Chapter 4 Numerical Dispersion and Stability

4.1 Introduction

- FDTD update equation introduced in Chap3 cause nonphysical (i.e. numerical) dispersion in simulated signals/waves

- Numerical dispersion varies with
  * wavelength or frequency
  * direction of wave propagation
  * $N_{\lambda,i}$ (grid discretization)

- Stability requirement for Δt relative to Δx, Δy, +/or Δz (Courant stability factor)

4.2 Derivation of the Numerical Dispersion Relation for Two-Dimensional Wave Propagation

→ Based on TM_z mode (3.13) w/ no sources and $\sigma = \sigma^* = 0$

→ Valid for any 2D TE or TM mode

TM_z mode equations for these conditions

$$\frac{\partial \mathcal{H}_x}{\partial t} = -\frac{1}{\mu}\frac{\partial \mathcal{E}_z}{\partial y} \quad (4.1a)$$

$$\frac{\partial \mathcal{H}_y}{\partial t} = +\frac{1}{\mu}\frac{\partial \mathcal{E}_z}{\partial x} \quad (4.1b)$$

$$\frac{\partial \mathcal{E}_z}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y}\right) \quad (4.1c)$$

→ Discretizing (4.1) and assuming a homogeneous media (ϵ + μ are constant) yields

$$\frac{H_x^{n+1/2}(i, j+1/2) - H_x^{n-1/2}(i, j+1/2)}{\Delta t} = -\frac{1}{\mu}\,\frac{E_z^{n}(i, j+1) - E_z^{n}(i, j)}{\Delta y} \quad (4.2a)$$

$$\frac{H_y^{n+1/2}(i+1/2, j) - H_y^{n-1/2}(i+1/2, j)}{\Delta t} = \frac{1}{\mu}\,\frac{E_z^{n}(i+1, j) - E_z^{n}(i, j)}{\Delta x} \quad (4.2b)$$

4.2 cont.

$$\frac{E_z^{n+1}(i,j) - E_z^{n}(i,j)}{\Delta t} = \frac{1}{\epsilon}\left[\frac{H_y^{n+1/2}(i+1/2,j) - H_y^{n-1/2}(i-1/2,j)}{\Delta x} - \frac{H_x^{n+1/2}(i,j+1/2) - H_x^{n+1/2}(i,j-1/2)}{\Delta y}\right] \quad (4.2c)$$

→ The next step is to assume a single frequency plane wave and substitute into (4.2)

Plane wave (general expression, apply appropriate indices)

$$E_z^{n}(I,J) = E_{z_0}\, e^{j(\omega n\Delta t - \tilde{k}_x I\Delta x - \tilde{k}_y J\Delta y)} \quad (4.3a)$$

$$H_x^{n}(I,J) = H_{x_0}\, e^{j(\omega n\Delta t - \tilde{k}_x I\Delta x - \tilde{k}_y J\Delta y)} \quad (4.3b)$$

$$H_y^{n}(I,J) = H_{y_0}\, e^{j(\omega n\Delta t - \tilde{k}_x I\Delta x - \tilde{k}_y J\Delta y)} \quad (4.3c)$$

* (I,J) generic location
* n generic time step
* $\tilde{k}_x$ + $\tilde{k}_y$ are numerical wave numbers in the x- and y-directions
* ω - angular frequency of plane wave

4.2 cont.

→ Putting (4.3) into (4.2a) yields

$$\frac{H_{x_0} e^{j(\omega(n+\frac{1}{2})\Delta t - \tilde{k}_x i\Delta x - \tilde{k}_y (j+\frac{1}{2})\Delta y)} - H_{x_0} e^{j(\omega(n-\frac{1}{2})\Delta t - \tilde{k}_x i\Delta x - \tilde{k}_y (j+\frac{1}{2})\Delta y)}}{\Delta t}$$

$$= -\frac{1}{\mu}\, \frac{E_{z0} e^{j(\omega n\Delta t - \tilde{k}_x i\Delta x - \tilde{k}_y (j+1)\Delta y)} - E_{z0} e^{j(\omega n\Delta t - \tilde{k}_x i\Delta x - \tilde{k}_y (j)\Delta y)}}{\Delta y}$$

→ Divide both sides by $e^{j(\omega n\Delta t - \tilde{k}_x i\Delta x - \tilde{k}_y j\Delta y)}$ to get

$$\frac{H_{x_0} e^{j(\omega \Delta t/2 - \tilde{k}_y \Delta y/2)} - H_{x_0} e^{j(-\omega \Delta t/2 - \tilde{k}_y \frac{\Delta y}{2})}}{\Delta t}$$

$$= -\frac{1}{\mu}\, \frac{E_{z0} e^{-j\tilde{k}_y \Delta y} - E_z e^{0}}{\Delta y}$$

→ Multiply both sides by $e^{+j\tilde{k}_y \Delta y/2}$ to get

$$\frac{H_{x_0}\left[e^{j\frac{\omega\Delta t}{2}} - e^{-j\frac{\omega\Delta t}{2}}\right]}{\Delta t} = \frac{+E_{z0}\left(e^{+j\frac{\tilde{k}_y\Delta y}{2}} - e^{-j\frac{\tilde{k}_y\Delta y}{2}}\right)}{\mu\Delta y}$$

4.2 cont.

Use Euler identity $e^{+jA} - e^{-jA} = 2j \sin A$

to get

$$\frac{H_{x0}\, 2j \sin\left(\frac{\omega \Delta t}{2}\right)}{\Delta t} = \frac{E_{z0}\, 2j \sin\left(\frac{\tilde{k}_y \Delta y}{2}\right)}{\mu \Delta y}$$

which simplifies to

$$H_{x0} = \frac{\Delta t\, E_{z0}}{\mu \Delta y} \frac{\sin\left(\frac{\tilde{k}_y \Delta y}{2}\right)}{\sin\left(\frac{\omega \Delta t}{2}\right)} \qquad (4.4a)$$

Similarly, using the update eqns (4.2b) + (4.2c) we get

$$H_{y0} = -\frac{\Delta t\, E_{z0}}{\mu \Delta x} \frac{\sin\left(\frac{\tilde{k}_x \Delta x}{2}\right)}{\sin\left(\frac{\omega \Delta t}{2}\right)} \qquad (4.4b)$$

and

$$E_{z0} \sin\left(\frac{\omega \Delta t}{2}\right) = \frac{\Delta t}{\epsilon}\left[\frac{H_{x0}}{\Delta y} \sin\left(\frac{\tilde{k}_y \Delta y}{2}\right) - \frac{H_{y0}}{\Delta x} \sin\left(\frac{\tilde{k}_x \Delta x}{2}\right)\right] \qquad (4.4c)$$

Substituting (4.4a) + (4.4b) into (4.4c), we get

$$\boxed{\left[\frac{1}{c\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right)\right]^2 = \left[\frac{1}{\Delta x}\sin\left(\frac{\tilde{k}_x \Delta x}{2}\right)\right]^2 + \left[\frac{1}{\Delta y}\sin\left(\frac{\tilde{k}_y \Delta y}{2}\right)\right]^2 \quad (4.5)}$$

where $c = \frac{1}{\sqrt{\mu\epsilon}}$ is the velocity of light in the material.

General numerical dispersion relation (2D)

→ Special case of square-grid $\Delta x = \Delta y \equiv \Delta$ where the Courant stability factor $S = c\Delta t/\Delta$ and defining the grid sampling density $N_\lambda = \frac{\lambda_0}{\Delta}$

gives

$$\frac{1}{S^2}\sin^2\left(\frac{\pi S}{N_\lambda}\right) = \sin^2\left(\frac{\Delta\tilde{k}\cos\phi}{2}\right) + \sin^2\left(\frac{\Delta\tilde{k}\sin\phi}{2}\right) \quad (4.6)$$

where $\tilde{k}_x = \tilde{k}\cos\phi$ and $\tilde{k}_y = \tilde{k}\sin\phi$ and ϕ is the angle of wave propagation wrt the positive x-axis

The equation of (4.6) can be reduced to the 1D case by letting $\phi = 0$

$$\frac{1}{S^2}\sin^2\left(\frac{\pi S}{N_\lambda}\right) = \sin^2\left(\frac{\Delta\tilde{k}}{2}\right)$$

$$\boxed{\begin{aligned} &\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right) = \sin\left(\frac{\Delta\tilde{k}}{2}\right) && (4.7a) \\ \text{OR}\quad &\tilde{k} = \frac{2}{\Delta}\sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)\right] && (4.7b) \end{aligned}}$$

General numerical dispersion relation (1D)

4.3 Extension to three Dimensions

The 3D General numerical dispersion relation is

$$\boxed{\left[\frac{1}{c\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right)\right]^2 = \left[\frac{1}{\Delta x}\sin\left(\frac{\tilde{k}_x\Delta x}{2}\right)\right]^2 + \left[\frac{1}{\Delta y}\sin\left(\frac{\tilde{k}_y\Delta y}{2}\right)\right]^2 + \left[\frac{1}{\Delta z}\sin\left(\frac{\tilde{k}_z\Delta z}{2}\right)\right]^2 \qquad (4.12)}$$

4.4 Comparison with the Ideal Dispersion Case

By definition, the wave numbers (in 3D) are related to frequency + velocity (or wavelength)

$$\left(\frac{2\pi}{\lambda}\right)^2 = \left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 \qquad (4.13)$$

$\rightarrow$ (4.13) can be obtained from (4.12) by letting $\Delta x, \Delta y, + \Delta z \rightarrow 0$ and using $\sin(A) \simeq A$ for $A \ll 1$

$$\left[\frac{1}{c\Delta t}\left(\frac{\omega\Delta t}{2}\right)\right]^2 \simeq \left[\frac{1}{\Delta x}\frac{\tilde{k}_x \Delta x}{2}\right]^2 + \left[\frac{1}{\Delta y}\frac{\tilde{k}_y \Delta y}{2}\right]^2 + \left[\frac{1}{\Delta z}\frac{\tilde{k}_z \Delta z}{2}\right]^2$$

Now mulitply thru by 2^2 to get

$$\left(\frac{\omega}{c}\right)^2 = \tilde{k}_x^2 + \tilde{k}_y^2 + \tilde{k}_z^2 = k_x^2 + k_y^2 + k_z^2$$

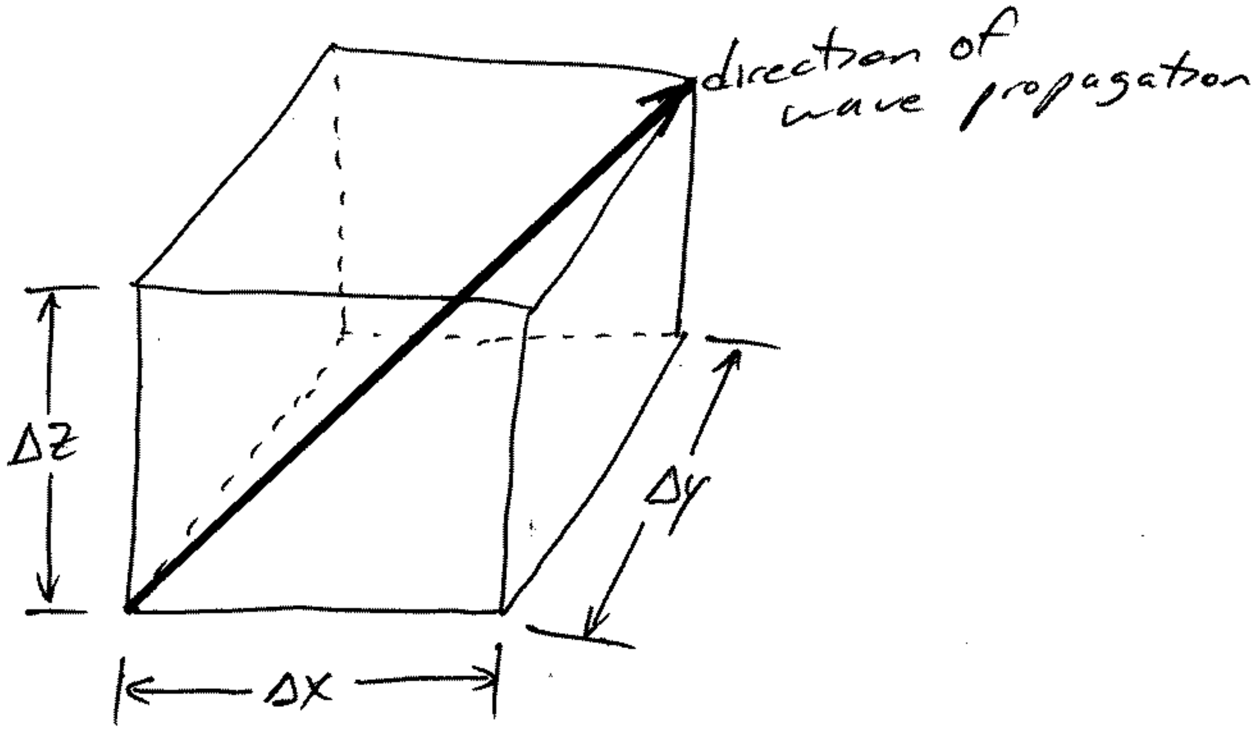
So, in the limit as $\Delta x, \Delta y, + \Delta z \rightarrow 0$

$$\tilde{k}_x = k_x$$
$$\tilde{k}_y = k_y$$
$$\tilde{k}_z = k_z$$

$\Rightarrow$ Reduce numerical dispersion by using small spatial steps

4.4 cont.

→ (4.1) also reduces to (4.13) if we choose $S = \frac{1}{\sqrt{3}}$ and the wave propagates along the diagonal of the 3D lattice



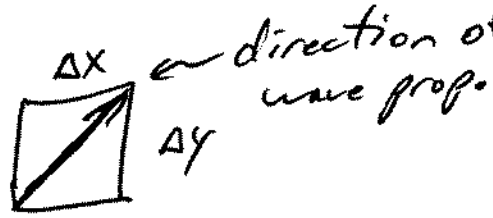
Here $\tilde{k}_x = \tilde{k}_y = \tilde{k}_z = \frac{\tilde{k}}{\sqrt{3}}$ and

$$\left.\begin{array}{l} \tilde{k}_x = \tilde{k}_y = \tilde{k}_z = \frac{\tilde{k}}{\sqrt{3}} \text{ and} \\ \tilde{k}_x^{\,2} + \tilde{k}_y^{\,2} + \tilde{k}_z^{\,2} = \tilde{k}^{\,2} = \left(\omega/c\right)^2 \end{array}\right\} \Rightarrow \text{No dispersion on diagonal!}$$

Similarly, in two dimensions, we get

$$\tilde{k}_x^{\,2} + \tilde{k}_y^{\,2} = \left(\omega/c\right)^2$$

when $S = 1/\sqrt{2}$ and



4.5 Anisotropy of the Numerical Phase Velocity

→ phase velocity change w/ direction of wave propagation in the 2D & 3D FDTD lattices/grids

4.5.1 Sample Values of Numerical Phase Velocity

→ 2D case w/ $\Delta x = \Delta y \equiv \Delta$ (square grid) has the dispersion equation

$$\frac{1}{S^2}\sin^2\left(\frac{\pi S}{N_\lambda}\right) = \sin^2\left(\frac{\Delta \tilde{k}\cos\phi}{2}\right) + \sin^2\left(\frac{\Delta \tilde{k}\sin\phi}{2}\right) \quad (4.6)$$

for $\phi = 0$,

$$\frac{1}{S^2}\sin^2\left(\frac{\pi S}{N_\lambda}\right) = \sin^2\left(\frac{\Delta \tilde{k}}{2}\right) + 0$$

$$\hookrightarrow \tilde{k} = \frac{2}{\Delta}\sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)\right] \quad (4.14a)$$

which can be used to find the numerical phase velocity

$$\tilde{v}_p = \frac{\omega}{\tilde{k}} = \frac{\pi}{N_\lambda \sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)\right]}\, c \quad (4.14b)$$

→ same holds true for $\phi = 90°, 180°, 270°$ (wave propagation along coordinate axes)

4.5.1 cont.

For $\phi = 45°, 135°, 225°, + 315°$, $|\cos\phi| = |\sin\phi| = \sqrt{2}/2$

and we get

$$\tilde{k} = \frac{2\sqrt{2}}{\Delta} \sin^{-1}\left[\frac{1}{S\sqrt{2}} \sin\left(\frac{\pi S}{N_\lambda}\right)\right] \qquad (4.15a)$$

$$\tilde{v}_p = \frac{\omega}{\tilde{k}} = \frac{\pi}{N_\lambda \sqrt{2} \sin^{-1}\left[\frac{1}{S\sqrt{2}} \sin\left(\frac{\pi S}{N_\lambda}\right)\right]} c \qquad (4.15b)$$

↳ wave propagation along diagonals

ex. If $S = 0.5$ and $N_\lambda = 20$

Coordinate axes	Diagonals
$\tilde{v}_p = 0.996892c$	$\tilde{v}_p = 0.998968c$

$$\frac{0.998968c}{0.996892c} = 1.00208$$

↑

0.2% difference!

Ex. 2D numerical phase velocity

$$vp\_major(S, N\lambda) := \frac{\pi}{N\lambda \cdot \operatorname{asin}\left(\frac{1}{S} \cdot \sin\left(\frac{\pi \cdot S}{N\lambda}\right)\right)}$$

$$vp\_diag(S, N\lambda) := \frac{\pi}{N\lambda \cdot \sqrt{2} \cdot \operatorname{asin}\left(\frac{1}{S \cdot \sqrt{2}} \cdot \sin\left(\frac{\pi \cdot S}{N\lambda}\right)\right)}$$

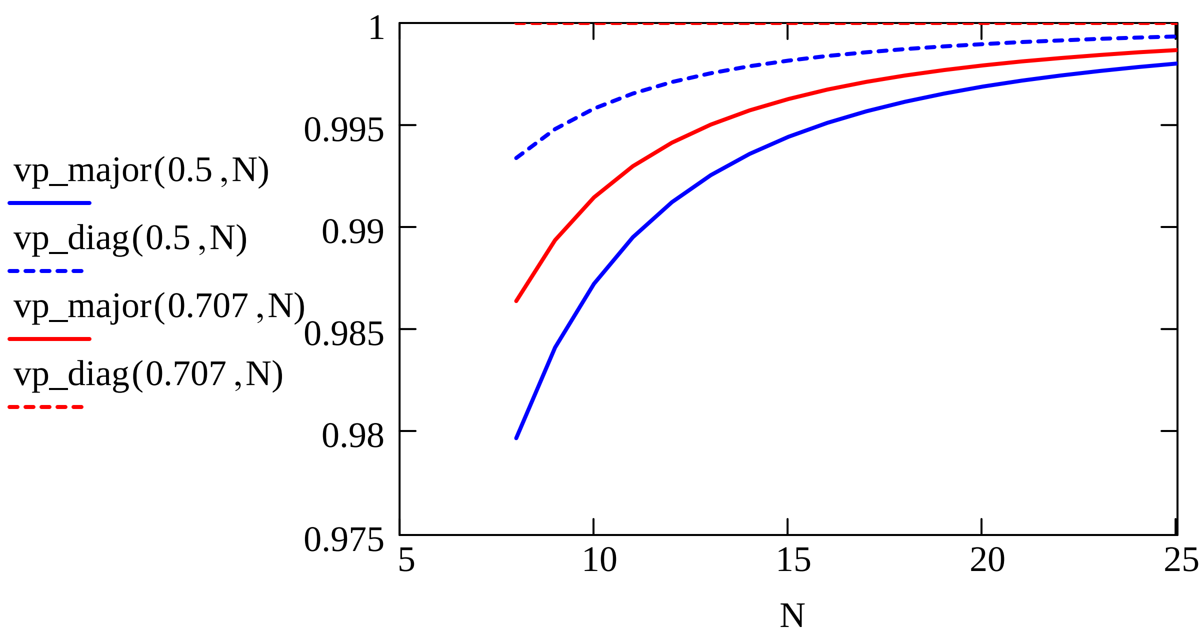
$vp\_major(0.5, 20) = 0.99689$ $vp\_major(0.7071, 20) = 0.99793$

$vp\_diag(0.5, 20) = 0.99897$ $vp\_diag(0.7071, 20) = 1$

$vp\_major(0.5, 10) = 0.98726$ $vp\_major(0.7071, 10) = 0.99149$

$vp\_diag(0.5, 10) = 0.99582$ $vp\_diag(0.7071, 10) = 1$

$N := 8 .. 25$



4.5.1 cont.

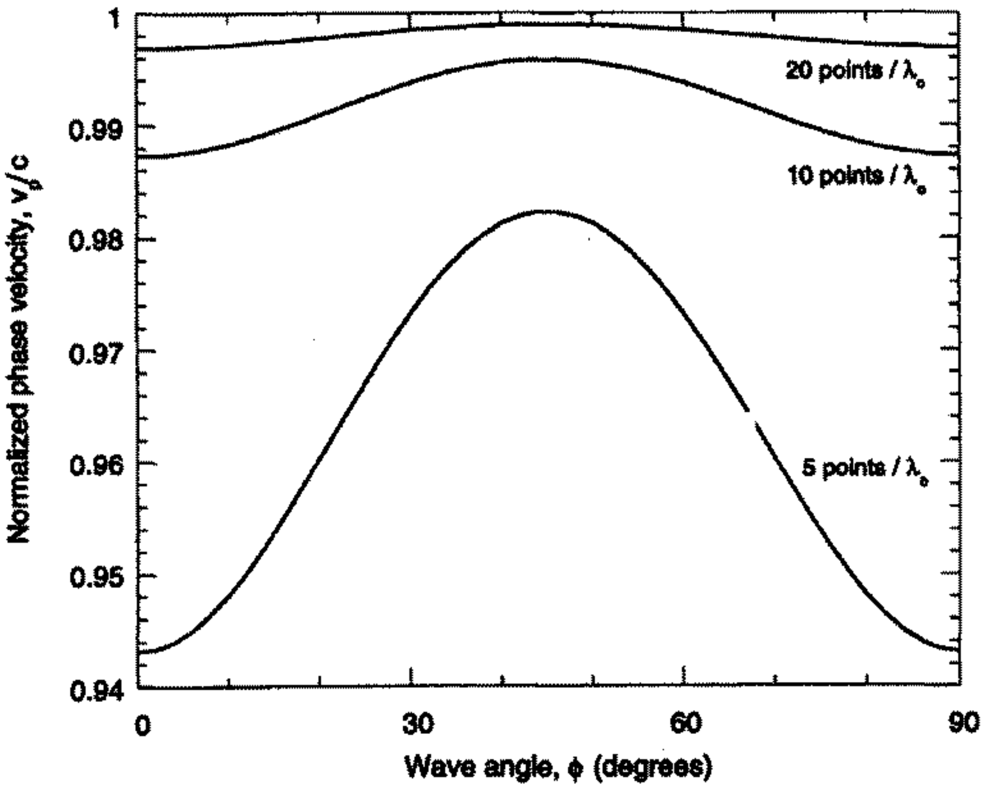


Fig. 4.2 Variation of numerical phase velocity with wave-propagation angle in a two-dimensional FDTD grid for three sampling densities of the square unit cells. $S = c\Delta t/\Delta = 0.5$ for all of the cases.

* error decreases w/ increasing N_λ

4.5.1 cont.

Key measures of $\tilde{v}_p$

$$\Delta \tilde{v}_{physical}\Big|_{N_\lambda} = \frac{\min[\tilde{v}_p(\phi)] - c}{c} \times 100\%$$

$$\Delta \tilde{v}_{aniso}\Big|_{N_\lambda} = \frac{\max[\tilde{v}_p(\phi)] - \min[\tilde{v}_p(\phi)]}{\min[\tilde{v}_p(\phi)]} \times 100\%$$

* $\Delta \tilde{v}_{physical}$ a good measure of how numerical waves lead or lag wrt c (i.e. physical wave)

* $\Delta \tilde{v}_{anis}$ a good measure of how the <u>wavefront</u> is distorted by $\tilde{v}_p$ being a function of ϕ

* errors are cumulative w/ distance through the grid
  i.e. error for 10λ is 10x larger than for λ

* there are ways to mitigate these problems (usually w/ some computational cost)

4.5.2 Intrinsic Grid Velocity Anisotropy

$\Delta \tilde{v}_{aniso} \leftarrow$ how much does $\tilde{v}$ vary wrt ϕ?

w/ $N_\lambda = 20$

$$S = \frac{1}{\sqrt{2}} \qquad \left.\begin{array}{l} \tilde{v}_p(0^\circ) = 0.997926c \\ \tilde{v}_p(45^\circ) = c \end{array}\right\} \Delta \tilde{v}_{aniso} = 0.208\%$$

$$S = 1/2 \qquad \left.\begin{array}{l} \tilde{v}_p(0^\circ) = 0.995859c \\ \tilde{v}(45^\circ) = 0.997937c \end{array}\right\} \Delta \tilde{v}_{aniso} = 0.208\%$$

Key points!

* $\Delta \tilde{v}_{aniso}$ only weakly dependent on S or Δt for $N_\lambda > 10$
* Δt choice does affect $\Delta \tilde{v}_{physical}$ which is due to both $(\Delta x, \Delta y, \Delta z)$ & Δt and the contributions can cancel or reinforce
* $\Delta \tilde{v}_{anis}$ only weakly dependent on time-marching scheme (2nd order, 4th order, ...)
* Numerical dispersion errors isotropic wrt propagation direction of wave

4.6 Complex-Valued Numerical Wavenumbers

* If N_λ too small $\tilde{k}$ becomes complex
* Can get superluminal waves

4.6.1 Case 1: Numerical Wave Prop. along the Principal Lattice Axes

$$\tilde{k} = \frac{2}{\Delta}\sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)\right] = \frac{2}{\Delta}\sin^{-1}(\zeta) \quad (4.29a)$$

where $\zeta = \frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)$

* $\tilde{k}$ goes from real to complex when

$$\zeta = 1$$

$$\hookrightarrow N_\lambda\Big|_{transition} = \frac{\pi S}{\sin^{-1}(S)}$$

* $\tilde{k}$ complex for $N_\lambda < N_\lambda\big|_{transition}$

ex. $S = 0.5$

$$N_\lambda\Big|_{transition} = \frac{\pi(0.5)}{\sin^{-1}(0.5)} = 3$$

4.6.1 cont.

* $N_\lambda > N_\lambda|_{trans}$

$$\tilde{k} = \tilde{k}_{real} = \frac{2}{\Delta}\sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)\right]$$

$$\tilde{k}_{imag} = 0$$

$$\hookrightarrow \quad \tilde{v}_p = \frac{\omega}{\tilde{k}_{real}} = \frac{\pi}{N_\lambda \sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\pi S}{N_\lambda}\right)\right]}\, c$$

wave multiplier (see eqn 2.21) $\quad e^{\tilde{k}_{imag}\Delta} \equiv e^{-\alpha\Delta} = e^{0} = 1$

* $N_\lambda < N_\lambda|_{trans}$

$$\tilde{k}_{real} = \pi/\Delta \qquad (4.36)$$

$$\tilde{k}_{imag} = -\frac{2}{\Delta}\ln\left(\zeta + \sqrt{\zeta^2 - 1}\right)$$

$$\tilde{v}_p = \frac{\omega}{\tilde{k}_{real}} = \frac{\omega}{(\pi/\Delta)} = \frac{2}{N_\lambda}\, c \qquad (4.37a)$$

$$e^{\tilde{k}_{imag}\Delta} \equiv e^{-\alpha\Delta} = \frac{1}{\zeta + \sqrt{\zeta^2 - 1}} \qquad (4.37b)$$

4.6.1 cont.

What is the maximum velocity?

From Nyquist $\lambda_{o,min} = c/f_{max} = \dfrac{c}{\left(\frac{1}{2\Delta t}\right)}$ ← 2 samples per cycle

$$= 2c\Delta t \qquad (4.38a)$$

The minimum grid sampling density is

$$N_{\lambda,min} = \frac{\lambda_{o,min}}{\Delta} = \frac{2c\Delta t}{\Delta} = 2S \qquad (4.38b)$$

using (4.37a)

$$\tilde{v}_{p,max} = \frac{2}{N_{\lambda,min}} c = c/S \qquad (4.39a)$$

$$= \frac{c}{\left(\frac{c\Delta t}{\Delta}\right)} = \frac{\Delta}{\Delta t} \qquad (4.39b)$$

At most, the wave can propagate one spatial cell (Δ) per time step Δt (ie. $\Delta = \tilde{v}_{p,max}\Delta t$) in the FDTD model

→ Similar results on the diagonal shown in section 4.6.2

4.7 Numerical Stability

→ Analysis based on 2D + 3D dispersion relations of (4.5) and (4.12)

→ complex-frequency analysis similar to that done in chapter 2

4.7.1 Complex-Frequency Analysis

→ discretized space + time (x_I, y_J, z_K, t_n)

→ complex numerical angular frequency $\tilde{\omega} = \tilde{\omega}_{real} + j\,\tilde{\omega}_{imag}$

Assume a field vector of the form

$$\bar{V}^n_{(I,J,K)} = \bar{V}_0\, e^{j[(\tilde{\omega}_{real} + j\tilde{\omega}_{imag})n\Delta t - \tilde{k}_x I\Delta x - \tilde{k}_y J\Delta y - \tilde{k}_z K\Delta z]}$$

$$= \bar{V}_0\, e^{-\tilde{\omega}_{imag} n\Delta t}\, e^{j[\tilde{\omega}_{real} n\Delta t - \tilde{k}_x I\Delta x - \tilde{k}_y J\Delta y - \tilde{k}_z K\Delta z]} \qquad (4.49)$$

↑ possible exponential decay/increase

↓ leads to

$$\left[\frac{1}{c\Delta t}\sin\left(\frac{\tilde{\omega}\Delta t}{2}\right)\right]^2 = \left[\frac{1}{\Delta x}\sin\left(\frac{\tilde{k}_x \Delta x}{2}\right)\right]^2 + \left[\frac{1}{\Delta y}\sin\left(\frac{\tilde{k}_y \Delta y}{2}\right)\right]^2 + \left[\frac{1}{\Delta z}\sin\left(\frac{\tilde{k}_z \Delta z}{2}\right)\right]^2 \qquad (4.50)$$

4.7.1 cont.

Solving for $\tilde{\omega}$ yields

$$\tilde{\omega} = \frac{2}{\Delta t} \sin^{-1}\left\{ \underbrace{c\Delta t \sqrt{\left[\frac{\sin\left(\frac{\tilde{k}_x \Delta x}{2}\right)}{\Delta x}\right]^2 + \left[\frac{\sin\left(\frac{\tilde{k}_y \Delta y}{2}\right)}{\Delta y}\right]^2 + \left[\frac{\sin\left(\frac{\tilde{k}_z \Delta z}{2}\right)}{\Delta z}\right]^2}}_{\xi} \right\} \qquad (4.51)$$

Now the range of ξ is

$$0 \leq \xi \leq c\Delta t \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}} = \xi_{\substack{upper \\ bound}} \qquad (4.52)$$

where $\xi_{\substack{upper \\ bound}}$ occurs when each sine term is equal to one, or

$$\tilde{k}_x = \pm \pi/\Delta x \;, \quad \tilde{k}_y = \pm \pi/\Delta y \;, \; \& \; \tilde{k}_z = \pm \pi/\Delta z$$

For $0 \leq \xi \leq 1$, $\tilde{\omega} = \tilde{\omega}_{real} \Rightarrow$ <u>Stable</u>

For $\xi > 1$, $\tilde{\omega} = \tilde{\omega}_{real} + j\tilde{\omega}_{imag} \Rightarrow$ <u>Unstable</u>

This can only occur if

$$\xi_{\substack{upper \\ bound}} = c\Delta t \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}} > 1$$

4.7.1 cont.

or

$$\Delta t > \frac{1}{c\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}} \qquad \underline{3D}$$

$$S = \frac{c\Delta t}{\Delta}$$

For example, if $\Delta x = \Delta y = \Delta z$ (cubic lattice)

$$\Delta t = \frac{\Delta}{c\sqrt{3}}$$

$$\underline{S_{max}} = \frac{c(\Delta/c\sqrt{3})}{\Delta} = \frac{1}{\sqrt{3}} = \underline{0.577} \qquad 3D$$

$$2D \qquad \Delta t > \frac{1}{c\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}}} \quad \overset{\text{square}}{\Longrightarrow} \quad S_{max} = \frac{1}{\sqrt{2}} = \underline{0.707}$$

$$1D \qquad \Delta t > \frac{\Delta x}{c} \quad \Longrightarrow \quad \underline{S_{max} = 1}$$