Chapter 7 Perfectly Matched Layer Absorbing Boundary Conditions

7.1 Introduction

→ use some absorbing medium/material to absorb waves at boundaries of FDTD modeling space

→ medium/material may or may not be physically realizable

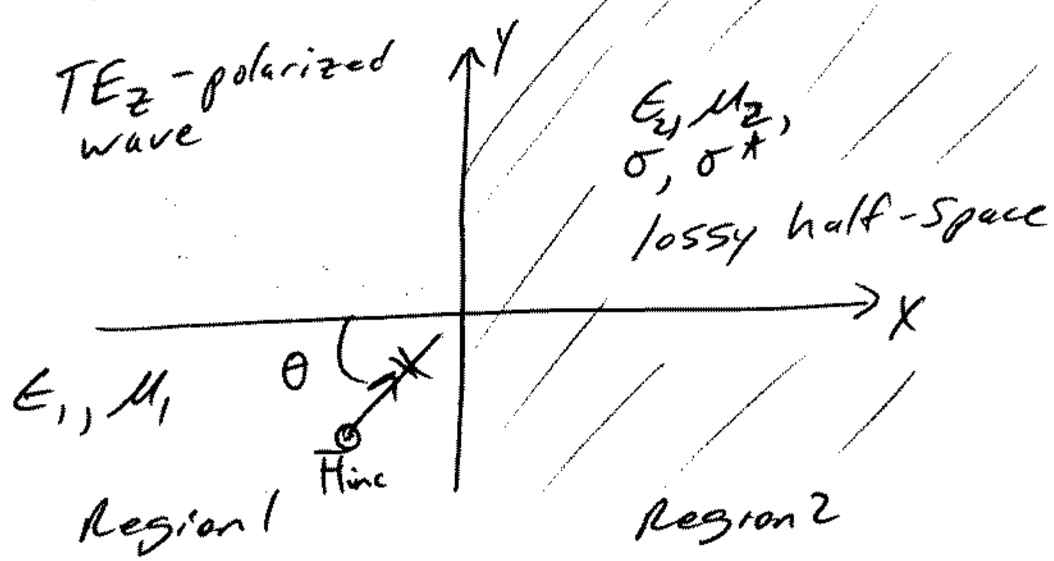
→ ideally a perfectly matched layer (PML) will absorb waves at any angle of incidence, polarization, or frequency. Also, it should work w/ or adjacent to modeled materials that are lossy, dispersive, inhomogeneous, anisotropic, non-linear, ... (cause big problems w/ analytic ABCs)

→ Berenger (1996) pioneered PMLs w/ a split-field formulation (12 differential scalar eqns instead of 6)

→ Other PMLs include a stretched-coordinate formulation, a convolutional PML (CPML), and a uniaxial PML (UPML).

7.2 Plane Wave Incident upon a Lossy Half-Space

→ background for PMLs



Assuming sinusoidal time-variation, the incident magnetic field of the wave is

$$\bar{H}_{inc} = \hat{a}_z H_0 e^{-j\beta_{1x}x} e^{-j\beta_{1y}y}$$

The total magnetic field in Region 1, including the reflection from the $x=0$ interface, is

$$\bar{H}_1 = \hat{a}_z H_0 \left(1 + \Gamma e^{j2\beta_{1x}x}\right) e^{-j\beta_{1x}x} e^{-j\beta_{1y}y}$$

with corresponding electric field of

$$\bar{E}_1 = \left[-\hat{a}_x \frac{\beta_{1y}}{\omega\epsilon_1}\left(1+\Gamma e^{j2\beta_{1x}x}\right) + \hat{a}_y \frac{\beta_{1x}}{\omega\epsilon_1}\left(1-\Gamma e^{j2\beta_{1x}x}\right)\right] H_0 e^{-j\beta_{1x}x} e^{-j\beta_{1y}y}$$

7.2 cont.

The transmitted fields into Region 2 are

$$\bar{H}_2 = \hat{a}_z H_o \tau e^{-j\beta_{2x}x} e^{-j\beta_{2y}y}$$

$$\bar{E}_2 = \left[-\hat{a}_x \frac{\beta_{2y}}{\omega\epsilon_2\left(1+\frac{\sigma}{j\omega\epsilon_2}\right)} + \hat{a}_y \frac{\beta_{2x}}{\omega\epsilon_2\left(1+\frac{\sigma}{j\omega\epsilon_2}\right)}\right] H_o \tau e^{-j\beta_{2x}x} e^{-j\beta_{2y}y}$$

Now let's define some coefficients/variables

Region 1

$\beta_{1x} = K_1 \cos\theta$ where $K_1 = \omega\sqrt{\mu_1\epsilon_1}$

$\beta_{1y} = K_1 \sin\theta$

What about in Region 2? After applying the $\bar{E}$ + $\bar{H}$ tangential boundary conditions at $x=0$, we get

$$\beta_{2y} = \beta_{1y} = K_1 \sin\theta$$

$$\beta_{2x} = \sqrt{K_2^2\left(1+\frac{\sigma}{j\omega\epsilon_2}\right)\left(1+\frac{\sigma^*}{j\omega\mu_2}\right) - \beta_{2y}^2}$$

where $K_2 = \omega\sqrt{\mu_2\epsilon_2}$

7.2 cont.

the reflection (Γ) and transmission (τ) coefficients are

$$\Gamma = \frac{\dfrac{\beta_{1x}}{\omega\epsilon_1} - \dfrac{\beta_{2x}}{\omega\epsilon_2(1+\sigma/j\omega\epsilon_2)}}{\dfrac{\beta_{1x}}{\omega\epsilon_1} + \dfrac{\beta_{2x}}{\omega\epsilon_2(1+\sigma/j\omega\epsilon_2)}} \qquad (7.5a,b)$$

$$\tau = 1+\Gamma$$

Defining the intrinsic wave impedances of the two regions as

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \quad \text{and} \quad \eta_2 = \sqrt{\frac{\mu_2(1+\sigma^*/j\omega\mu_2)}{\epsilon_2(1+\sigma/j\omega\epsilon_2)}}$$

At $\theta=0$ (normal incidence) where $\beta_{1y}=\beta_{2y}=0$

$$\Gamma = \frac{\eta_1-\eta_2}{\eta_1+\eta_2}$$

then $\underline{\underline{\Gamma=0 \ @ \ \theta=0}}$ if $\epsilon_1=\epsilon_2$, $\mu_1=\mu_2$, and $\dfrac{\sigma^*}{\mu_1}=\sigma/\epsilon_1$ ($k_1=k_2$ & $\eta_1=\eta_2$)

7.2 cont.

and $\beta_{2x} = \left(1 + \frac{\sigma}{j\omega\epsilon_1}\right)\kappa_1 = \kappa_1 - j\sigma\eta_1$

What about the transmitted fields?

$$\bar{E}_2 = \hat{a}_y \eta_1 H_0 e^{-j\kappa_1 x} e^{-\sigma\eta_1 x}$$

exponential decay!

$$\bar{H}_2 = \hat{a}_z H_0 e^{-j\kappa_1 x} e^{-\sigma\eta_1 x}$$

dispersionless as well

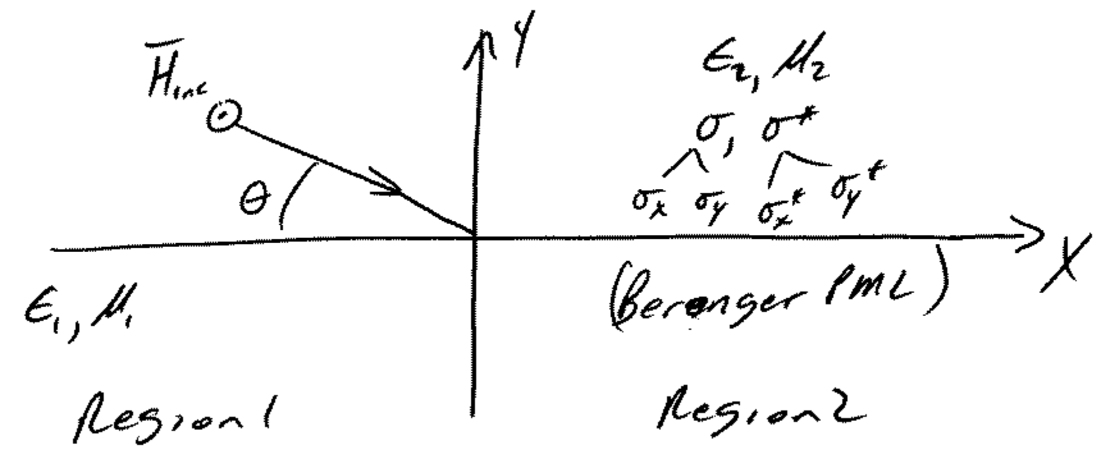
7.3 Plane Wave Incident Upon Berenger's PML Medium

→ idea is to replicate the $\Gamma = 0$ for normal incidence performance of the lossy media of section 7.2 for all angles of incidence

→ Berenger introduced additional degrees of freedom by doing field-splitting

→ improvement of ~70dB over 2nd order Mur ABC ← Huge!

7.3.1 Two-Dimensional TE_z Case



Now the regular governing diff. eq'ns for the TE_z case w/ no sources are

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\frac{\partial H_z}{\partial y} - \frac{\sigma}{\epsilon}E_x$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x} - \frac{\sigma}{\epsilon}E_y$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\frac{\partial E_x}{\partial y} - \frac{1}{\mu}\frac{\partial E_y}{\partial x} - \frac{\sigma^*}{\mu}H_z$$

7.3.1 cont.

Berenger modified them to be in Region 2

$$\epsilon_2 \frac{\partial E_x}{\partial t} + \sigma_y E_x = \frac{\partial H_z}{\partial y} \qquad (7.11a)$$

$$\epsilon_2 \frac{\partial E_y}{\partial t} + \sigma_x E_y = -\frac{\partial H_z}{\partial x} \qquad (7.11b)$$

$$\mu_2 \frac{\partial H_{zx}}{\partial t} + \sigma_x^* H_{zx} = -\frac{\partial E_y}{\partial x} \qquad (7.11c)$$

$$\mu_2 \frac{\partial H_{zy}}{\partial t} + \sigma_y^* H_{zy} = \frac{\partial E_x}{\partial y} \qquad (7.11d)$$

where $H_z = H_{zx} + H_{zy}$ (7.12)

Note: $\sigma \rightarrow \sigma_x + \sigma_y$

$\sigma^* \rightarrow \sigma_x^* + \sigma_y^*$

& magnetic time derivative eqn split into 2 parts

Possibilities: $\sigma_y = \sigma_y^* = 0 \rightarrow$ medium attenuates x-prop. waves (E_x & H_{zx}) but lossless for y-prop. waves (E_y & H_{zy})

$\sigma_x = \sigma_x^* = 0 \rightarrow$ vice versa

$\rightarrow$ leads to idea of getting $\Gamma_x = \Gamma_y = 0$ per Section 7.2

7.3.1 cont.

Next, put these diff. eqn in time-harmonic (phasor) form

$$j\omega\epsilon_z\left(1+\frac{\sigma_y}{j\omega\epsilon_z}\right)E_x = \frac{\partial}{\partial y}\left(H_{zx}+H_{zy}\right) \qquad (7.13a)$$

$$j\omega\epsilon_z\left(1+\frac{\sigma_x}{j\omega\epsilon_z}\right)E_y = -\frac{\partial}{\partial x}\left(H_{zx}+H_{zy}\right) \qquad (7.13b)$$

$$j\omega\mu_z\left(1+\frac{\sigma_x^*}{j\omega\mu_z}\right)H_{zx} = -\frac{\partial E_y}{\partial x} \qquad (7.13c)$$

$$j\omega\mu_z\left(1+\frac{\sigma_y^*}{j\omega\mu_z}\right)H_{zy} = \frac{\partial E_x}{\partial y} \qquad (7.13d)$$

To simplify the notation, let

$$S_x = 1+\frac{\sigma_x}{j\omega\epsilon_z} \quad \text{and} \quad S_y = 1+\frac{\sigma_y}{j\omega\epsilon_z}$$

$$S_x^* = 1+\frac{\sigma_x^*}{j\omega\mu_z} \quad \text{and} \quad S_y^* = 1+\frac{\sigma_y^*}{j\omega\mu_z}$$

Next, find a the plane-wave that propagates into the Berenger PML

→ Take $\frac{\partial}{\partial y}$ of (7.13a) $\quad j\omega\epsilon_z S_y \frac{\partial E_x}{\partial y} = \frac{\partial^2}{\partial y^2}\left(H_{zx}+H_{zy}\right)$

→ Take $\frac{\partial}{\partial x}$ of (7.13b) $\quad j\omega\epsilon_z S_x \frac{\partial E_y}{\partial x} = -\frac{\partial^2}{\partial x^2}\left(H_{zx}+H_{zy}\right)$

7.3.1 cont.

Sub (7.13c) + (7.13d) into these equations

$$j\omega\epsilon_2 S_y \, j\omega\mu_2 S_y^* H_{zy} = \frac{\partial^2}{\partial y^2}\left(H_{zx} + H_{zy}\right)$$

$$j\omega\epsilon_2 S_x\left(-j\omega\mu_2 S_x^* H_{zx}\right) = -\frac{\partial^2}{\partial x^2}\left(H_{zx} + H_{zy}\right)$$

↓ Re-arrange

$$-\omega^2\mu_2\epsilon_2 H_{zx} = -\frac{1}{S_x^*}\frac{1}{S_x}\frac{\partial^2}{\partial x^2}\left(H_{zx} + H_{zy}\right)$$

$$-\omega^2\mu_2\epsilon_2 H_{zy} = -\frac{1}{S_y^*}\frac{1}{S_y}\frac{\partial^2}{\partial y^2}\left(H_{zx} + H_{zy}\right)$$

adding the equations + using $H_z = H_{zx} + H_{zy}$

$$\boxed{\frac{1}{S_x^* S_x}\frac{\partial^2(H_z)}{\partial x^2} + \frac{1}{S_y^* S_y}\frac{\partial^2(H_z)}{\partial y^2} + \omega^2\mu_2\epsilon_2 H_z = 0} \quad (7.17)$$

wave equation

A solution to the wave equation (in PML region) is

$$H_z = H_0 \tau e^{-j\sqrt{S_x S_x^*}\,\beta_{2x} x} e^{-j\sqrt{S_y S_y^*}\,\beta_{2y} y} \quad (7.18)$$

where $(\beta_{2x})^2 + (\beta_{2y})^2 = K_2^2 = \omega^2\mu_2\epsilon_2$

The corresponding electric fields

$$E_x = -H_0 \tau \frac{\beta_{2y}}{\omega\epsilon_2}\sqrt{\frac{s_y^*}{s_y}}\, e^{-j\sqrt{s_x s_x^*}\,\beta_{2x} x}\, e^{-j\sqrt{s_y s_y^*}\,\beta_{2y} y} \qquad (7.20a)$$

$$E_y = H_0 \tau \frac{\beta_{2x}}{\omega\epsilon_2}\sqrt{\frac{s_x^*}{s_x}}\, e^{-j\sqrt{s_x s_x^*}\,\beta_{2x} x}\, e^{-j\sqrt{s_y s_y^*}\,\beta_{2y} y} \qquad (7.20b)$$

Now, once again the tangential $\bar{E}$ + $\bar{H}$ boundary conditions must be satisfied at $x=0$.

For this to occur, $\underline{s_y = s_y^* = 1 \ (\sigma_y = \sigma_y^* = 0)}$,

and $\underline{\beta_{2y} = \beta_{1y} = k_1 \sin\theta}$.

The reflection coefficient is then

$$\Gamma = \frac{\left(\dfrac{\beta_{1x}}{\omega\epsilon_1} - \dfrac{\beta_{2x}}{\omega\epsilon_2}\sqrt{\dfrac{s_x^*}{s_x}}\right)}{\left(\dfrac{\beta_{1x}}{\omega\epsilon_1} + \dfrac{\beta_{2x}}{\omega\epsilon_2}\sqrt{\dfrac{s_x^*}{s_x}}\right)} \qquad (7.21a)$$

and the transmission coefficient is

$$\underline{\tau = 1 + \Gamma} \qquad (7.21b)$$

7.3.1 cont.

Reflectionless Matching Condition

→ want to get $\Gamma = 0$

→ let $\epsilon_1 = \epsilon_2$, $\mu_1 = \mu_2$, and $s_x = s_x^*$

Then

$$s_x = 1 + \frac{\sigma_x}{j\omega\epsilon_1} = 1 + \frac{\sigma_x^*}{j\omega\mu_1} = s_x^*$$

$$\hookrightarrow \quad \frac{\sigma_x}{\epsilon_1} = \frac{\sigma_x^*}{\mu_1}$$

and $k_1 = k_2 = \omega\sqrt{\mu_1\epsilon_1}$

From (7.19), with $\beta_{1y} = \beta_{2y} = k_1 \sin\theta$,

$$\beta_{2x} = \left[\overset{k_1^2}{\cancel{k_2^2}} - \overset{k_1^2\sin^2\theta}{\cancel{\beta_{2y}^2}}\right]^{1/2}$$

$$= \left[k_1^2 - k_1^2\sin^2\theta\right]^{1/2} = k_1\cos\theta$$

but $k_1\cos\theta = \beta_{1x}$ (7.4a)

so $\beta_{2x} = \beta_{1x}$ under these conditions

7.3.1 cont.

Plugging this into the eqn for Γ yields

$$\Gamma = \frac{\left(\dfrac{\beta_{1x}}{\omega\epsilon_1} - \dfrac{\overset{\beta_{1x}}{\cancel{\beta_{2x}}}}{\omega\,\underset{\epsilon_1}{\cancel{\epsilon_2}}}\underset{1}{\cancel{\sqrt{\dfrac{s_x^*}{s_x}}}}\right)}{\left(\dfrac{\beta_{1x}}{\omega\epsilon_1} + \dfrac{\beta_{1x}}{\omega\epsilon_1}(1)\right)}$$

$$\underline{\underline{\Gamma = 0}}$$ $\leftarrow$ Not dependent on angle θ or frequency!

$$\underline{\mathcal{T} = 1 + \Gamma = 1}$$

$\rightarrow$ The transmitted fields are then

$$H_z = H_0\, e^{-j s_x \beta_{1x} x}\, e^{-j\beta_{1y} y}$$

$$= \underbrace{H_0\, e^{-j\beta_{1x} x - j\beta_{1y} y}}_{\text{Same as incident wave}}\ \underbrace{e^{-\sigma_x x \eta_1 \cos\theta}}_{\text{PML attenuation}} \qquad (7.22a)$$

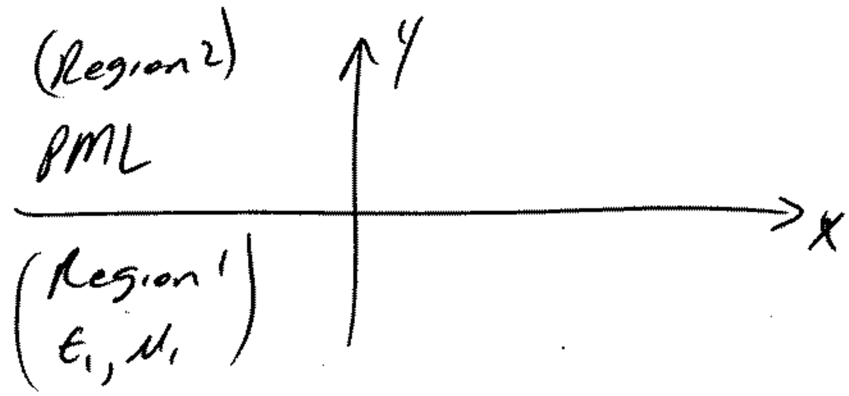
$$E_x = -H_0 \eta_1 \sin\theta\, e^{-j\beta_{1x} x - j\beta_{1y} y}\, e^{-\sigma_x x \eta_1 \cos\theta} \qquad (7.22b)$$

$$E_y = H_0 \eta_1 \cos\theta\, e^{-j\beta_{1x} x - j\beta_{1y} y}\, e^{-\sigma_x x \eta_1 \cos\theta} \qquad (7.22c)$$

* Note that the PML attenuation get smaller as $\theta \to 90°$ ($\cos\theta \to 0$). However, the wave near grazing incidence stays in a finite thickness PML longer.

* Also, note that $e^{-\sigma_x x \eta_1 \cos\theta}$ is frequency independent.

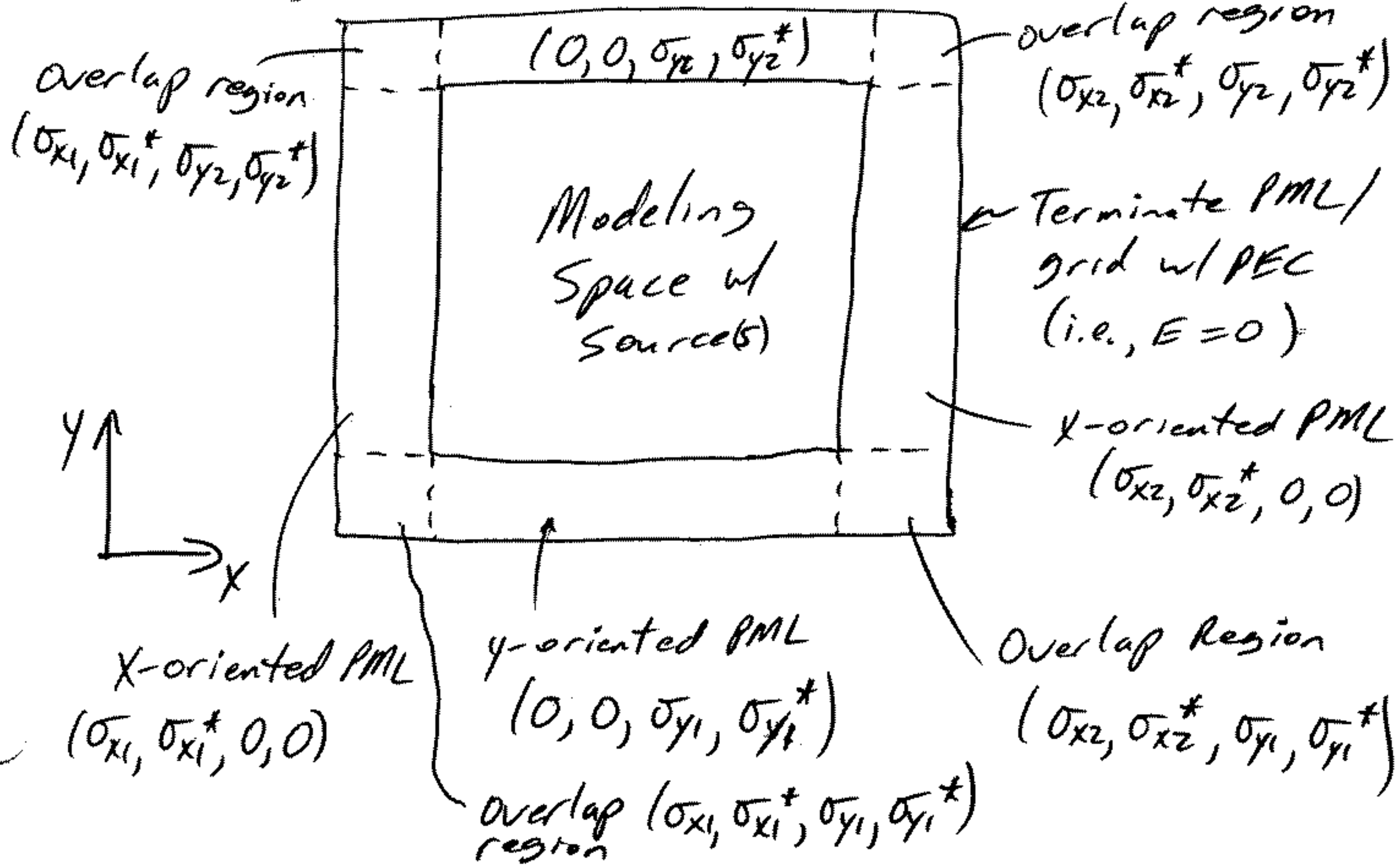
* All of the preceding can be repeated for a PML going in the y-direction.



Here, we select or get $S_x = S_x^* = 1$ (i.e, $\sigma_x = \sigma_x^* = 0$), $\epsilon_1 = \epsilon_2$, $\mu_1 = \mu_2$, and $S_y = S_y^*$ (i.e., $\frac{\sigma_y}{\epsilon_1} = \frac{\sigma_y^*}{\mu_1}$) to get $\underline{\Gamma = 0}$

7.3.1 cont.

Looking at an overall 2D TE_z grid



→ Note, utilize both x- + y-oriented PML losses in overlap regions.

→ Numerical dispersion problems do NOT have a significant effect on PML performance

→ Practical concern (deal w/ later) is how to select σ_x, σ_x^*, σ_y, + σ_y^* in a discrete grid (hint: taper)

7.3.2 Two-Dimensional TM_z case

→ Here the field components are E_z, H_x, + H_y and we split E_z into E_{zx} + E_{zy} where $E_z = E_{zx} + E_{zy}$.

→ The applicable differential equations are

$$\mu_2 \frac{\partial H_x}{\partial t} + \sigma_y^* H_x = -\frac{\partial E_z}{\partial y} \qquad (7.23a)$$

$$\mu_2 \frac{\partial H_y}{\partial t} + \sigma_x^* H_y = \frac{\partial E_z}{\partial x} \qquad (7.23b)$$

$$\epsilon_2 \frac{\partial E_{zx}}{\partial t} + \sigma_x E_{zx} = \frac{\partial H_y}{\partial x} \qquad (7.23c)$$

$$\epsilon_2 \frac{\partial E_{zy}}{\partial t} + \sigma_y E_{zy} = \frac{-\partial H_x}{\partial y}$$

→ Very similar derivation process leads to identical PML conditions:

$\epsilon_1 = \epsilon_2$, $\mu_1 = \mu_2$, and

x-oriented

$S_y = S_y^* = 1$ $(\sigma_y = \sigma_y^* = 0)$

$S_x = S_x^*$ $\left(\frac{\sigma_x}{\epsilon_1} = \frac{\sigma_x^*}{\mu_1}\right)$

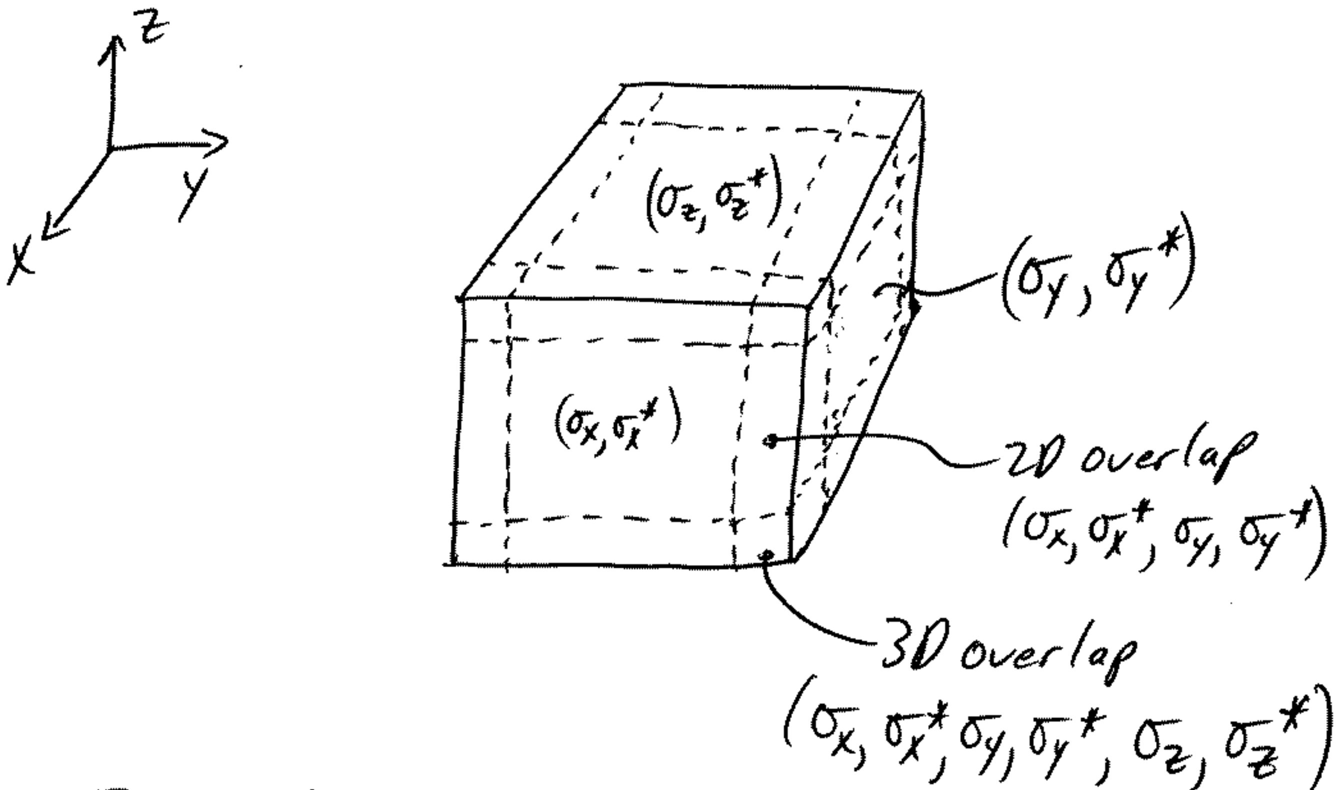
y-oriented

$S_x = S_x^* = 1$ $(\sigma_x = \sigma_x^* = 0)$

$S_y = S_y^*$ $\left(\frac{\sigma_y}{\epsilon_1} = \frac{\sigma_y^*}{\mu_1}\right)$

7.3.3 Three-Dimensional Case

To extend Berenger's PML to the 3-D case, all 6 field components (e.g. $E_x, E_y, E_z, H_x, H_y, + H_z$) must be split w/in the PML. This doubles the number of scalar differential equations derived from Ampere's + Faraday's Laws from 6 to <u>12</u>. (see next page)



In each case,

$$\frac{\sigma_w}{\epsilon} = \frac{\sigma_w^*}{\mu}$$

$\leftarrow$ $\epsilon + \mu$ are for adjacent modeling space on interior

for a <u>PML</u>.

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7.3.3 cont.

- Assume **no** electric or magnetic current sources
- Start with Ampere's and Faraday's Laws

$$\varepsilon\frac{\partial\bar{\mathcal{E}}}{\partial t}+\sigma\bar{\mathcal{E}}=\bar{\nabla}\times\bar{\mathcal{H}}$$

and

$$\mu\frac{\partial\bar{\mathcal{H}}}{\partial t}+\sigma^{*}\bar{\mathcal{H}}=-(\bar{\nabla}\times\bar{\mathcal{E}})$$

respectively.

- From Ampere's Law

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_y\right)\mathcal{E}_{xy}=\frac{\partial}{\partial y}\left(\mathcal{H}_{zx}+\mathcal{H}_{zy}\right) \tag{7.24a}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_z\right)\mathcal{E}_{xz}=-\frac{\partial}{\partial z}\left(\mathcal{H}_{yx}+\mathcal{H}_{yz}\right) \tag{7.24b}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_z\right)\mathcal{E}_{yz}=\frac{\partial}{\partial z}\left(\mathcal{H}_{xy}+\mathcal{H}_{xz}\right) \tag{7.24c}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_x\right)\mathcal{E}_{yx}=-\frac{\partial}{\partial x}\left(\mathcal{H}_{zx}+\mathcal{H}_{zy}\right) \tag{7.24d}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_x\right)\mathcal{E}_{zx}=\frac{\partial}{\partial x}\left(\mathcal{H}_{yx}+\mathcal{H}_{yz}\right) \tag{7.24e}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma_y\right)\mathcal{E}_{zy}=-\frac{\partial}{\partial y}\left(\mathcal{H}_{xy}+\mathcal{H}_{xz}\right) \tag{7.24f}$$

7.3.3 cont.

- From Faraday's Law

$$\left(\mu\frac{\partial}{\partial t}+\sigma_y^*\right)\mathcal{H}_{xy}=-\frac{\partial}{\partial y}\left(\mathcal{E}_{zx}+\mathcal{E}_{zy}\right) \tag{7.25a}$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_z^*\right)\mathcal{H}_{xz}=\frac{\partial}{\partial z}\left(\mathcal{E}_{yx}+\mathcal{E}_{yz}\right) \tag{7.25b}$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_z^*\right)\mathcal{H}_{yz}=-\frac{\partial}{\partial z}\left(\mathcal{E}_{xy}+\mathcal{E}_{xz}\right) \tag{7.25c}$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_x^*\right)\mathcal{H}_{yx}=\frac{\partial}{\partial x}\left(\mathcal{E}_{zx}+\mathcal{E}_{zy}\right) \tag{7.25d}$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_x^*\right)\mathcal{H}_{zx}=-\frac{\partial}{\partial x}\left(\mathcal{E}_{yx}+\mathcal{E}_{yz}\right) \tag{7.25e}$$

$$\left(\mu\frac{\partial}{\partial t}+\sigma_y^*\right)\mathcal{H}_{zy}=\frac{\partial}{\partial y}\left(\mathcal{E}_{xy}+\mathcal{E}_{xz}\right) \tag{7.25f}$$

7.4 Stretched-Coordinate Formulation of Berenger's PML

→ Chew/Weedon also Rappaport developed

→ allows mapping into non-Cartesian systems

→ more efficient than Berenger's PML formulation (12 → 6 eqns)

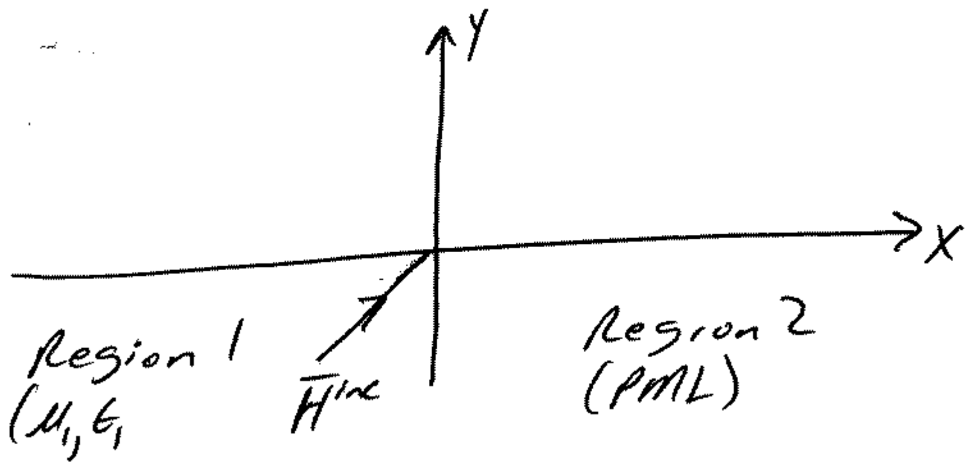
7.5 An Anisotropic PML Absorbing Medium

→ Avoids using split coordinates (more memory effic.)

→ uses tensors (3x3 matrices) for permittivity + permeability

→ achieves same/similar performance to Berenger's PML

7.5.1 Perfectly Matched Uniaxial Medium



$$\bar{H}^{inc} = \bar{H}_0 \, e^{-j\beta_{1x}x - j\beta_{1y}y}$$ ← arbitrary polarization

In region 2, the permittivity + permeability tensors are

$$\bar{\bar{\epsilon}}_2 = \epsilon_2 \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} + \bar{\bar{\mu}}_2 = \mu_2 \begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \quad (7.35)$$

Note: $\epsilon_{yy} = \epsilon_{zz}$ + $\mu_{yy} = \mu_{zz}$

7.5.1 cont.

→ Want Region 2 fields to also be plane-waves & to satisfy Maxwell's Curl eqns:

$$\bar{\beta}_2 \times \bar{E} = \omega \bar{\bar{\mu}}_2 \bar{H} \quad \& \quad \bar{\beta}_2 \times \bar{H} = -\omega \bar{\bar{\epsilon}}_2 \bar{E} \quad (7.36)$$

where $\bar{\beta}_2 = \hat{a}_x \beta_{2x} + \hat{a}_y \beta_{2y}$ is the wavevector in Region 2. Substituting for $\bar{E}$ in the first, yields

$$\bar{\beta}_2 \times \left(\bar{\bar{\epsilon}}_2^{-1} \bar{\beta}_2 \right) \times \bar{H} + \omega^2 \bar{\bar{\mu}}_2 \bar{H} = 0$$

which is a wave equation. This can be written in matrix form as

$$\begin{bmatrix} K_2^2 c - (\beta_{2y})^2 b^{-1} & \beta_{2x}\beta_{2y} b^{-1} & 0 \\ \beta_{2x}\beta_{2y} b^{-1} & K_2^2 d - (\beta_{2x})^2 b^{-1} & 0 \\ 0 & 0 & K_2^2 d - (\beta_{2x})^2 b^{-1} - (\beta_{2y})^2 a^{-1} \end{bmatrix} \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \\ \bar{H}_z \end{bmatrix} = 0 \quad (7.38)$$

where $K_2 = \omega\sqrt{\mu_2 \epsilon_2}$ or $K_2^2 = \omega^2 \mu_2 \epsilon_2$.

The dispersion relation for Region 2 can be found from the determinant of the matrix. Solving for β_{2x}, yields 4 eigenmode solutions that can be decoupled into fwd & bwd prop.

TE$_z$ + TM$_z$ modes with dispersion relations

$$k_2^2 - (\beta_{2x})^2 b^{-1} d^{-1} - (\beta_{2y})^2 a^{-1} d^{-1} = 0 \quad (\text{TE}_z \ \bar{H}_x = \bar{H}_y = 0) \quad (7.39)$$

and

$$k_2^2 - (\beta_{2x})^2 b^{-1} d^{-1} - (\beta_{2y})^2 b^{-1} c^{-1} = 0 \quad (\text{TM}_z \ \bar{H}_z = 0) \quad (7.40)$$

TE$_z$ case

In Region 1 (per section 7.2), the total fields are

$$\bar{H}_1 = \hat{a}_z H_0 \left(1 + \Gamma e^{2j\beta_{1x}x}\right) e^{-j\beta_{1x}x - j\beta_{1y}y} \quad (7.41a)$$

$$\bar{E}_1 = \left[-\hat{a}_x \frac{\beta_{1y}}{\omega\epsilon_1}\left(1 + \Gamma e^{2j\beta_{1x}x}\right) + \hat{a}_y \frac{\beta_{1x}}{\omega\epsilon_1}\left(1 + \Gamma e^{2j\beta_{1x}x}\right)\right] \times H_0 e^{-j\beta_{1x}x - j\beta_{1y}y} \quad (7.41b)$$

and in Region 2, the transmitted waves are

$$\bar{H}_2 = \hat{a}_z H_0 \tau e^{-j\beta_{2x}x - j\beta_{2y}y} \quad (7.42a)$$

$$\bar{E}_2 = \left[-\hat{a}_x \frac{\beta_{2y}}{\omega\epsilon_2 a} + \hat{a}_y \frac{\beta_{2x}}{\omega\epsilon_2 b}\right] H_0 \tau e^{-j\beta_{2x}x - j\beta_{2y}y} \quad (7.42b)$$

from $\bar{\bar{\epsilon}}_2$

7.5.1 cont.

TE$_z$ case cont.

Enforcing the tangential field conditions at $x=0$, leads to

$$\boxed{\Gamma = \frac{\beta_{1x} - \beta_{2x} b^{-1}}{\beta_{1x} + \beta_{2x} b^{-1}}} \quad \text{(7.43a)} \qquad \text{and} \qquad \boxed{\tau = 1+\Gamma = \frac{2\beta_{1x}}{\beta_{1x} + \beta_{2x} b^{-1}}} \quad \text{(7.43b)}$$

and $\boxed{\beta_{2y} = \beta_{1y}}$ (7.44)

Using (7.44) in the TE$_z$ dispersion relation + solving for β_{2x} gives

$$\beta_{2x} = \sqrt{k_2^2 bd - \beta_{1y}^2 a^{-1} b} \qquad (7.45)$$

To get $\Gamma = 0$, select $\underbrace{\epsilon_1 = \epsilon_2, \mu_1 = \mu_2}_{\text{implies } k_1 = k_2}$, $d = b$, and $a^{-1} = b$.

With these choices

$$\beta_{2x} = b\sqrt{k_1^2 - \beta_{1y}^2} = b\,\beta_{1x} \qquad (7.46)$$

and $$\Gamma_{TE_z} = \frac{\beta_{1x} - (b\beta_{1x})b^{-1}}{\beta_{1x} + (b\beta_{1x})b^{-1}} = \underline{\underline{0}}$$

7.5.1 cont.

TM$_z$ case

Apply duality, i.e., replace b w/ d (& vice versa) and "a" w/ "c" in the reflection coefficient

(b in $\bar{\bar{\epsilon}}_2$, d in $\bar{\bar{\mu}}_2$, a in $\bar{\bar{\epsilon}}_2$, c in $\bar{\bar{\mu}}_2$)

$$\Gamma_{TM_z} = \frac{\beta_{1x} - \beta_{2x} d^{-1}}{\beta_{1x} + \beta_{2x} d^{-1}} \quad \& \quad \tau = 1 + \Gamma = \frac{2\beta_{1x}}{\beta_{1x} + \beta_{2x} d^{-1}}$$

Again, for phase matching $\beta_{1y} = \beta_{2y}$ and using the TM$_z$ dispersion relation

$$k_2^2 - (\beta_{2x})^2 b^{-1} d^{-1} - (\cancel{\beta_{2y}}^{\beta_{1y}})^2 b^{-1} c^{-1} = 0$$

$$k_2^2 bd - \beta_{2x}^2 - \beta_{1y}^2 c^{-1} d = 0$$

$$\beta_{2x} = \sqrt{k_2^2 bd - \beta_{1y}^2 c^{-1} d}$$

Now, if we set $\underline{\epsilon_1 = \epsilon_2}$, $\underline{\mu_1 = \mu_2}$, $\underline{b = d}$, & $\underline{c^{-1} = d}$

$$\downarrow$$

$$k_1 = k_2$$

$$\beta_{2x} = \sqrt{k_1^2 d^2 - \beta_{1y}^2 d^2} = d\sqrt{k_1^2 - \beta_{1y}^2} = d\,\beta_{1x}$$

$$\Gamma_{TM_z} = \frac{\beta_{1x} - (d\beta_{1x}) d^{-1}}{\beta_{1x} + (d\beta_{1x}) d^{-1}} = \underline{\underline{0}}$$

7.5.1 cont.

Combining these results (want $\Gamma = 0$ for both TE_z + TM_z mode, i.e. any arbitrary field), we get

$$\epsilon_1 = \epsilon_2, \; \mu_1 = \mu_2, \; b = d, \; \underbrace{c^{-1} = d}_{=b}, \; + \; \underbrace{a^{-1} = b}_{=d}$$

$$\Rightarrow a = c$$

Therefore, the matrices for the $\bar{\bar{\epsilon}}_2$ + $\bar{\bar{\mu}}_2$ tensors are identical, and can be written as

$$\boxed{\bar{\bar{\epsilon}}_2 = \epsilon_1 \bar{\bar{S}}, \quad \bar{\bar{\mu}}_2 = \mu_1 \bar{\bar{S}}, \text{ where } \bar{\bar{S}} = \begin{bmatrix} S_x^{-1} & 0 & 0 \\ 0 & S_x & 0 \\ 0 & 0 & S_x \end{bmatrix}} \quad (7.47)$$

where S_x indicates that we have an x-directed PML (attenuation in x-direction).

Now, in both cases, we have $\underline{\beta_{2x} = S_x \beta_{1x}}$. (7.48)

To make the $e^{-j\beta_{2x} x}$ attenuative (+ similar to like terms in Berenger's PML), select

$$\boxed{S_x = 1 + \frac{\sigma_x}{j\omega\epsilon_1} = 1 - j\frac{\sigma_x}{\omega\epsilon_1}}$$

7.5.1 cont.

This yields

$$\beta_{2x} = \left(1 - j\frac{\sigma_x}{\omega\epsilon_1}\right)\beta_{1x} = \beta_{1x} - j\frac{\sigma_x \beta_{1x}}{\omega\epsilon_1}$$

and the corresponding exponential term is

$$e^{-j\beta_{2x}x} = e^{-j\beta_{1x}x}\, e^{-\frac{\sigma_x \beta_{1x}}{\omega\epsilon_1}x} = \underbrace{e^{-j\beta_{1x}x}}_{\text{same phase as in Region 1}}\, \underbrace{e^{-\sigma_x x \eta_1 \cos\theta}}_{\text{PML attenuation}}$$

Once again, the attenuation is <u>frequency independent</u>, but does depend on angle of incidence θ and σ_x selection.

<u>TE_z mode</u>

$$\overline{H}_2 = \hat{a}_z H_0\, e^{-j\beta_{1x}x - j\beta_{1y}y}\, e^{-\sigma_x x \eta_1 \cos\theta} \qquad (7.49a)$$

Identical to Berenger PML (7.22a)

$$\overline{E}_2 = \left(-\hat{a}_x s_x \eta_1 \sin\theta + \hat{a}_y \eta_1 \cos\theta\right) H_0\, e^{-j\beta_{1x}x - j\beta_{1y}y}\, e^{-\sigma_x x \eta_1 \cos\theta} \qquad (7.49b)$$

extra term vs. (7.22b)

Identical to Berenger PML (7.22c)

7.5.2 Relationship to Berenger's Split-Field PML

→ same wave equations (see (7.17) + (7.37))

→ same dispersion relations

→ same tangential field components

→ different normal electric field (E_x) by a factor of s_x

$$\text{Berenger PML} \quad E_{1x}(x=0) = E_{2x}(x=0)$$

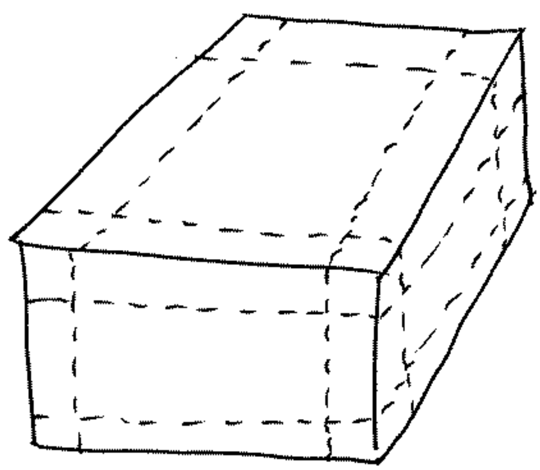
$$\text{UPML} \quad D_{1x}(x=0) = D_{2x}(x=0) = s_x^{-1} E_x$$

Remembering that normal boundary conditions are governed by Gauss' Law, this implies that Berenger's PML and the UPML obey different Gauss' Law.

7.5.3 A Generalized Three-Dimensional Formulation

→ want PML to work in y- + z-directions as well as x-direction

→ also, need to handle corners + edges where there is overlap



To accomplish this, define

$$S_x = \kappa_x + \frac{\sigma_x}{j\omega\epsilon}$$

$$S_y = \kappa_y + \frac{\sigma_y}{j\omega\epsilon}$$

$$S_z = \kappa_z + \frac{\sigma_z}{j\omega\epsilon}$$

lowercase Greek letter Kappa (extra generality, used to attenuate evanescent incident waves where β_{1w} is complex)

to be elements in the tensor $\bar{\bar{S}}$ which is given by

7.5.3 cont.

$$\bar{\bar{S}} = \begin{bmatrix} S_x^{-1} & 0 & 0 \\ 0 & S_x & 0 \\ 0 & 0 & S_x \end{bmatrix} \begin{bmatrix} S_y^{-1} & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_y \end{bmatrix} \begin{bmatrix} S_z^{-1} & 0 & 0 \\ 0 & S_z & 0 \\ 0 & 0 & S_z \end{bmatrix}$$

$$= \begin{bmatrix} S_y S_z S_x^{-1} & 0 & 0 \\ 0 & S_x S_z S_y^{-1} & 0 \\ 0 & 0 & S_x S_y S_z^{-1} \end{bmatrix} \quad (7.54)$$

which is applied to Faraday's + Ampere's Laws (Maxwell's curl eqns) as

$$\bar{\nabla} \times \bar{H} = j\omega\epsilon\bar{\bar{S}}\bar{E} \qquad (7.53a)$$

$$\bar{\nabla} \times \bar{E} = -j\omega\mu\bar{\bar{S}}\bar{H} \qquad (7.53b)$$

→ what can be done is to apply the UPML or anisotropic PML to the whole FDTD modeling grid/lattice, but change the K_w + σ_w values so that the interior is non-lossy (regular media)

7.5.3 cont.

The choices for κ_w & σ_w to accomplish this are:

$$\kappa_x(x) = \begin{cases} \kappa_x'(x) & x \le x_{min},\ x \ge x_{max} \quad (\text{PML}) \\ 1 & x_{min} < x < x_{max} \quad (\text{interior}) \end{cases}$$

$$\sigma_x(x) = \begin{cases} \sigma_x'(x) & x \le x_{min},\ x \ge x_{max} \quad (\text{PML}) \\ 0 & x_{min} < x < x_{max} \quad (\text{interior}) \end{cases} \qquad (7.56a)$$

→ Similar distributions vs. y & z for κ_y + σ_y and κ_z + σ_z, respectively. (7.56b) & (7.56c).

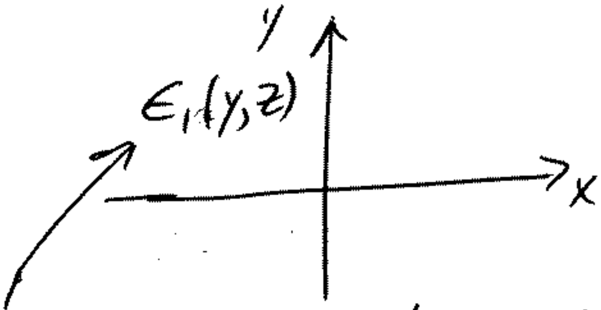
→ In the interior, $s_x = 1$, $s_y = 1$, $s_z = 1$

$$\overline{\overline{S}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ → right back to regular Maxwell's eqns

7.5.4 Inhomogeneous Media

→ What if the modeling space is inhomogeneous?

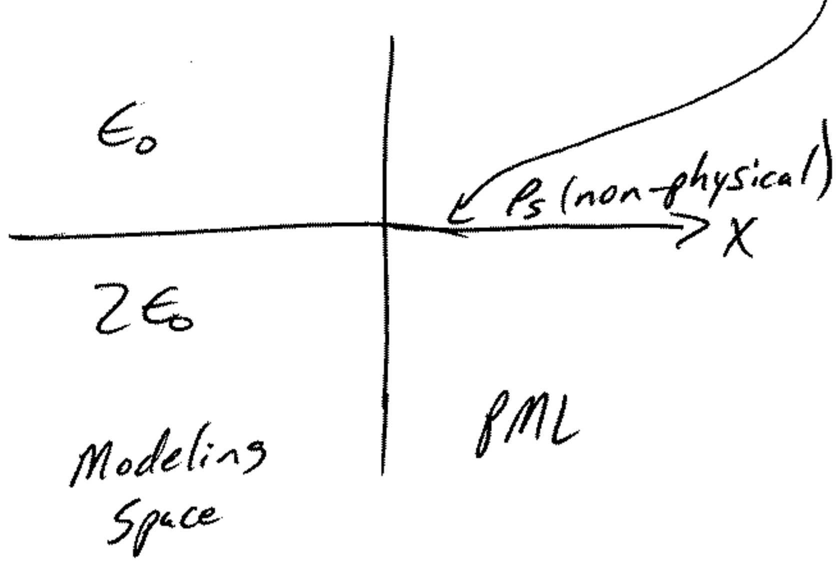
e.g.



can change in the transverse directions
(piece-wise constant in FDTD grid)

→ Must be careful, due to change in Gauss' Law for PML region, charge (non-physical) can build-up at interface(s).

e.g.



→ To avoid problems S_w must be independent of the transverse coordinates

e.g. For $S_x = \kappa_x + \frac{\sigma_x}{j\omega\epsilon}$, $\frac{\sigma_x}{\epsilon} = \underline{\text{constant}}$

7.5.4 cont.

How?

Method 1: change σ_w in transverse directions

so that $\frac{\sigma_w}{\epsilon}$ = constant

→ big pain if there are lots of changes

e.g. $$\frac{\sigma_x(x,y)}{\epsilon(x,y)} = \text{constant}$$

Method 2: normalize σ_w by ϵ_r (can be/is function of transverse coordinates)

so that

$$S_W = K_w + \frac{\sigma_w'}{j\omega\epsilon_0}$$

$\sigma_w' = \frac{\sigma_w}{\epsilon_r}$ constant in transverse directions

makes UPML, material independent

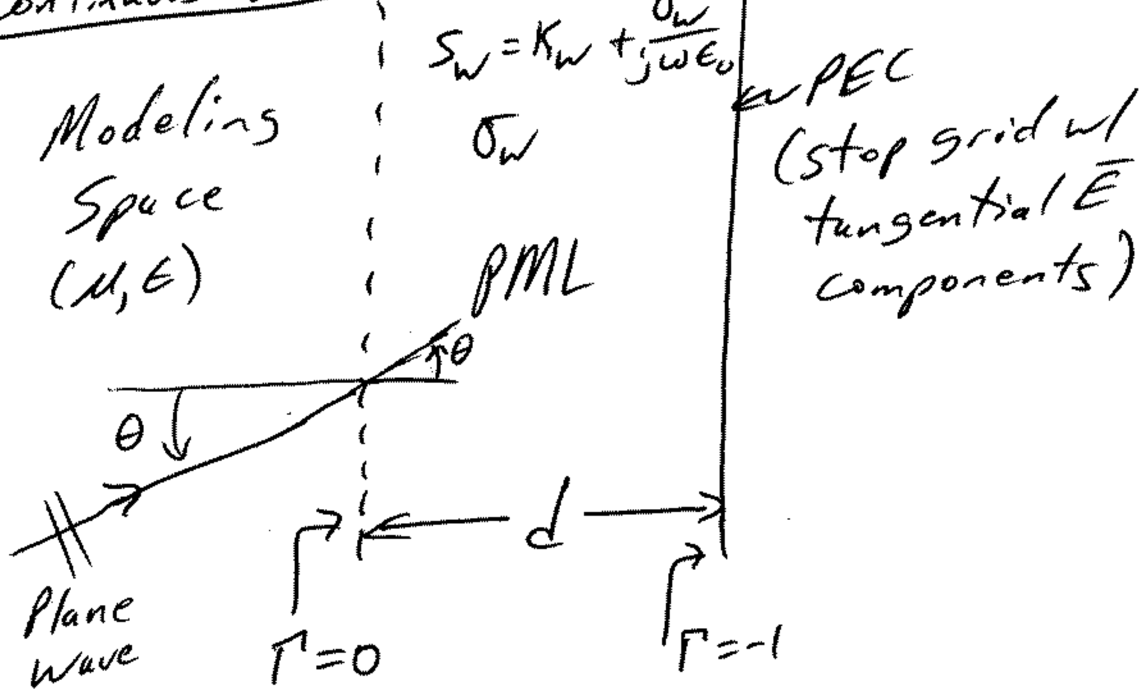
ϵ in $\nabla \times \bar{H} = j\omega\epsilon\bar{\bar{s}}\bar{E}$ still a function

* this is how the UPML is implemented in section 7.8.

In (7.81(a,b,c)), the 'prime' is dropped from the UPML conductivity, i.e. $\sigma_w' \rightarrow \sigma_w$.

7.6 Theoretical Performance of the PML

7.6.1 Continuous Space



Per (7.49a) + (7.49b)'s $e^{-\sigma_x x \eta_1 \cos\theta}$ terms, we can infer that the wave will attenuate by $e^{-\sigma_w d \eta \cos\theta}$ reaching the PEC wall, will be 100% reflected, and attenuate another $e^{-\sigma_w d \eta \cos\theta}$ reaching the PML-Modeling space interface for a net attenuation of

$$R(\theta) = e^{-2\sigma_w \eta d \cos\theta} \qquad (7.58)$$

called "reflection error"

* want $R(\theta)$ as small as possible
* $R(\theta) = 1$ at $\theta = 90°$ (grazing incidence)

7.6.2 Discrete Space

- In theory, we can use any σ, σ^*, or σ_w at the interface w/ the PML
- In an FDTD grid/lattice, a large jump from σ, σ^*, or $\sigma_w = 0$ to some value will give unwanted reflections.
- Nothing in the PML theory says that σ, σ^*, or σ_w must be constant. Therefore, Berenger proposed tapering them from small values near the interface to larger values near the PEC wall, e.g. $\sigma_w(w)$, $\sigma_x(x)$, ..., wrt the normal direction. This makes the reflection error

$$R(\theta) = e^{-2\eta\cos\theta\int_0^d \sigma_w(w)\,dw} \qquad (7.59)$$

- many tapers for $\sigma_w(w)$ (+ $K_w(w)$ for UPML) have been investigated, best results found using polynomial tapers ← Note: w is depth into PML in w-direction

$$\sigma_w(w) = \left(\frac{w}{d}\right)^m \sigma_{w,max}$$ ← starts at 0 + goes up to $\sigma_{w,max}$

(7.60)

$$K_w(w) = 1 + (K_{w,max} - 1)\left(\frac{w}{d}\right)^m$$ ← starts at 1 + goes up to $K_{w,max}$

7.6.2 cont.

Putting the polynomial $\sigma_w(w)$ into $R(\theta)$ yields

$$R(\theta) = e^{-2\eta\cos\theta \int_0^d \sigma_{w,max} (w/d)^m dw}$$

$$= e^{-2\eta \cos\theta \frac{\sigma_{w,max}}{d^m} \left.\frac{w^{m+1}}{m+1}\right|_0^d}$$

$$R(\theta) = e^{-2\eta\, \sigma_{w,max}\, d \cos\theta / (m+1)} \qquad (7.61)$$

Question: What values of $\sigma_{w,max}$ and m are optimal for a given d?

→ Trade-off between larger $\sigma_{w,max}$ and m and increasing discretization errors w/ UPML in FDTD grid

→ Theoretically, for a given reflection coefficient error at normal incidence $R(0) = R(\theta = 0)$

$$\sigma_{w,max} = \frac{-(m+1)\ln[R(0)]}{2\eta d} \qquad (7.62)$$

7.6.2 cont.

→ Experimentally, it was demonstrated that $R(0) \simeq e^{-16}$ for $d = 10\Delta$ and $R(0) \simeq e^{-8}$ for $d = 5\Delta$ gave optimal results (most attenuation) with:

$$3 \le m \le 4$$

and
$$\sigma_{w,opt} \simeq \frac{0.8(m+1)}{\eta\Delta} \qquad (7.66)$$

→ Per the discussion in section 7.5.4, it is convenient to normalize σ_w per (7.57) to deal with more general (e.g. inhomogeneous, dispersive, +/or non-linear) media

$$(7.57) \quad S_x = K_x + \frac{\sigma_x'}{j\omega\epsilon_0} \Rightarrow \sigma_x' = \frac{\sigma_x}{\epsilon_r}$$

So

$$\sigma_{w,max} = \sigma_{w,opt}' = \frac{0.8(m+1)}{\eta_0 \Delta \sqrt{\epsilon_{r,eff}\,\mu_{r,eff}}} \qquad (7.67)$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 376.7303\,\Omega$ and $\epsilon_{r,eff}$ + $\mu_{r,eff}$ are chosen using mean/ave ϵ_r, μ_r in modeling space

7.7 Complex Frequency-shifted Tensor

⇒ skip

7.8 Efficient Implementation of UPML in FDTD

→ need to convert from phasor representation to time-domain to get FDTD update equations

→ will use an auxiliary differential equation (ADE) approach to implement UPML

7.8.1 Ampere's Law for UPML (3D)

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = j\omega\epsilon \begin{bmatrix} \frac{S_y S_z}{S_x} & 0 & 0 \\ 0 & \frac{S_x S_z}{S_y} & 0 \\ 0 & 0 & \frac{S_x S_y}{S_z} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{7.80}$$

where $S_w = K_w + \frac{\sigma_w}{j\omega\epsilon_0}$ ← Note: σ_w is Normalized per (7.57) "prime" has been dropped

Problem — can't go directly back to time-domain via Fourier transform since the F.T.

$$x(t) * v(t) \longleftrightarrow X(\omega)V(\omega)$$

tells us we would get a convolution (expensive numerically)

7.8.1 cont.

For the ADE approach, define auxiliary variables

$$D_x = \epsilon \frac{s_z}{s_x} E_x$$

$$D_y = \epsilon \frac{s_x}{s_y} E_y \qquad (7.82)$$

$$D_z = \epsilon \frac{s_y}{s_z} E_z \qquad \leftarrow \text{co-located w/ corresponding } \bar{E}$$

Then, Ampere's law can be re-written using these auxiliary variables as

$$\begin{bmatrix} \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} \\[2ex] \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} \\[2ex] \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} \end{bmatrix} = j\omega \begin{bmatrix} \overset{\kappa_y + \frac{\sigma_y}{j\omega\epsilon_0}}{\cancel{s_y}} & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \qquad (7.83)$$

Use the F.T. pair

$$\frac{\partial x(t)}{\partial t} \longleftrightarrow j\omega X(\omega)$$

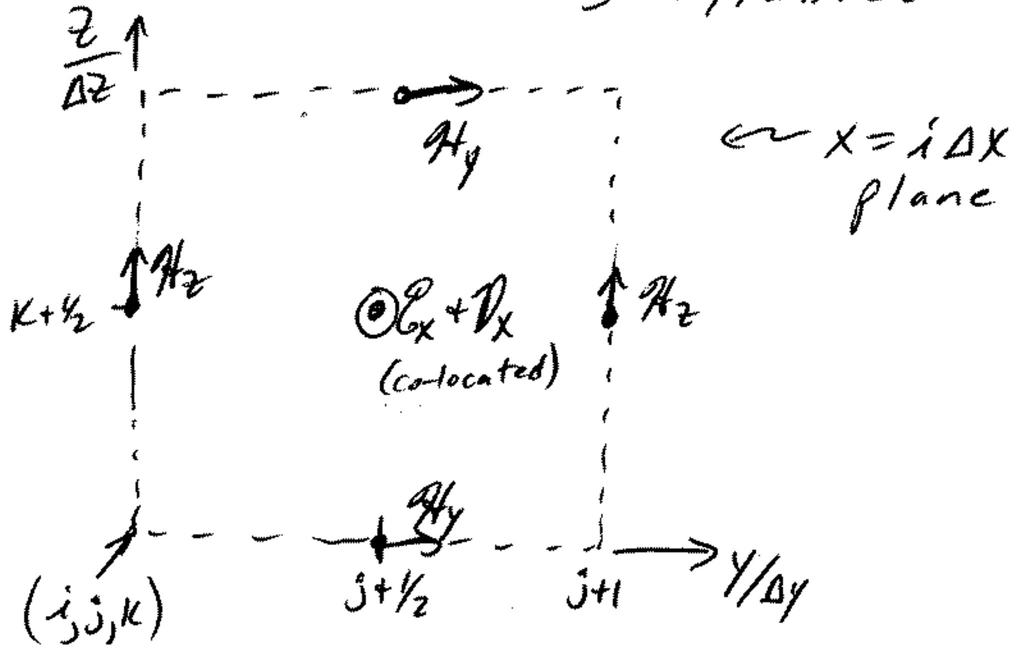
to go back to the time domain

7.8.1 cont.

$$\begin{bmatrix} \dfrac{\partial \mathcal{H}_z}{\partial y} - \dfrac{\partial \mathcal{H}_y}{\partial z} \\ \dfrac{\partial \mathcal{H}_x}{\partial z} - \dfrac{\partial \mathcal{H}_z}{\partial x} \\ \dfrac{\partial \mathcal{H}_y}{\partial x} - \dfrac{\partial \mathcal{H}_x}{\partial y} \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} K_y & 0 & 0 \\ 0 & K_z & 0 \\ 0 & 0 & K_x \end{bmatrix} \begin{bmatrix} \mathcal{D}_x \\ \mathcal{D}_y \\ \mathcal{D}_z \end{bmatrix} + \frac{1}{\epsilon_0} \begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{bmatrix} \begin{bmatrix} \mathcal{D}_x \\ \mathcal{D}_y \\ \mathcal{D}_z \end{bmatrix} \qquad (7.84)$$

Let's find updates for $\mathcal{D}_x$ + $\mathcal{E}_x$ components.

From the standard Yee grid/lattice



3.6.1 continued

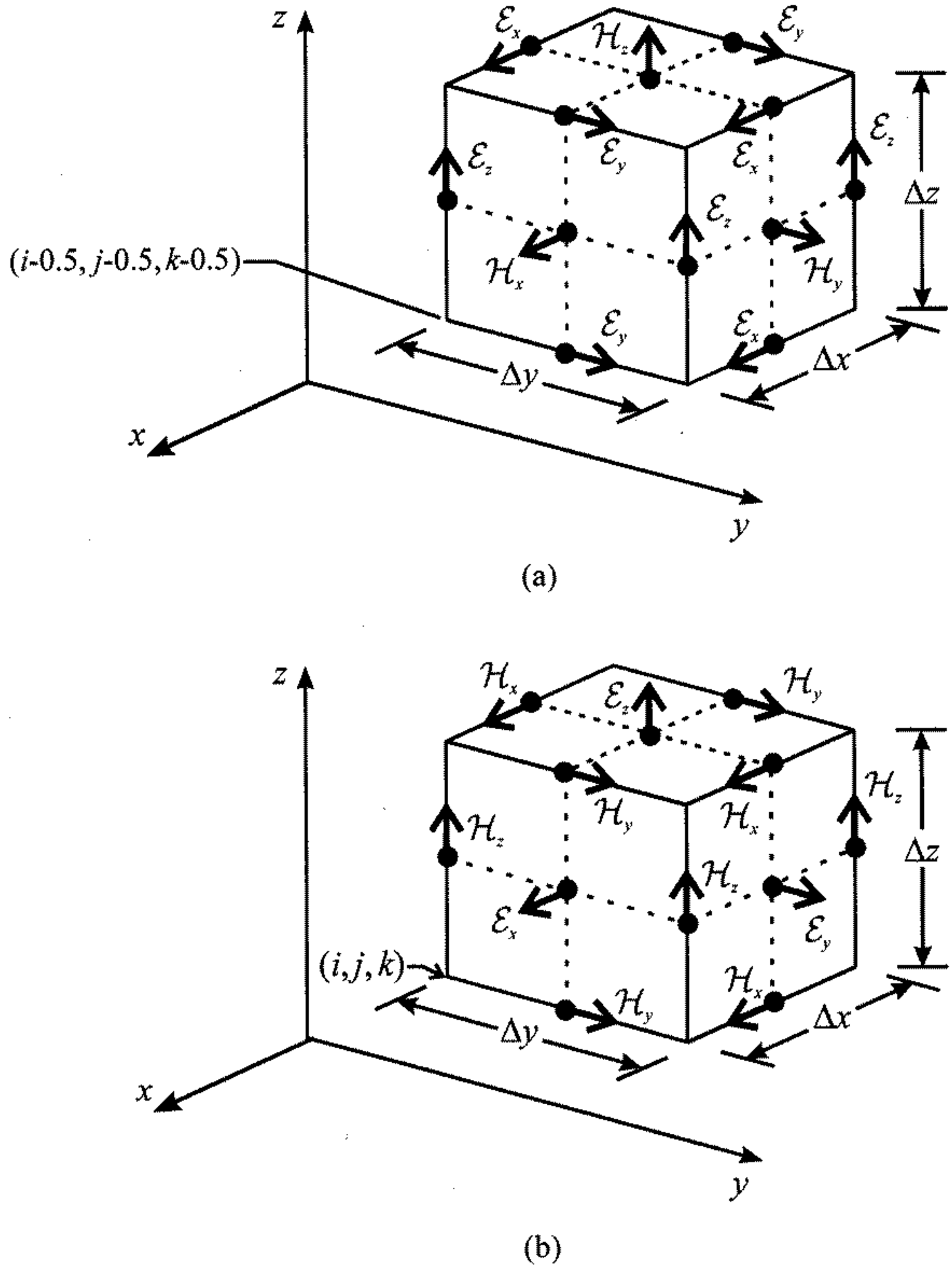


Figure 1 Unit cell of 3D spatial Yee grid/lattice with faces centered on (a) magnetic and (b) electric field components.

The diff. eqn. for D_x is:

$$\kappa_y \frac{\partial D_x}{\partial t} + \frac{\sigma_y}{\epsilon_0} D_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

$\downarrow$ Discretize about $t=(n+\frac{1}{2})\Delta t$ & $(i\Delta x, (j+\frac{1}{2})\Delta y, (k+\frac{1}{2})\Delta z)$

simple ave.

$$\kappa_y\left(\frac{D_x^{n+1}(i,j+\frac{1}{2},k+\frac{1}{2}) - D_x^{n}(i,j+\frac{1}{2},k+\frac{1}{2})}{\Delta t}\right) + \frac{\sigma_y}{\epsilon_0}\left(\frac{D_x^{n+1}(i,j+\frac{1}{2},k+\frac{1}{2}) + D_x^{n}(i,j+\frac{1}{2},k+\frac{1}{2})}{2}\right)$$

$$= \left(\frac{H_z^{n+\frac{1}{2}}(i,j+1,k+\frac{1}{2}) - H_z^{n+\frac{1}{2}}(i,j,k+\frac{1}{2})}{\Delta y}\right) - \left(\frac{H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k+1) - H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k)}{\Delta z}\right)$$

Solving for $D_x^{n+1}()$ yields:

$$\boxed{\begin{aligned} D_x^{n+1}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) &= \left(\frac{2\epsilon_0\kappa_y - \sigma_y\Delta t}{2\epsilon_0\kappa_y + \sigma_y\Delta t}\right) D_x^{n}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) \\ &+ \left(\frac{2\epsilon_0\Delta t}{2\epsilon_0\kappa_y + \sigma_y\Delta t}\right)\left[\left(\frac{H_z^{n+\frac{1}{2}}(i,j+1,k+\frac{1}{2}) - H_z^{n+\frac{1}{2}}(i,j,k+\frac{1}{2})}{\Delta y}\right) - \left(\frac{H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k+1) - H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k)}{\Delta z}\right)\right]\end{aligned}} \tag{7.85}$$

* Values for κ_y and σ_y depend on where in the PML this component is located, e.g. on left and right sides they are tapered, elsewhere $\kappa_y = 1$ and $\sigma_y = 0$.

7.8.1 cont.

Now, that we have the aux. variable D_x updated, what is E_x?

In the phasor domain, we defined

$$D_x = \epsilon \frac{K_z + \frac{\sigma_z}{j\omega\epsilon_0}}{K_x + \frac{\sigma_x}{j\omega\epsilon_0}} E_x = \epsilon \frac{s_z}{s_x} E_x$$

$$\left(K_x + \frac{\sigma_x}{j\omega\epsilon_0}\right) D_x = \epsilon\left(K_z + \frac{\sigma_z}{j\omega\epsilon_0}\right) E_x$$

$$\left(j\omega K_x + \frac{\sigma_x}{\epsilon_0}\right) D_x = \epsilon\left(j\omega K_z + \frac{\sigma_z}{\epsilon_0}\right) E_x$$

Doing an inverse F.T. gives

$$K_x \frac{\partial \mathcal{D}_x}{\partial t} + \frac{\sigma_x}{\epsilon_0} \mathcal{D}_x = \epsilon K_z \frac{\partial \mathcal{E}_x}{\partial t} + \frac{\sigma_z \epsilon}{\epsilon_0} \mathcal{E}_x$$

$\downarrow$ Discretizing

$$K_x\left(\frac{D_x^{n+1}(i, j+1/2, k+1/2) - D_x^{n}(\,)}{\Delta t}\right) + \frac{\sigma_x}{\epsilon_0}\left(\frac{D_x^{n+1}(\,) + D_x^{n}(\,)}{2}\right)$$

$$= \epsilon_0 K_z \left(\frac{E_x^{n+1}(\,) - E_x^{n}(\,)}{\Delta t}\right) + \frac{\sigma_z \epsilon}{\epsilon_0}\left(\frac{E_x^{n+1}(\,) + E_x^{n}(\,)}{2}\right)$$

7.8.1 cont.

Solving for E_x^{n+1} yields

$$\boxed{E_x^{n+1}(\,) = \left(\frac{2\epsilon_0 \kappa_z - \sigma_z \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t}\right) E_x^n(\,) + \frac{1}{(2\epsilon_0 \kappa_z + \sigma_z \Delta t)\epsilon}\left[(2\epsilon_0 \kappa_x + \sigma_x \Delta t) D_x^{n+1}(\,) - (2\epsilon_0 \kappa_x - \sigma_x \Delta t) D_x^n(\,)\right] \quad (7.88)}$$

→ updates for D_y & E_y and D_z & E_z are similar

Similar process for Faraday's Law using

$$B_x = \mu \frac{s_z}{s_x} H_x$$

$$B_y = \mu \frac{s_x}{s_y} H_y$$

$$B_z = \mu \frac{s_y}{s_z} H_z$$

in

$$\begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix} = -j\omega\mu \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

7.8.1 cont.

to get

$$\begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix} = -j\omega \begin{bmatrix} S_y & 0 & 0 \\ 0 & S_z & 0 \\ 0 & 0 & S_x \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

⋮

Update eqns for B & H

→ stability under Courant limits for UPML update has been shown

→ UPML satisfies Gauss' Law

→ In all cases, the values of σ_w and K_w depend on location, e.g. $\sigma_w = 0$ and $K_w = 1$ in the modeling space (if PML used throughout FDTD grid)

7.8.2 Computer Implementation of the UPML

For E_x (+ D_x) in a general 3D UPML, we have the following update equations

$$\boxed{\begin{aligned} D_x^{n+1}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) &= \left[\frac{2\epsilon_0 \kappa_y - \sigma_y \Delta t}{2\epsilon_0 \kappa_y + \sigma_y \Delta t}\right] D_x^{n}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) \\ &+ \left(\frac{2\epsilon_0 \Delta t}{2\epsilon_0 \kappa_y + \sigma_y \Delta t}\right)\left[\left(\frac{H_z^{n+1/2}(i, j+1, k+\tfrac{1}{2}) - H_z^{n+1/2}(i, j, k+\tfrac{1}{2})}{\Delta y}\right) \right. \\ &\qquad \left. - \left(\frac{H_y^{n+1/2}(i, j+\tfrac{1}{2}, k+1) - H_y^{n+1/2}(i, j+\tfrac{1}{2}, k)}{\Delta z}\right)\right] \end{aligned}}$$

and

$$\boxed{\begin{aligned} E_x^{n+1}(i, j+\tfrac{1}{2}, k+\tfrac{1}{2}) &= \left(\frac{2\epsilon_0 \kappa_z - \sigma_z \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t}\right) E_x^{n}(\,) \\ &+ \left(\frac{2\epsilon_0 \kappa_x + \sigma_x \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t}\right) \frac{D_x^{n+1}(\,)}{\epsilon} \\ &- \left(\frac{2\epsilon_0 \kappa_x - \sigma_x \Delta t}{2\epsilon_0 \kappa_z + \sigma_z \Delta t}\right) \frac{D_x^{n}(\,)}{\epsilon} \end{aligned}}$$

where σ_x + κ_x can vary wrt x
σ_y + κ_y " " " y
σ_z + κ_z " " " z

Remember $\sigma_w = 0$ (no attenuation of prop. waves) + $K_w = 1$ (no attenuation of evanescent/reactive fields) within the working volume (modeling space).

For D_x + E_x, we evaluate

$K_x + \sigma_x$ @ $x = i\Delta x$ per $\sigma_x(x) = \frac{|x - x_{interface}|^m}{d^m}\sigma_{x,max}$

and, if used, $K_x(x) = 1 + (K_{x,max} - 1)\frac{|x - x_{interface}|^m}{d^m}$

when $\left.\begin{array}{l} x = i\Delta x \leq x_{min} \\ + \ x = i\Delta x \geq x_{max} \end{array}\right\} \leftarrow$ edges/interface between UPML + working volume

Similarly

$K_y + \sigma_y$ are evaluated at $(j+\frac{1}{2})\Delta y$

$K_z + \sigma_z$ are evaluated at $(k+\frac{1}{2})\Delta z$

when in the UPML region.

7.8.2 cont.

Looking at the update equations, we see that six distinct coefficients can be defined

$$C1(j+\tfrac{1}{2}) = \frac{2\epsilon_0 K_y(j+\tfrac{1}{2}) - \sigma_y(j+\tfrac{1}{2})\Delta t}{2\epsilon_0 K_y(j+\tfrac{1}{2}) + \sigma_y(j+\tfrac{1}{2})\Delta t}$$

$$C2(j+\tfrac{1}{2}) = \frac{2\epsilon_0 \Delta t}{2\epsilon_0 K_y(j+\tfrac{1}{2}) + \sigma_y(j+\tfrac{1}{2})\Delta t}$$

$$C3(k+\tfrac{1}{2}) = \frac{2\epsilon_0 K_z(k+\tfrac{1}{2}) - \sigma_z(k+\tfrac{1}{2})\Delta t}{2\epsilon_0 K_z(k+\tfrac{1}{2}) + \sigma_z(k+\tfrac{1}{2})\Delta t}$$

$$C4(k+\tfrac{1}{2}) = \frac{1}{2\epsilon_0 K_z(k+\tfrac{1}{2}) + \sigma_z(k+\tfrac{1}{2})\Delta t}$$

$$C5(i) = 2\epsilon_0 K_x(i) + \sigma_x(i)\Delta t$$

$$C6(i) = 2\epsilon_0 K_x(i) - \sigma_x(i)\Delta t$$

Note; w/in working vol.

$C1(\,) = 1$

$C2(\,) = \Delta t$

$C3(\,) = 1$

$C4(\,) = \frac{1}{2\epsilon_0}$

$C5(\,) = 2\epsilon_0$

$C6(\,) = 2\epsilon_0$

7.8.2 cont.

Assuming the Yee cell shown in Chap 3 notes and wanting to place tangential E field components on the PEC walls outside the UPML

$E_x(I, J, K=1) = E_x(I, J, K_{max}) = 0$ (Top + Bottom)

$E_x(I, J=1, K) = E_x(I, J_{max}, K) = 0$ (Left + Right)

in the x-direction $1 \le I \le I_{max} - 1$ ← $DX/2$ in from E_y + E_z components on Front + Back of volume

Pseudocode for E_x + D_x updates

```
DO 10  K = 2, KMAX-1
  DO 20  J = 2, JMAX-1
    DO 30  I = 1, IMAX-1
          DOLD = DX(I,J,K)   <- save old value of DX
Dx^n()   ->
Dx^n+1() ->  DX(I,J,K) = C1(J) * DX(I,J,K)
                  + C2(J) * ((Hz(I,J,K) - HZ(I,J-1,K))
Ex^n+1() ->  EX(I,J,K) = C3(K)*EX(I,J,K) + C4(K)*C5(I)*DX(I,J,K)/E
                  - C4(K)*C6(I)*DXOLD/E
    30 CONTINUE
  20 CONTINUE
10 CONTINUE
```