electric susceptance

9.2.1 cont.

$$\epsilon_r = \underbrace{\overbrace{1 + \chi_t}^{\epsilon_{\infty p}} + \frac{\epsilon_{sp} - \epsilon_{\infty p}}{1 + \omega^2 \tau_p^2}}_{\epsilon_r'} - j \underbrace{\frac{(\epsilon_{sp} - \epsilon_{\infty p})}{1 + \omega^2 \tau_p^2} \omega \tau_p}_{\epsilon_r''}$$

$$\epsilon = \epsilon_r' \epsilon_0$$

$$= \epsilon_0 \left[\epsilon_{\infty p} + \frac{\epsilon_{sp} - \epsilon_{\infty p}}{1 + \omega^2 \tau_p^2} \right]$$

ϵ_r' called effective relative permittivity

$$\frac{\sigma'}{\omega} = \epsilon_r'' \epsilon_0 \rightarrow \sigma' = \epsilon_r'' \epsilon_0 \omega$$

$$\sigma' = \frac{\epsilon_{sp} - \epsilon_{\infty p}}{1 + \omega^2 \tau_p^2} \omega^2 \tau_p \epsilon_0$$

effective conductivity due to ϵ_r''

$$\sigma = \sigma_0 + \frac{\epsilon_{sp} - \epsilon_{\infty p}}{1 + \omega^2 \tau_p^2} \omega^2 \tau_p \epsilon_0$$

overall effective conductivity

9.2.1 cont.

A good example of a Debye media is soil. The frequency dependence of ϵ + σ in soils is mainly due to the water content over frequencies up to ~20 GHz. They are modeled as (due to a single pole)

$$\epsilon_r(\omega) = \epsilon_{r,\infty} + \frac{\epsilon_{r,s} - \epsilon_{r,\infty}}{1+\omega^2\tau^2}$$

$$\sigma(\omega) = \sigma_0 + \frac{\epsilon_{r,s} - \epsilon_{r,\infty}}{1+\omega^2\tau^2}\,\omega^2\tau\epsilon_0$$

Some typical values are:

Soil	$\epsilon_{r,s}$	$\epsilon_{r,\infty}$	σ_0 (S/m)	τ (ps)
dry (5%)	4	2.5	0.005	9
medium (10%)	8	3.5	0.02	9
wet (20%)	15	5	0.05	9

τ is for water @ 70°F

Soil_debye_plots.pdf

Plot the effective relative permittivity and conductivity of soils with Debye dependence.

$\tau := 9\cdot 10^{-12}$ s $\qquad \varepsilon 0 := 8.854\cdot 10^{-12}$ F/m

$\varepsilon rs\_dry := 4 \qquad \varepsilon rh\_dry := 2.5 \qquad \sigma 0\_dry := 0.005$ S/m

$\varepsilon rs\_med := 8 \qquad \varepsilon rh\_med := 3.5 \qquad \sigma 0\_med := 0.02$ S/m

$\varepsilon rs\_wet := 15 \qquad \varepsilon rh\_wet := 5 \qquad \sigma 0\_wet := 0.05$ S/m

$k := 1 .. 3000 \qquad f_k := k\cdot 10^{-2}\cdot 10^{9} \qquad \omega_k := 2\cdot\pi\cdot f_k$

$$\varepsilon r\_dry_k := \varepsilon rh\_dry + \frac{\varepsilon rs\_dry - \varepsilon rh\_dry}{1 + (\omega_k)^2\cdot\tau^2}$$

$$\varepsilon r\_med_k := \varepsilon rh\_med + \frac{\varepsilon rs\_med - \varepsilon rh\_med}{1 + (\omega_k)^2\cdot\tau^2}$$

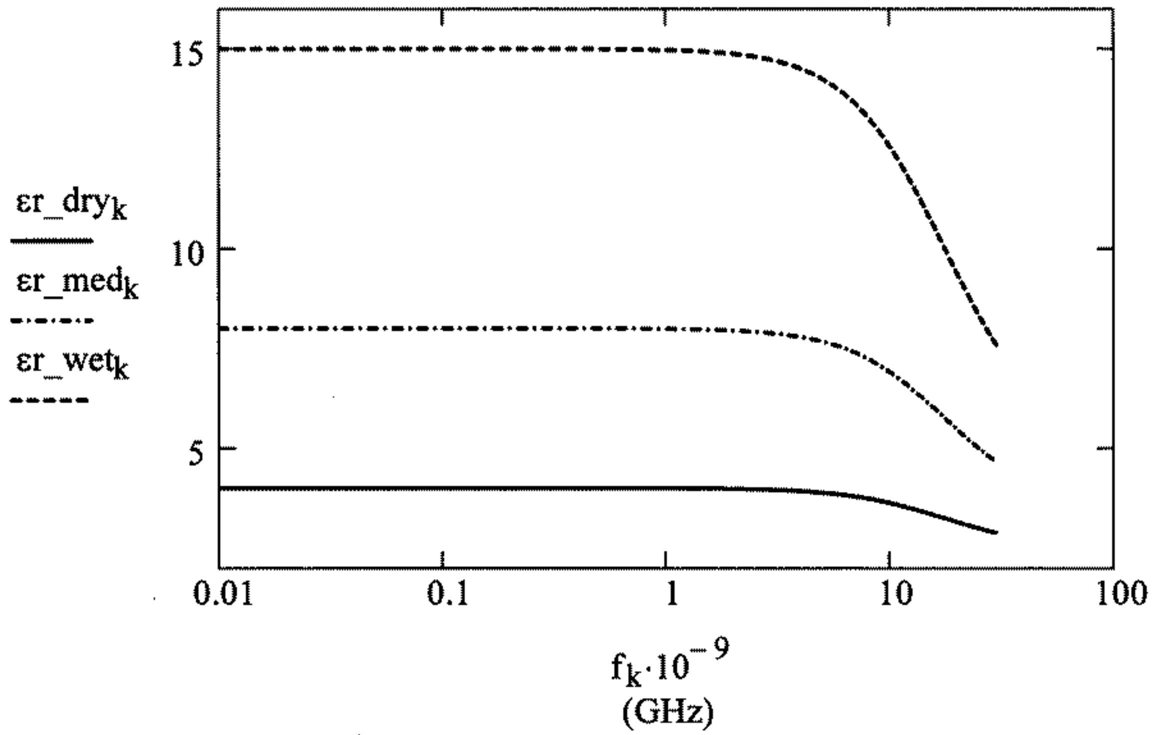
$$\varepsilon r\_wet_k := \varepsilon rh\_wet + \frac{\varepsilon rs\_wet - \varepsilon rh\_wet}{1 + (\omega_k)^2\cdot\tau^2}$$

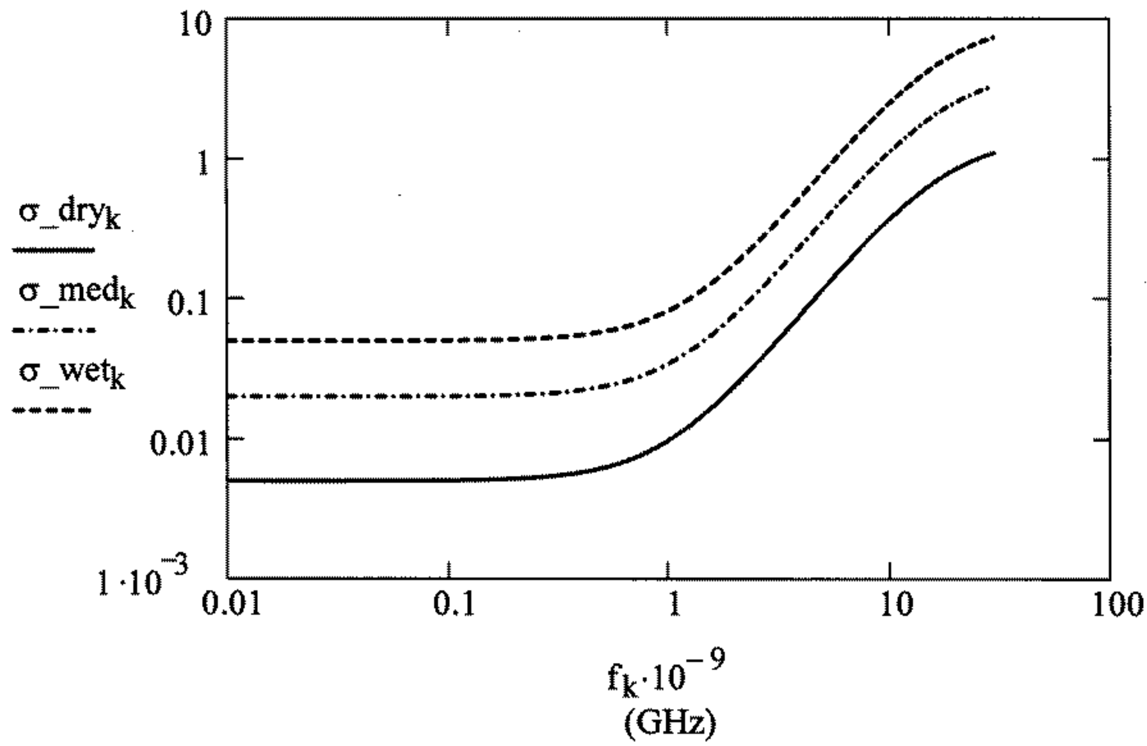
$$\sigma\_dry_k := \sigma 0\_dry + \frac{\varepsilon rs\_dry - \varepsilon rh\_dry}{1 + (\omega_k)^2\cdot\tau^2}\cdot\left[(\omega_k)^2\cdot\tau\cdot\varepsilon 0\right]$$

$$\sigma\_med_k := \sigma 0\_med + \frac{\varepsilon rs\_med - \varepsilon rh\_med}{1 + (\omega_k)^2\cdot\tau^2}\cdot\left[(\omega_k)^2\cdot\tau\cdot\varepsilon 0\right]$$

$$\sigma\_wet_k := \sigma 0\_wet + \frac{\varepsilon rs\_wet - \varepsilon rh\_wet}{1 + (\omega_k)^2\cdot\tau^2}\cdot\left[(\omega_k)^2\cdot\tau\cdot\varepsilon 0\right]$$

Debye dependence continued





9.2.2 Lorentz Media

Here the material depends on complex conjug. pole pairs

$$\chi_p(\omega) = \frac{\Delta\epsilon_p \omega_p^2}{\omega_p^2 + 2j\omega\delta_p - \omega^2}$$ ← single pole pair

where $\Delta\epsilon_p = \epsilon_{s,p} - \epsilon_{\infty,p}$

ω_p - frequency of pole pair

δ_p - damping coefficient

$$\chi_p(t) = \frac{\Delta\epsilon_p \omega_p^2}{\sqrt{\omega_p^2 - \delta_p^2}} e^{-\delta_p t} \sin\left(\sqrt{\omega_p^2 - \delta_p^2}\, t\right) u(t)$$

For multiple pole pairs

$$\epsilon(\omega) = \epsilon_\infty + \sum_{p=1}^{P} \frac{\Delta\epsilon_p \omega_p^2}{\omega_p^2 + 2j\omega\delta_p - \omega^2}$$

9.2.3 Drude Media

- optical regime
- accounts for EM wave interaction w/ metals internal electron motion

$$\chi_p(\omega) = \frac{-\omega_p^2}{\omega^2 - j\omega\gamma_p}$$ ← single pole

where ω_p is the Drude pole frequency
γ_p is the inverse of the pole relaxation time

$$\chi_p(t) = \frac{\omega_p^2}{\gamma_p}\left(1 - e^{-\gamma_p t}\right) u(t)$$

For multiple poles

$$\epsilon(\omega) = \epsilon_\infty - \sum_{p=1}^{P} \frac{\omega_p^2}{\omega^2 - j\omega\gamma_p}$$

9.4 Auxiliary Differential Equation Method, Linear Material Case

→ avoids complex number calculations

9.4.1 Formulation for Multiple Debye Poles

Ampere's Law can be written as

$$\bar{\nabla} \times \bar{\mathcal{H}} = \epsilon_0 \epsilon_{r\infty} \frac{\partial \bar{\mathcal{E}}}{\partial t} + \underset{\uparrow \text{really } \sigma_0}{\sigma} \bar{\mathcal{E}} + \sum_{p=1}^{P} \bar{\mathcal{J}}_p \quad \swarrow \text{polarization currents}$$

Now the goal is to get $\bar{\mathcal{J}}_p$ updates that we can use in the update equations for $\bar{\mathcal{E}}$ derived from Ampere's Law.

Going back to the phasor domain

$$\bar{\nabla} \times \bar{H} = \underbrace{\sigma_0 \bar{E}}_{\substack{\text{conduction}\\ \text{current}\\ \text{(static)}}} + \underbrace{j\omega\epsilon \bar{E}}_{\substack{\text{displacement}\\ \text{current}}}$$

using $\epsilon(\omega) = \epsilon_r(\omega)\,\epsilon_0 = \left[\epsilon_{r\infty} + \sum_{p=1}^{P} \frac{\Delta\epsilon_p}{1 + j\omega\tau_p}\right]\epsilon_0$

9.4.1 cont.

$$\bar\nabla \times \bar H = \sigma_0 \bar E + j\omega\epsilon_0\left[\epsilon_{r\infty} + \sum_{p=1}^{P} \frac{\Delta\epsilon_p}{1+j\omega\tau_p}\right]\bar E$$

$$= \sigma_0 \bar E + j\omega\epsilon_0\epsilon_{r\infty}\bar E + \underbrace{\sum_{p=1}^{P}\frac{\Delta\epsilon_p}{1+j\omega\tau_p}\, j\omega\epsilon_0\bar E}_{\sum J_p}$$

Using $\frac{\partial \bar{\mathcal{E}}}{\partial t} \longleftrightarrow j\omega\bar{\mathcal{E}}$, do an inverse Fourier transform to get

$$\bar\nabla\times\bar{\mathcal{H}} = \epsilon_0\epsilon_{r\infty}\frac{\partial\bar{\mathcal{E}}}{\partial t} + \sigma_0\bar{\mathcal{E}} + \sum_{p=1}^{P}\bar{\mathcal{J}}_p$$

Therefore, in the phasor domain, a single pole Debye current

$$\bar J_p = \frac{j\omega\epsilon_0\Delta\epsilon_p}{1+j\omega\tau_p}\bar E = \epsilon_0\Delta\epsilon_p\left(\frac{j\omega}{1+j\omega\tau_p}\right)\bar E$$

multiply both sides by $(1+j\omega\tau_p)$

$$\bar J_p + j\omega\tau_p\bar J_p = \epsilon_0\Delta\epsilon_p\, j\omega\bar E$$

Doing an inverse Fourier Transform

$$\bar{\mathcal{J}}_p + \tau_p\frac{\partial\bar{\mathcal{J}}_p}{\partial t} = \epsilon_0\Delta\epsilon_p\frac{\partial\bar{\mathcal{E}}}{\partial t} \leftarrow \text{ADE!}$$

9.4.1 cont.

Notes: * $\bar{J}_p$ takes care of both the changing (w/ frequency) portion of σ + ϵ

* $\bar{J}_p$ co-located w/ $\bar{E}$ nodes in FDTD grid

Discretizing about $t = (n+\frac{1}{2})\Delta t$ using the standard central-difference FDTD approximation yields

$$\left(\frac{J_p\big|_{()}^{n+1} + J_p\big|_{()}^{n}}{2}\right) + \tau_p\left(\frac{J_p\big|_{()}^{n+1} - J_p\big|_{()}^{n}}{\Delta t}\right) = \epsilon_0 \Delta\epsilon_p \left(\frac{E\big|_{()}^{n+1} - E\big|_{()}^{n}}{\Delta t}\right)$$

$$J_p\big|_{()}^{n+1}\left(\frac{1}{2} + \frac{\tau_p}{\Delta t}\right) - J_p\big|_{()}^{n}\left(\frac{\tau_p}{\Delta t} - \frac{1}{2}\right) = \epsilon_0 \Delta\epsilon_p \left(\frac{E\big|_{()}^{n+1} - E\big|_{()}^{n}}{\Delta t}\right)$$

$$J_p\big|_{()}^{n+1} = \left(\frac{\frac{\tau_p}{\Delta t} - \frac{1}{2}}{\frac{\tau_p}{\Delta t} + \frac{1}{2}}\right) J_p\big|_{()}^{n} + \left(\frac{\epsilon_0 \Delta\epsilon_p}{\frac{\tau_p}{\Delta t} + \frac{1}{2}}\right)\left(\frac{E\big|_{()}^{n+1} - E\big|_{()}^{n}}{\Delta t}\right)$$

$$J_p^{n+1} = \left(\frac{1 - \frac{\Delta t}{2\tau_p}}{1 + \frac{\Delta t}{2\tau_p}}\right) J_p^{n} + \left(\frac{\epsilon_0 \Delta\epsilon_p \Delta t / \tau_p}{1 + \Delta t / 2\tau_p}\right)\left(\frac{E\big|_{()}^{n+1} - E\big|_{()}^{n}}{\Delta t}\right)$$

Define $k_p = \left(\frac{1 - \Delta t / 2\tau_p}{1 + \frac{\Delta t}{2\tau_p}}\right)$ + $\beta_p = \frac{\epsilon_0 \Delta\epsilon_p \Delta t / \tau_p}{1 + \Delta t / 2\tau_p}$

$$\boxed{J_p^{n+1} = k_p J_p^{n} + \beta_p \left(\frac{E^{n+1} - E^{n}}{\Delta t}\right)} \quad (9.39)$$

classic chicken + the egg dilemma, how do we get E^{n+1}?

9.4.1 cont.

Assume/approximate

$$J_p^{n+1/2} \simeq \frac{J_p^{n+1} + J_p^n}{2} = \frac{K_p J_p^n + \beta_p\left(\frac{E^{n+1}-E^n}{\Delta t}\right) + J_p^n}{2}$$

$$= \tfrac{1}{2}\left[(1+K_p) J_{p()}^n + \frac{\beta_p}{\Delta t}\left(E_{()}^{n+1} - E_{()}^n\right)\right]$$

Now, go back to Ampere's Law + discretize wrt time about $t=(n+\tfrac{1}{2})\Delta t$

$$\bar{\nabla} \times \bar{H}_{()}^{n+1/2} = \epsilon_0 \epsilon_{r\infty}\left(\frac{E_{()}^{n+1} - E_{()}^n}{\Delta t}\right) + \sigma\left(\frac{E_{()}^{n+1} + E_{()}^n}{2}\right)$$

$$+ \sum_{p=1}^{P} J_p^{n+1/2}$$

substitute in above expression

$$\bar{\nabla} \times \bar{H}_{()}^{n+1/2} = \epsilon_0 \epsilon_{r\infty}\left(\frac{\bar{E}_{()}^{n+1} - \bar{E}_{()}^n}{\Delta t}\right) + \sigma\left(\frac{\bar{E}_{()}^{n+1} + \bar{E}_{()}^n}{2}\right)$$

$$+ \sum_{p=1}^{P} \tfrac{1}{2}\left[(1+K_p)\bar{J}_p^n + \frac{\beta_p}{\Delta t}\left(\bar{E}^{n+1} - \bar{E}^n\right)\right]$$

Now, solve for $\bar{E}^{n+1}$ interms of the other variables (either constants or past values)

9.4.1 cont.

$$\bar{E}^{n+1} = \left[\frac{2\epsilon_0 \epsilon_{r\infty} + \sum_{p=1}^{P} \beta_p - \sigma \Delta t}{2\epsilon_0 \epsilon_{r\infty} + \sum_{p=1}^{P} \beta_p + \sigma \Delta t}\right] \bar{E}^n + \left(\frac{2\Delta t}{2\epsilon_0 \epsilon_{r\infty} + \sum_{p=1}^{P} \beta_p + \sigma \Delta t}\right)\left[\bar{\nabla} \times \bar{H}^{n+1/2} - \frac{1}{2}\sum_{p=1}^{P}(1+k_p)\bar{J}_p^n\right] \quad (9.43)$$

General Procedure

n = 1, Nmax % time loop

i = 1, Iexmax

j = 2, Jexmax

k = 2, Kexmax

$Exold = E_x^n(i,j,k)$ % save old value

$E_x^{n+1}(i,j,k)$ = see above (9.43)

$p = 1, P$

$J_{x(i,j,k)}^{n+1}$ = see (9.39) + use Exold for $E_x^n(i,j,k)$

continue p

continue k

continue j

continue i

9.4.1 cont.

Repeat for E_y $J_{p,y}$

Repeat for E_z + $J_{p,z}$

Regular H_x updates + loops

Regular H_y updates + loops

Regular H_z updates + loops

Based on Faraday's Law $\bar{\nabla} \times \bar{E} =$ (No $\epsilon(\omega)$ to worry about)

continue n

Similar procedure shown in sections 9.4.2 + 9.4.3 for Lorentz + Drude media