# EE692 Applied EM- FDTD Method
# Chapter 3 Introduction to Maxwell's Equations and the Yee Algorithm

## 3.2 Maxwell's Equations in Three Dimensions

- Assume **no** electric or magnetic current sources
- Faraday's Law

$$\frac{\partial \bar{\mathcal{B}}}{\partial t} = -\bar{\nabla} \times \bar{\mathcal{E}} - \bar{\mathcal{M}}$$

$$\frac{\partial}{\partial t} \iint_S \bar{\mathcal{B}} \bullet d\bar{s} = -\oint_L \bar{\mathcal{E}} \bullet d\bar{l} - \iint_S \bar{\mathcal{M}} \bullet d\bar{s} \qquad (3.1)$$

where $\bar{\mathcal{M}}$ is an equivalent magnetic current density (V/m$^2$).

- Ampere's Law

$$\frac{\partial \bar{\mathcal{D}}}{\partial t} = \bar{\nabla} \times \bar{\mathcal{H}} - \bar{\mathcal{J}}$$

$$\frac{\partial}{\partial t} \iint_S \bar{\mathcal{D}} \bullet d\bar{s} = \oint_L \bar{\mathcal{H}} \bullet d\bar{l} - \iint_S \bar{\mathcal{J}} \bullet d\bar{s} \qquad (3.2)$$

- Gauss' Law

$$\bar{\nabla} \bullet \bar{\mathcal{D}} = 0$$

$$\oiint_S \bar{\mathcal{D}} \bullet d\bar{s} = 0 \qquad (3.3)$$

- Gauss' Law (equivalent for magnetic fields)

$$\bar{\nabla} \bullet \bar{\mathcal{B}} = 0$$

$$\oiint_S \bar{\mathcal{B}} \bullet d\bar{s} = 0 \qquad (3.4)$$

## 3.2 continued

- For simple media (i.e., homogeneous, isotropic, & linear), the following constitutive relations hold

$$\bar{\mathcal{B}} = \mu \bar{\mathcal{H}} = \mu_r \mu_0 \bar{\mathcal{H}}$$
$$\bar{\mathcal{D}} = \varepsilon \bar{\mathcal{E}} = \varepsilon_r \varepsilon_0 \bar{\mathcal{E}} \tag{3.5}$$

- Define the electric and equivalent magnetic current densities as

$$\bar{\mathcal{J}} = \bar{\mathcal{J}}_{\text{source}} + \sigma \bar{\mathcal{E}}$$
$$\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{source}} + \sigma^* \bar{\mathcal{H}} \tag{3.6}$$

where $\sigma$ is the conductivity (S/m) and $\sigma^*$ is the equivalent magnetic loss ($\Omega$/m), allows for electrically and magnetically lossy materials.

- With these relations, Ampere's and Faraday's Laws become

$$\frac{\partial \bar{\mathcal{H}}}{\partial t} = -\frac{1}{\mu}\left(\bar{\nabla} \times \bar{\mathcal{E}}\right) - \frac{1}{\mu}\left(\bar{\mathcal{M}}_{\text{source}} + \sigma^* \bar{\mathcal{H}}\right) \tag{3.7}$$

$$\frac{\partial \bar{\mathcal{E}}}{\partial t} = \frac{1}{\varepsilon}\bar{\nabla} \times \bar{\mathcal{H}} - \frac{1}{\varepsilon}\left(\bar{\mathcal{J}}_{\text{source}} + \sigma \bar{\mathcal{E}}\right) \tag{3.8}$$

respectively. Since Gauss' Laws are NOT truly independent of Faraday's and Ampere's Laws, they are not necessary for setting up FDTD update equations for the electric and magnetic fields. However, they must be obeyed when setting up the FDTD grid(s).

## 3.2 continued

- Equations (3.7) and (3.8) can be split (3 equations/each) out into vector-component scalar differential equations. These equations are interdependent. While other orthogonal coordinate systems can be used, usually we use Cartesian coordinates for the FDTD method.

From Faraday's Law

$$\frac{\partial \mathcal{H}_x}{\partial t} = \frac{1}{\mu}\left[\frac{\partial \mathcal{E}_y}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial y} - \left(\mathcal{M}_{\text{source},x} + \sigma^* \mathcal{H}_x\right)\right]$$

$$\frac{\partial \mathcal{H}_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial \mathcal{E}_z}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial z} - \left(\mathcal{M}_{\text{source},y} + \sigma^* \mathcal{H}_y\right)\right]$$

$$\frac{\partial \mathcal{H}_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial \mathcal{E}_x}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial x} - \left(\mathcal{M}_{\text{source},z} + \sigma^* \mathcal{H}_z\right)\right] \qquad (3.9)$$

From Ampere's Law

$$\frac{\partial \mathcal{E}_x}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} - \left(\mathcal{J}_{\text{source},x} + \sigma \mathcal{E}_x\right)\right]$$

$$\frac{\partial \mathcal{E}_y}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x} - \left(\mathcal{J}_{\text{source},y} + \sigma \mathcal{E}_y\right)\right]$$

$$\frac{\partial \mathcal{E}_z}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} - \left(\mathcal{J}_{\text{source},z} + \sigma \mathcal{E}_z\right)\right] \qquad (3.10)$$

## 3.3 Reduction to Two Dimensions

- Some problems do not vary with respect to one dimension. Therefore, they reduce to two dimensions.
- For example, say the problem is infinite in the *z*-direction with no variations in material or structure cross-section. Then, there are no spatial changes with respect to *z*, i.e., $\frac{\partial()}{\partial z} = 0$. Then,

From Faraday's Law

$$\frac{\partial \mathcal{H}_x}{\partial t} = \frac{1}{\mu}\left[ -\frac{\partial \mathcal{E}_z}{\partial y} - \left( \mathcal{M}_{\text{source},x} + \sigma^* \mathcal{H}_x \right) \right] \tag{3.11a}$$

$$\frac{\partial \mathcal{H}_y}{\partial t} = \frac{1}{\mu}\left[ \frac{\partial \mathcal{E}_z}{\partial x} - \left( \mathcal{M}_{\text{source},y} + \sigma^* \mathcal{H}_y \right) \right] \tag{3.11b}$$

$$\frac{\partial \mathcal{H}_z}{\partial t} = \frac{1}{\mu}\left[ \frac{\partial \mathcal{E}_x}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial x} - \left( \mathcal{M}_{\text{source},z} + \sigma^* \mathcal{H}_z \right) \right] \tag{3.11c}$$

From Ampere's Law

$$\frac{\partial \mathcal{E}_x}{\partial t} = \frac{1}{\varepsilon}\left[ \frac{\partial \mathcal{H}_z}{\partial y} - \left( \mathcal{J}_{\text{source},x} + \sigma \mathcal{E}_x \right) \right] \tag{3.12a}$$

$$\frac{\partial \mathcal{E}_y}{\partial t} = \frac{1}{\varepsilon}\left[ -\frac{\partial \mathcal{H}_z}{\partial x} - \left( \mathcal{J}_{\text{source},y} + \sigma \mathcal{E}_y \right) \right] \tag{3.12b}$$

$$\frac{\partial \mathcal{E}_z}{\partial t} = \frac{1}{\varepsilon}\left[ \frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} - \left( \mathcal{J}_{\text{source},z} + \sigma \mathcal{E}_z \right) \right] \tag{3.12c}$$

- Examining these equations, we see that they can be grouped into two groups of three inter-related equations each. Each group has a unique combination of three (total) electric & magnetic components.

## 3.3.1 TM$_z$ Mode

- This group includes $\mathcal{H}_x$, $\mathcal{H}_y$, and $\mathcal{E}_z$ where the magnetic field components are transverse to the *z*-direction.
- The applicable differential equations are

$$\frac{\partial \mathcal{H}_x}{\partial t} = \frac{1}{\mu}\left[-\frac{\partial \mathcal{E}_z}{\partial y} - \left(\mathcal{M}_{\text{source},x} + \sigma^* \mathcal{H}_x\right)\right] \quad (3.13a)$$

$$\frac{\partial \mathcal{H}_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial \mathcal{E}_z}{\partial x} - \left(\mathcal{M}_{\text{source},y} + \sigma^* \mathcal{H}_y\right)\right] \quad (3.13b)$$

$$\frac{\partial \mathcal{E}_z}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} - \left(\mathcal{J}_{\text{source},z} + \sigma \mathcal{E}_z\right)\right] \quad (3.13c)$$

- $\mathcal{E}_z$ <u>can not</u> exist near metallic or PEC surfaces that infinite in extent in the *z*-direction and still satisfy the tangential electric field boundary conditions.
- This implies that surface waves or ‘creeping’ waves will not be found in this mode.

### 3.3.2 TE$_z$ Mode

- This group includes $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{H}_z$ where the electric field components are transverse to the *z*-direction.
- The applicable differential equations are

$$\frac{\partial \mathcal{E}_x}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_z}{\partial y} - \left(\mathcal{J}_{\text{source},x} + \sigma \mathcal{E}_x\right)\right] \quad (3.14a)$$

$$\frac{\partial \mathcal{E}_y}{\partial t} = \frac{1}{\varepsilon}\left[-\frac{\partial \mathcal{H}_z}{\partial x} - \left(\mathcal{J}_{\text{source},y} + \sigma \mathcal{E}_y\right)\right] \quad (3.14b)$$

$$\frac{\partial \mathcal{H}_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial \mathcal{E}_x}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial x} - \left(\mathcal{M}_{\text{source},z} + \sigma^* \mathcal{H}_z\right)\right] \quad (3.14c)$$

- In this mode, the electric field components ($\mathcal{E}_x$ and $\mathcal{E}_y$) <u>can</u> exist near metallic or PEC surfaces that are infinite in extent in the *z*-direction.
- This implies that surface waves or ‘creeping’ waves are possible in this mode.

## 3.4 Reduction to One Dimension

- Some problems do not vary with respect to two dimensions. Therefore, these problems reduce to one dimension.
- For example, say the problem is infinite in the *y*- & *z*-directions with no variations in material or structure cross-section. Then, there are no spatial changes with respect to *y* & *z*, i.e., $\frac{\partial()}{\partial y} = \frac{\partial()}{\partial z} = 0$.

## 3.4.1 *x*-directed, *z*-polarized TEM Mode

- Eliminate all the $\frac{\partial()}{\partial y}$ terms in the TM$_z$ mode equations (3.13). Also, assume no source in the *x*-direction and that $\mathcal{H}_x(t=0)=0$ to get

$$\frac{\partial \mathcal{H}_x}{\partial t} = \frac{1}{\mu}\left[-\left(\mathcal{M}_{\text{source},x} + \sigma^* \mathcal{H}_x\right)\right] = 0 \qquad (3.15a)$$

$$\frac{\partial \mathcal{H}_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial \mathcal{E}_z}{\partial x} - \left(\mathcal{M}_{\text{source},y} + \sigma^* \mathcal{H}_y\right)\right] \qquad (3.15b)/(3.16a)$$

$$\frac{\partial \mathcal{E}_z}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_y}{\partial x} - \left(\mathcal{J}_{\text{source},z} + \sigma \mathcal{E}_z\right)\right] \qquad (3.15c)/(3.16b)$$

- Only have $\mathcal{H}_y$ and $\mathcal{E}_z$ (hence z-polarization) field components and wave propagation in ±*x*-directions.

## 3.4.2 *x*-directed, *y*-polarized TEM Mode

Eliminate all the $\frac{\partial()}{\partial y}$ terms in the TE$_z$ mode equations (3.14). Also, assume no source in the x-direction and that $\mathcal{E}_x(t=0)=0$ to get

$$\frac{\partial \mathcal{E}_x}{\partial t} = \frac{1}{\varepsilon}\left[-\left(\mathcal{J}_{\text{source},x} + \sigma \mathcal{E}_x\right)\right] = 0 \qquad (3.17a)$$

$$\frac{\partial \mathcal{E}_y}{\partial t} = \frac{1}{\varepsilon}\left[-\frac{\partial \mathcal{H}_z}{\partial x} - \left(\mathcal{J}_{\text{source},y} + \sigma \mathcal{E}_y\right)\right] \qquad (3.17b)/(3.18a)$$

$$\frac{\partial \mathcal{H}_z}{\partial t} = \frac{1}{\mu}\left[-\frac{\partial \mathcal{E}_y}{\partial x} - \left(\mathcal{M}_{\text{source},z} + \sigma^* \mathcal{H}_z\right)\right] \qquad (3.17c)/(3.18b)$$

- Only have $\mathcal{E}_y$ (hence *y*-polarization) and $\mathcal{H}_z$ field components and wave propagation in ±*x*-directions.

## 3.5 Equivalence to the Wave Equation in One Dimension

- Assuming the source terms in the $x$-directed, $z$-polarized TEM Mode equations (3.16) are zero, and that σ = σ* = 0. The differential equations reduce to

$$\frac{\partial \mathcal{H}_y}{\partial t} = \frac{1}{\mu}\frac{\partial \mathcal{E}_z}{\partial x} \text{ and } \frac{\partial \mathcal{E}_z}{\partial t} = \frac{1}{\varepsilon}\frac{\partial \mathcal{H}_y}{\partial x}.$$

- Taking another partial derivative with respect to time yields

$$\frac{\partial^2 \mathcal{H}_y}{\partial t^2} = \frac{1}{\mu}\frac{\partial^2 \mathcal{E}_z}{\partial x \partial t} \text{ and } \frac{\partial^2 \mathcal{E}_z}{\partial t^2} = \frac{1}{\varepsilon}\frac{\partial^2 \mathcal{H}_y}{\partial x \partial t}.$$

- Taking another partial derivative with respect to $x$ yields

$$\frac{\partial^2 \mathcal{H}_y}{\partial x \partial t} = \frac{1}{\mu}\frac{\partial^2 \mathcal{E}_z}{\partial t^2} \text{ and } \frac{\partial^2 \mathcal{E}_z}{\partial x \partial t} = \frac{1}{\varepsilon}\frac{\partial^2 \mathcal{H}_y}{\partial t^2}.$$

- By cross-substituting, we then find the one-dimensional, single variable, scalar wave equations

$$\frac{\partial^2 \mathcal{H}_y}{\partial t^2} = \frac{1}{\mu\varepsilon}\frac{\partial^2 \mathcal{H}_y}{\partial x^2} = c^2 \frac{\partial^2 \mathcal{H}_y}{\partial x^2} \quad (3.19c)$$

and

$$\frac{\partial^2 \mathcal{E}_z}{\partial t^2} = \frac{1}{\mu\varepsilon}\frac{\partial^2 \mathcal{E}_z}{\partial x^2} = c^2 \frac{\partial^2 \mathcal{E}_z}{\partial x^2} \quad (3.20c)$$

where $c = 1/\sqrt{\mu\varepsilon}$ is the speed of light in the material.

## 3.6 The Yee Algorithm

### 3.6.1 Basic Ideas

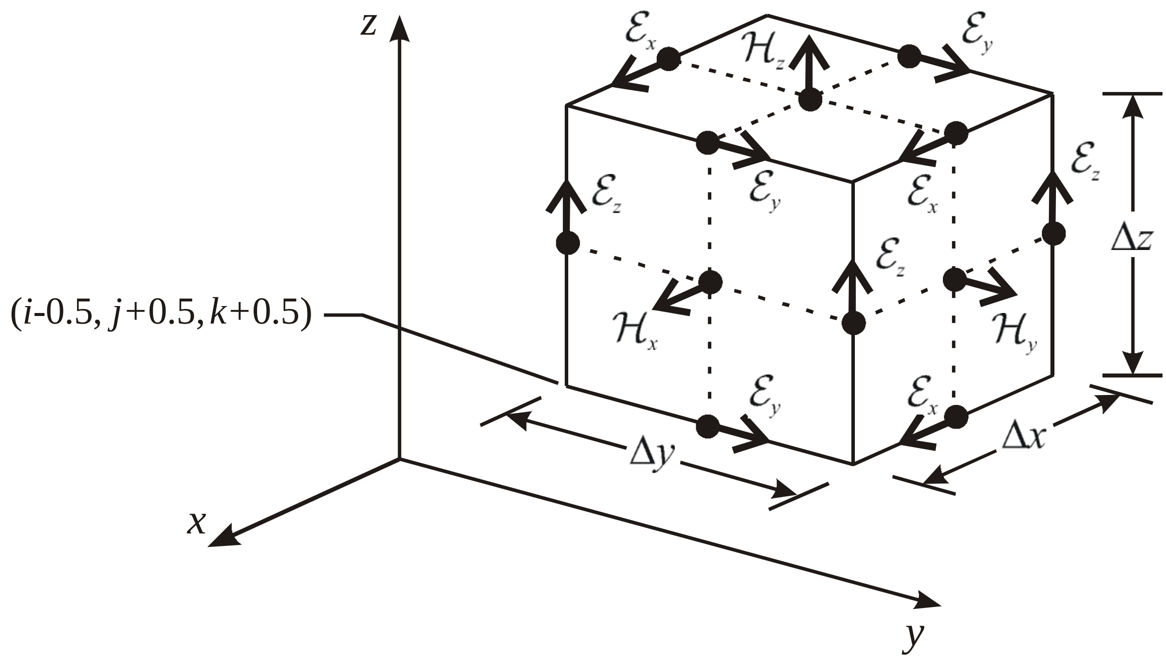
- Seminal paper (hand-out) was published in May of 1966.
- Yee's FDTD equations assumed σ = σ* = 0 and no $\bar{\mathcal{M}}$.
- Algorithm does NOT use scalar wave equations, but instead is based on coupled/inter-related differential equations (i.e., the curl equations of Ampere's and Faraday's Laws). Therefore, we have both electric and magnetic field components.
- Having both electric and magnetic field components gives tremendous flexibility in modeling shapes (e.g., can handle thin wires, slots, corners where the electric field varies as $1/r^2$, …) as well as changing material properties (changes can be made on a cell-by-cell basis).
- A unit cell in the grid/lattice that Yee selected is shown in Figure 1. Note how the field components are all centered spatially with respect to one another in the *x*-, *y*-, and *z*-directions.
- Figure 1 also illustrates how the integral forms of Faraday's and Ampere's Laws (omit currents and assume simple media)

$$\oint_L \bar{\mathcal{E}} \bullet d\bar{l} = -\frac{\partial}{\partial t} \iint_S \mu \bar{\mathcal{H}} \bullet d\bar{s}$$

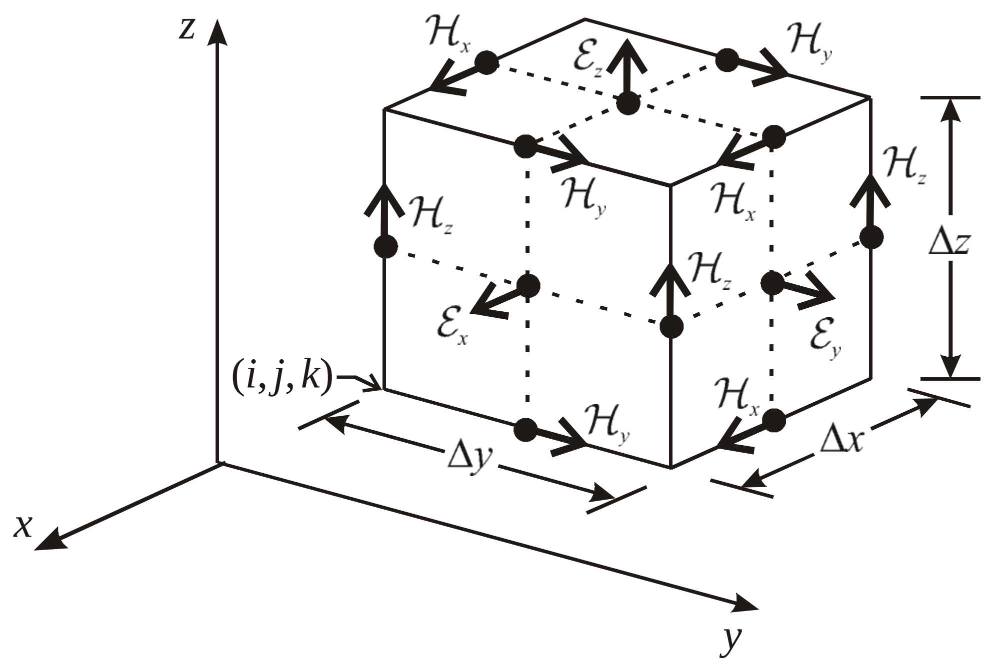
$$\oint_L \bar{\mathcal{H}} \bullet d\bar{l} = \frac{\partial}{\partial t} \iint_S \varepsilon \bar{\mathcal{E}} \bullet d\bar{s}$$

are satisfied. For Ampere's Law, note how $\mathcal{E}_x$ is centered on the front face of the Figure 1(b) and how $\mathcal{H}_y$ and $\mathcal{H}_z$ circulate around the contour surrounding the surface of the front face of the cube.

## 3.6.1 continued




(a)




(b)

Figure 1 Unit cell of 3D spatial Yee grid/lattice with faces centered on (a) magnetic and (b) electric field components.

## 3.6.1 continued

- Overall both the differential and integral forms of Maxwell's equations are modeled well by the unit cells shown (including Gauss' Laws).
- The discretization of the spatial derivatives will be accomplished using second-order accurate central-difference approximations.
- The Yee cells make it easy to satisfy tangential electric and magnetic boundary conditions, if the material boundaries align with the Cartesian axes.
- Tremendous flexibility in modeling different materials (e.g., $\varepsilon$, $\mu$, $\sigma$, and $\sigma^*$). These material properties can be specified on a cell-by-cell basis which gives a 'staircase' or step approximation to changes in the material/structure.
- Note that there is no divergence for either the electric or magnetic field components in the Yee cells. Gauss' Laws are implicitly satisfied.
- The field quantities are interleaved in both space **and time** (see Figure 2). Note how the electric field components are placed at integer multiples of the time step $\Delta t$ while the magnetic field components are placed halfway in between.
- This allows for an algorithm which successively updates the electric and magnetic field components in a fashion referred to as 'time-steping', 'leapfrog', 'recursive', or 'time-marching'. E.g.,
  - @ times $(n+0.5)\Delta t$, update the $\mathcal{H}$ components using the prior values & spatially adjacent $\mathcal{E}$ components from time $n\Delta t$.
  - @ times $(n+1)\Delta t$, update the $\mathcal{E}$ components using the prior values & spatially adjacent $\mathcal{H}$ components from time $(n+0.5)\Delta t$.

**3.6.1 continued**

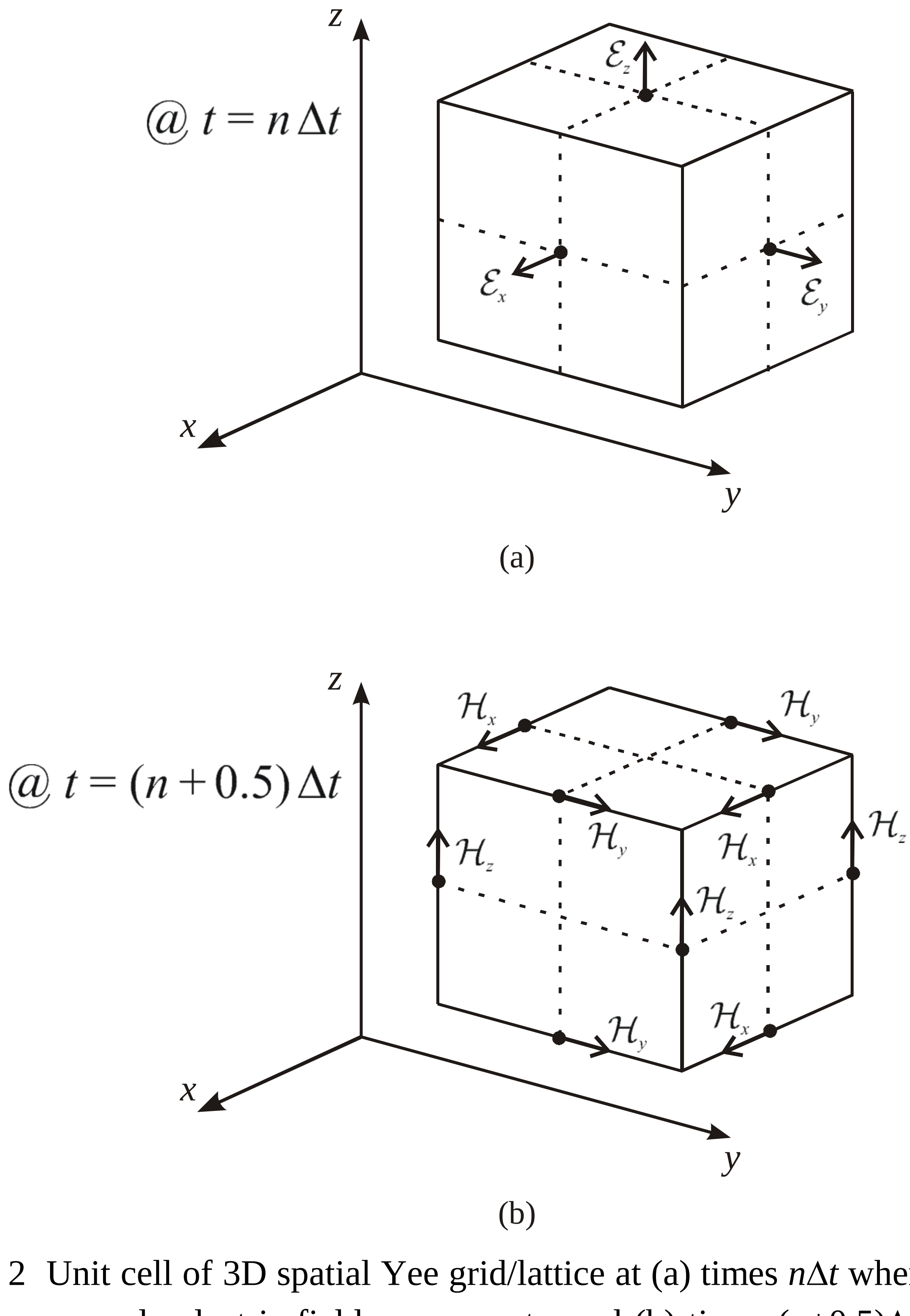



Figure 2 Unit cell of 3D spatial Yee grid/lattice at (a) times $n\Delta t$ where there are only electric field components, and (b) times $(n+0.5)\Delta t$ where there are only magnetic field components.

## 3.6.1 continued

- As part of setting up these update equations, the discretization of the temporal derivatives is also accomplished using second-order accurate central-difference approximations.
- Note that the Yee FDTD algorithm has no matrix inversions. Unlike MoM and finite-elements techniques, each field component is individually and explicitly calculated.
- The Yee FDTD algorithm is nondissipative, i.e., waves do not decay due to numerical/non-physical reasons.

## 3.6.2 Finite Differences and Notation

- As part of setting up these update equations, we will adopt a shorthand notation for expressing the spatial locations and time. E.g.,

$$\mathcal{U}(x = i\Delta x, y = j\Delta y, z = k\Delta z, t = n\Delta t) = U^n(i, j, k)$$

where $i$, $j$, and $k$ are integer indices associated with the $x$-, $y$-, and $z$-coordinate directions and $n$ is an integer index associated with time. Here, $\Delta x, \Delta y, \Delta z,$ and $\Delta t$ are the spatial and temporal step sizes.

- Doing a Taylor's series expansion of $\mathcal{U}(x, y, z, t)$ about $x = i\Delta x$ gives

$$\frac{\partial \mathcal{U}(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial x} = \frac{U^n(i+0.5, j, k) - U^n(i-0.5, j, k)}{\Delta x} + O\left[(\Delta x)^2\right]$$

Note the similarity to the results of Chapter 2. The key difference is that the steps forward and backward were $\Delta x / 2$ instead of a full $\Delta x$.

- The second-order accurate central-difference approximation to the spatial derivative with respect to $x$ is then

$$\frac{\partial \mathcal{U}(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial x} \approx \frac{U^n(i+0.5, j, k) - U^n(i-0.5, j, k)}{\Delta x}$$

- Similarly, the second-order accurate central-difference approximation to the derivative with respect to time $t = n\Delta t$ is then

$$\frac{\partial \mathcal{U}(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial t} \approx \frac{U^{n+0.5}(i, j, k) - U^{n-0.5}(i, j, k)}{\Delta t}$$

### 3.6.3 Finite-Difference Expressions for Maxwell's Equations in Three Dimensions

- For some unknown reason, the text authors have reversed the usual convention that electric field components are placed at times $t = n\Delta t$ while the magnetic field components are placed at times $t = (n+0.5)\Delta t$. I will follow the usual convention. Remember the relative locations of the field components (both spatially and temporally) are important, not the specific indices.
- Assume material properties $(\mu, \varepsilon, \sigma, \sigma^*)$ are time-invariant.
- To begin deriving the update equations, we'll start with the scalar differential equation (3.10a) for the $x$-component of Ampere's Law

$$\frac{\partial \mathcal{E}_x}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} - \left(\mathcal{J}_{\text{source},x} + \sigma \mathcal{E}_x\right)\right]$$

- Per Figure 1(b), $\mathcal{E}_x$ (on the front face) is located at the spatial location ($x = i\Delta x$, $y = (j+0.5)\Delta y$, $z = (k+0.5)\Delta z$) and the time derivative is approximated about time $t = (n+0.5)\Delta t$. When discretized, this equation becomes

$$\frac{E_x^{n+1}(i, j+0.5, k+0.5) - E_x^{n}(i, j+0.5, k+0.5)}{\Delta t} =$$

$$\frac{1}{\varepsilon_{(i,j+0.5,k+0.5)}}\begin{bmatrix} \dfrac{H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i, j, k+0.5)}{\Delta y} \\ -\dfrac{H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i, j+0.5, k)}{\Delta z} \\ -J_{\text{source},x}^{n+0.5}(i, j+0.5, k+0.5) \\ +\sigma(i, j+0.5, k+0.5)E_x^{n+0.5}(i, j+0.5, k+0.5) \end{bmatrix}$$

## 3.6.3 continued

- Now the field component $E_x^{n+0.5}(i, j+0.5, k+0.5)$ is not available in the grid. Therefore, it is approximated, using linear interpolation, as

$$E_x^{n+0.5}(i, j+0.5, k+0.5) \approx \frac{E_x^{n+1}(i, j+0.5, k+0.5) + E_x^{n}(i, j+0.5, k+0.5)}{2}$$

- Using this approximation and solving for $E_x^{n+1}$, yields the update equation

$$E_x^{n+1}(i, j+0.5, k+0.5) = \left( \frac{1 - \frac{\sigma_{(i,j+0.5,k+0.5)}\Delta t}{2\varepsilon_{(i,j+0.5,k+0.5)}}}{1 + \frac{\sigma_{(i,j+0.5,k+0.5)}\Delta t}{2\varepsilon_{(i,j+0.5,k+0.5)}}} \right) E_x^{n}(i, j+0.5, k+0.5)$$

$$+ \left( \frac{\frac{\Delta t}{\varepsilon_{(i,j+0.5,k+0.5)}}}{1 + \frac{\sigma_{(i,j+0.5,k+0.5)}\Delta t}{2\varepsilon_{(i,j+0.5,k+0.5)}}} \right) \left[ \begin{array}{l} \frac{H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i, j, k+0.5)}{\Delta y} \\ - \frac{H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i, j+0.5, k)}{\Delta z} \\ - J_{\text{source},x}^{n+0.5}(i, j+0.5, k+0.5) \end{array} \right] \quad (3.29a)$$

- Similarly, the update equations for $\mathcal{E}_y$, located on the right face of Figure 1(b) at ($x$ = ($i$-0.5)$\Delta x$, $y$ = ($j$+1)$\Delta y$, $z$ = ($k$+0.5)$\Delta z$), and $\mathcal{E}_z$, located on the top face of Figure 1(b) at ($x$ = ($i$-0.5)$\Delta x$, $y$ = ($j$+0.5)$\Delta y$, $z$ = ($k$+1)$\Delta z$), with the time derivatives approximated about time $t = (n+0.5)\Delta t$ are found from Ampere's Law using the scalar differential equations (3.10b) and (3.10c) as

## 3.6.3 continued

$$E_y^{n+1}(i-0.5, j+1, k+0.5) = \left( \frac{1 - \dfrac{\sigma_{(i-0.5, j+1, k+0.5)} \Delta t}{2\varepsilon_{(i-0.5, j+1, k+0.5)}}}{1 + \dfrac{\sigma_{(i-0.5, j+1, k+0.5)} \Delta t}{2\varepsilon_{(i-0.5, j+1, k+0.5)}}} \right) E_y^n(i-0.5, j+1, k+0.5)$$

$$+ \left( \frac{\dfrac{\Delta t}{\varepsilon_{(i-0.5, j+1, k+0.5)}}}{1 + \dfrac{\sigma_{(i-0.5, j+1, k+0.5)} \Delta t}{2\varepsilon_{(i-0.5, j+1, k+0.5)}}} \right) \left[ \begin{array}{l} \dfrac{H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j+1, k)}{\Delta z} \\ - \dfrac{H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i-1, j+1, k+0.5)}{\Delta x} \\ - J_{\text{source},y}^{n+0.5}(i-0.5, j+1, k+0.5) \end{array} \right]$$

(3.29b)

and

$$E_z^{n+1}(i-0.5, j+0.5, k+1) = \left( \frac{1 - \dfrac{\sigma_{(i-0.5, j+0.5, k+1)} \Delta t}{2\varepsilon_{(i-0.5, j+0.5, k+1)}}}{1 + \dfrac{\sigma_{(i-0.5, j+0.5, k+1)} \Delta t}{2\varepsilon_{(i-0.5, j+0.5, k+1)}}} \right) E_z^n(i-0.5, j+0.5, k+1)$$

$$+ \left( \frac{\dfrac{\Delta t}{\varepsilon_{(i-0.5, j+0.5, k+1)}}}{1 + \dfrac{\sigma_{(i-0.5, j+0.5, k+1)} \Delta t}{2\varepsilon_{(i-0.5, j+0.5, k+1)}}} \right) \left[ \begin{array}{l} \dfrac{H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i-1, j+0.5, k+1)}{\Delta x} \\ - \dfrac{H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j, k+1)}{\Delta y} \\ - J_{\text{source},z}^{n+0.5}(i-0.5, j+0.5, k+1) \end{array} \right]$$

(3.29c)

## 3.6.3 continued

- In a similar fashion, the update equations for the magnetic field components are found using Faraday's Law using the scalar differential equations (3.11a) - (3.11c) for $\mathcal{H}_x$, located on the upper right edge of Figure 1(b) at ($x = (i$-0.5)$\Delta x$, $y = (j$+1)$\Delta y$, $z = (k$+1)$\Delta z$), $\mathcal{H}_y$, located on the upper front edge of Figure 1(b) at ($x = i\Delta x$, $y = (j$+0.5)$\Delta y$, $z = (k$+1)$\Delta z$), and $\mathcal{H}_z$, located on the right front edge of Figure 1(b) at ($x = i\Delta x$, $y = (j$+1)$\Delta y$, $z = (k$+0.5)$\Delta z$) with the time derivatives approximated about time $t = n\Delta t$.

- The update equations for the magnetic field components are

$$H_x^{n+0.5}(i-0.5,\, j+1, k+1) = \left( \frac{1 - \dfrac{\sigma^*_{(i-0.5,\, j+1, k+1)}\Delta t}{2\mu_{(i-0.5,\, j+1, k+1)}}}{1 + \dfrac{\sigma^*_{(i-0.5,\, j+1, k+1)}\Delta t}{2\mu_{(i-0.5,\, j+1, k+1)}}} \right) H_x^{n-0.5}(i-0.5,\, j+1, k+1)$$

$$+ \left( \frac{\dfrac{\Delta t}{\mu_{(i-0.5,\, j+1, k+1)}}}{1 + \dfrac{\sigma^*_{(i-0.5,\, j+1, k+1)}\Delta t}{2\mu_{(i-0.5,\, j+1, k+1)}}} \right) \left[ \begin{array}{l} \dfrac{E_y^n(i-0.5,\, j+1, k+1.5) - E_y^n(i-0.5,\, j+1, k+0.5)}{\Delta z} \\ - \dfrac{E_z^n(i-0.5,\, j+1.5, k+1) - E_z^n(i-0.5,\, j+0.5, k+1)}{\Delta y} \\ - M^n_{\text{source},x}(i-0.5,\, j+1, k+1) \end{array} \right]$$

(3.30a)

## 3.6.3 continued

$$H_y^{n+0.5}(i,j+0.5,k+1)=\left(\frac{1-\frac{\sigma^*_{(i,j+0.5,k+1)}\Delta t}{2\mu_{(i,j+0.5,k+1)}}}{1+\frac{\sigma^*_{(i,j+0.5,k+1)}\Delta t}{2\mu_{(i,j+0.5,k+1)}}}\right)H_y^{n-0.5}(i,j+0.5,k+1)$$
$$+\left(\frac{\frac{\Delta t}{\mu_{(i,j+0.5,k+1)}}}{1+\frac{\sigma^*_{(i,j+0.5,k+1)}\Delta t}{2\mu_{(i,j+0.5,k+1)}}}\right)\left[\begin{array}{l}\frac{E_z^n(i+0.5,j+0.5,k+1)-E_z^n(i-0.5,j+0.5,k+1)}{\Delta x}\\ -\frac{E_x^n(i,j+0.5,k+1.5)-E_x^n(i,j+0.5,k+0.5)}{\Delta z}\\ -M^n_{\text{source},y}(i,j+0.5,k+1)\end{array}\right]$$

(3.30b)

and

$$H_z^{n+0.5}(i,j+1,k+0.5)=\left(\frac{1-\frac{\sigma^*_{(i,j+1,k+0.5)}\Delta t}{2\mu_{(i,j+1,k+0.5)}}}{1+\frac{\sigma^*_{(i,j+1,k+0.5)}\Delta t}{2\mu_{(i,j+1,k+0.5)}}}\right)H_z^{n-0.5}(i,j+1,k+0.5)$$
$$+\left(\frac{\frac{\Delta t}{\mu_{(i,j+1,k+0.5)}}}{1+\frac{\sigma^*_{(i,j+1,k+0.5)}\Delta t}{2\mu_{(i,j+1,k+0.5)}}}\right)\left[\begin{array}{l}\frac{E_x^n(i,j+1.5,k+0.5)-E_x^n(i,j+0.5,k+0.5)}{\Delta y}\\ -\frac{E_y^n(i+0.5,j+1,k+0.5)-E_y^n(i-0.5,j+1,k+0.5)}{\Delta x}\\ -M^n_{\text{source},z}(i,j+1,k+0.5)\end{array}\right]$$

(3.30c)

## 3.6.4 Space Region with a Continuous Variation of Material Properties

- The implementation of the update equations for the electric and magnetic field components in a region where the material properties change continuously or frequently can be done directly as given in (3.29) and (3.30). However, from a computational standpoint, it would be grossly inefficient to re-calculate the time-invariant coefficients at each time step.
- Instead, these coefficients are usually pre-processed and stored in memory.
- For the electric field updates, the relevant coefficients at a general location $(i, j, k)$ are

$$C_{a(i,j,k)} = \left(1 - \frac{\sigma_{(i,j,k)}\Delta t}{2\varepsilon_{(i,j,k)}}\right) \Bigg/ \left(1 + \frac{\sigma_{(i,j,k)}\Delta t}{2\varepsilon_{(i,j,k)}}\right) \tag{3.31a}$$

$$C_{b1(i,j,k)} = \left(\frac{\Delta t}{\varepsilon_{(i,j,k)}\Delta_1}\right) \Bigg/ \left(1 + \frac{\sigma_{(i,j,k)}\Delta t}{2\varepsilon_{(i,j,k)}}\right) \tag{3.31b}$$

$$C_{b2(i,j,k)} = \left(\frac{\Delta t}{\varepsilon_{(i,j,k)}\Delta_2}\right) \Bigg/ \left(1 + \frac{\sigma_{(i,j,k)}\Delta t}{2\varepsilon_{(i,j,k)}}\right) \tag{3.31c}$$

where the coefficient location matches the relevant electric field component and $\Delta_1$ & $\Delta_2$ denote the two possible spatial increments (out of $\Delta x$, $\Delta y$, and $\Delta z$). For the case of a cubic lattice where $\Delta x = \Delta y = \Delta z = \Delta$, $C_{b1} = C_{b2} = C_b$.

## 3.6.4 continued

- For the magnetic field updates, the relevant coefficients at a general location $(i, j, k)$ are

$$D_{a(i,j,k)} = \left(1 - \frac{\sigma^*_{(i,j,k)}\Delta t}{2\mu_{(i,j,k)}}\right) \Bigg/ \left(1 + \frac{\sigma^*_{(i,j,k)}\Delta t}{2\mu_{(i,j,k)}}\right) \quad (3.31a)$$

$$D_{b1(i,j,k)} = \left(\frac{\Delta t}{\mu_{(i,j,k)}\Delta_1}\right) \Bigg/ \left(1 + \frac{\sigma^*_{(i,j,k)}\Delta t}{2\mu_{(i,j,k)}}\right) \quad (3.31b)$$

$$D_{b2(i,j,k)} = \left(\frac{\Delta t}{\mu_{(i,j,k)}\Delta_2}\right) \Bigg/ \left(1 + \frac{\sigma^*_{(i,j,k)}\Delta t}{2\mu_{(i,j,k)}}\right) \quad (3.31c)$$

where the coefficient location matches the relevant electric field component and $\Delta_1$ & $\Delta_2$ denote the two possible spatial increments (out of $\Delta x$, $\Delta y$, and $\Delta z$). For the case of a cubic lattice where $\Delta x = \Delta y = \Delta z = \Delta$, $D_{b1} = D_{b2} = D_b$.

- Using these coefficients, the electric field update equations, for a cubic lattice, become

$$E_x^{n+1}(i, j+0.5, k+0.5) = C_{a,E_x(i,j+0.5,k+0.5)}\, E_x^n(i, j+0.5, k+0.5)$$
$$+C_{b,E_x(i,j+0.5,k+0.5)} \begin{bmatrix} \left(H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i, j, k+0.5)\right) \\ -\left(H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i, j+0.5, k)\right) \\ -J_{\text{source},x}^{n+0.5}(i, j+0.5, k+0.5)\Delta \end{bmatrix}$$

(3.33a)

## 3.6.4 continued

$$E_y^{n+1}(i-0.5, j+1, k+0.5) = C_{a,E_y(i-0.5,j+1,k+0.5)}\, E_y^n(i-0.5, j+1, k+0.5)$$
$$+C_{b,E_y(i-0.5,j+1,k+0.5)} \begin{bmatrix} \left(H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j+1, k)\right) \\ -\left(H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i-1, j+1, k+0.5)\right) \\ - J_{\text{source},y}^{n+0.5}(i-0.5, j+1, k+0.5)\,\Delta \end{bmatrix}$$

(3.33b)

and

$$E_z^{n+1}(i-0.5, j+0.5, k+1) = C_{a,E_z(i-0.5,j+0.5,k+1)} E_z^n(i-0.5, j+0.5, k+1)$$
$$+C_{b,E_z(i-0.5,j+0.5,k+1)} \begin{bmatrix} \left(H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i-1, j+0.5, k+1)\right) \\ -\left(H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j, k+1)\right) \\ - J_{\text{source},z}^{n+0.5}(i-0.5, j+0.5, k+1)\,\Delta \end{bmatrix}$$

(3.33c)

- Similarly, using these coefficients, the magnetic field update equations, for a cubic lattice, become

$$H_x^{n+0.5}(i-0.5, j+1, k+1) = D_{a,H_x(i-0.5,j+1,k+1)}\, H_x^{n-0.5}(i-0.5, j+1, k+1)$$
$$+D_{b,H_x(i-0.5,j+1,k+1)} \begin{bmatrix} \left(E_y^n(i-0.5, j+1, k+1.5) - E_y^n(i-0.5, j+1, k+0.5)\right) \\ -\left(E_z^n(i-0.5, j+1.5, k+1) - E_z^n(i-0.5, j+0.5, k+1)\right) \\ - M_{\text{source},x}^n(i-0.5, j+1, k+1)\,\Delta \end{bmatrix}$$

(3.34a)

## 3.6.4 continued

$$H_y^{n+0.5}(i,j+0.5,k+1) = D_{a,H_y(i,j+0.5,k+1)}\,H_y^{n-0.5}(i,j+0.5,k+1) + D_{b,H_y(i,j+0.5,k+1)}\begin{bmatrix} \left(E_z^n(i+0.5,j+0.5,k+1) - E_z^n(i-0.5,j+0.5,k+1)\right) \\ -\left(E_x^n(i,j+0.5,k+1.5) - E_x^n(i,j+0.5,k+0.5)\right) \\ -M_{\text{source},y}^n(i,j+0.5,k+1)\,\Delta \end{bmatrix}$$

(3.34b)

and

$$H_z^{n+0.5}(i,j+1,k+0.5) = D_{a,H_z(i,j+1,k+0.5)}\,H_z^{n-0.5}(i,j+1,k+0.5) + D_{b,H_z(i,j+1,k+0.5)}\begin{bmatrix} \left(E_x^n(i,j+1.5,k+0.5) - E_x^n(i,j+0.5,k+0.5)\right) \\ -\left(E_y^n(i+0.5,j+1,k+0.5) - E_y^n(i-0.5,j+1,k+0.5)\right) \\ -M_{\text{source},z}^n(i,j+1,k+0.5)\,\Delta \end{bmatrix}$$

(3.34c)

- Using these field update equations, the total computer memory required is approximately (code will have overhead)

  $$18N = (6\text{ components} + 2\text{ coefficients/component} \times 6\text{ components})N$$

  where $N$ is the total number of Yee cells in the model.

## 3.6.5 Space region with a Finite Number of Distinct Media

- In free space or materials where the material properties are mostly homogeneous, the total computer memory required is reduced to approximately $6N$ or $12N$.
- One approach is to define the needed constant coefficients before the time loop and split the electric and magnetic field spatial update loops (i.e., loops with respect to $i$, $j$, & $k$) up to reflect the different material regions.
  - The advantage of this approach is that you get a very efficient & fast code.
  - The disadvantage is that every problem must be custom coded.
  - Total computer memory required is reduced to approximately $6N$
- Another approach is to define an integer pointer arrays MEDIA($i$, $j$, $k$) where the stored integer $m$ is used to reference the appropriate coefficients, e.g., $C_a(m)$, $C_b(m)$. $D_a(m)$, and $D_b(m)$.
  - The advantage of this approach is that you get a more efficient & faster code that is quite general.
  - The disadvantage is that it is not quite as efficient or fast.
  - Total computer memory required is reduced to approximately

$$12N = (6\,\text{components} + 1\,\text{coefficient/component} \times 6\,\text{components})N$$

  - In this case, the applicable update equations are

## 3.6.5 continued

$$m = \text{MEDIA}_{E_x}(i, j+0.5, k+0.5)$$
$$E_x^{n+1}(i, j+0.5, k+0.5) = C_a(m)\, E_x^n(i, j+0.5, k+0.5)$$
$$+C_b(m)\begin{bmatrix} \left(H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i, j, k+0.5)\right) \\ -\left(H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i, j+0.5, k)\right) \\ -\, J_{\text{source},x}^{n+0.5}(i, j+0.5, k+0.5)\,\Delta \end{bmatrix} \tag{3.35a}$$

$$m = \text{MEDIA}_{E_y}(i-0.5, j+1, k+0.5)$$
$$E_y^{n+1}(i-0.5, j+1, k+0.5) = C_a(m)\, E_y^n(i-0.5, j+1, k+0.5)$$
$$+C_b(m)\begin{bmatrix} \left(H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j+1, k)\right) \\ -\left(H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i-1, j+1, k+0.5)\right) \\ -\, J_{\text{source},y}^{n+0.5}(i-0.5, j+1, k+0.5)\,\Delta \end{bmatrix} \tag{3.35b}$$

$$m = \text{MEDIA}_{E_z}(i-0.5, j+0.5, k+1)$$
$$E_z^{n+1}(i-0.5, j+0.5, k+1) = C_a(m) E_z^n(i-0.5, j+0.5, k+1)$$
$$+C_b(m)\begin{bmatrix} \left(H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i-1, j+0.5, k+1)\right) \\ -\left(H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j, k+1)\right) \\ -\, J_{\text{source},z}^{n+0.5}(i-0.5, j+0.5, k+1)\,\Delta \end{bmatrix} \tag{3.35c}$$

for the electric field components.

## 3.6.5 continued

$$m = \mathrm{MEDIA}_{H_x}(i-0.5, j+1, k+1)$$

$$H_x^{n+0.5}(i-0.5, j+1, k+1) = D_a(m)\, H_x^{n-0.5}(i-0.5, j+1, k+1)$$

$$+D_b(m)\begin{bmatrix} \left(E_y^n(i-0.5, j+1, k+1.5) - E_y^n(i-0.5, j+1, k+0.5)\right) \\ -\left(E_z^n(i-0.5, j+1.5, k+1) - E_z^n(i-0.5, j+0.5, k+1)\right) \\ -M_{\mathrm{source},x}^n(i-0.5, j+1, k+1)\,\Delta \end{bmatrix}$$

(3.36a)

$$m = \mathrm{MEDIA}_{H_y}(i, j+0.5, k+1)$$

$$H_y^{n+0.5}(i, j+0.5, k+1) = D_a(m)\, H_y^{n-0.5}(i, j+0.5, k+1)$$

$$+D_b(m)\begin{bmatrix} \left(E_z^n(i+0.5, j+0.5, k+1) - E_z^n(i-0.5, j+0.5, k+1)\right) \\ -\left(E_x^n(i, j+0.5, k+1.5) - E_x^n(i, j+0.5, k+0.5)\right) \\ -M_{\mathrm{source},y}^n(i, j+0.5, k+1)\,\Delta \end{bmatrix}$$

(3.36b)

and

$$m = \mathrm{MEDIA}_{H_y}(i, j+1, k+0.5)$$

$$H_z^{n+0.5}(i, j+1, k+0.5) = D_a(m)\, H_z^{n-0.5}(i, j+1, k+0.5)$$

$$+D_b(m)\begin{bmatrix} \left(E_x^n(i, j+1.5, k+0.5) - E_x^n(i, j+0.5, k+0.5)\right) \\ -\left(E_y^n(i+0.5, j+1, k+0.5) - E_y^n(i-0.5, j+1, k+0.5)\right) \\ -M_{\mathrm{source},z}^n(i, j+1, k+0.5)\,\Delta \end{bmatrix}$$

(3.36c)

for the magnetic field components.

## 3.6.6 Space Region with Nonpermeable Media

- In this case, the media is non-magnetic (i.e., $\mu = \mu_0$ and $\sigma^* = 0$). The electric field component update equations (3.29) are unchanged. However, the magnetic field component update equations (3.30) become

$$H_x^{n+0.5}(i-0.5, j+1, k+1) = H_x^{n-0.5}(i-0.5, j+1, k+1) + \left(\frac{\Delta t}{\mu_0}\right)\left[\begin{array}{l} \dfrac{E_y^n(i-0.5, j+1, k+1.5) - E_y^n(i-0.5, j+1, k+0.5)}{\Delta z} \\ -\dfrac{E_z^n(i-0.5, j+1.5, k+1) - E_z^n(i-0.5, j+0.5, k+1)}{\Delta y} \\ -M_{\text{source},x}^n(i-0.5, j+1, k+1) \end{array}\right]$$

$$H_y^{n+0.5}(i, j+0.5, k+1) = H_y^{n-0.5}(i, j+0.5, k+1) + \left(\frac{\Delta t}{\mu_0}\right)\left[\begin{array}{l} \dfrac{E_z^n(i+0.5, j+0.5, k+1) - E_z^n(i-0.5, j+0.5, k+1)}{\Delta x} \\ -\dfrac{E_x^n(i, j+0.5, k+1.5) - E_x^n(i, j+0.5, k+0.5)}{\Delta z} \\ -M_{\text{source},y}^n(i, j+0.5, k+1) \end{array}\right]$$

$$H_z^{n+0.5}(i, j+1, k+0.5) = H_z^{n-0.5}(i, j+1, k+0.5) + \left(\frac{\Delta t}{\mu_0}\right)\left[\begin{array}{l} \dfrac{E_x^n(i, j+1.5, k+0.5) - E_x^n(i, j+0.5, k+0.5)}{\Delta y} \\ -\dfrac{E_y^n(i+0.5, j+1, k+0.5) - E_y^n(i-0.5, j+1, k+0.5)}{\Delta x} \\ -M_{\text{source},z}^n(i, j+1, k+0.5) \end{array}\right]$$

## 3.6.6 continued

- One possibility for efficiently implementing the FDTD updates is to introduce proportional electric fields and magnetic equivalent currents,

$$\hat{E} = \left(\frac{\Delta t}{\mu_0 \Delta}\right)\bar{E} \text{ and } \hat{M} = \left(\frac{\Delta t}{\mu_0}\right)\bar{M} \qquad (3.37)$$

where $\Delta x = \Delta y = \Delta z = \Delta$, in the FDTD update equations, and defining a scaled update coefficient

$$\hat{C}_b(m) = \left(\frac{\Delta t}{\mu_0 \Delta}\right) C_b(m) \qquad (3.38)$$

- Then, the updates of (3.35) and (3.36) can be re-written as

$$m = \text{MEDIA}_{E_x}(i, j+0.5, k+0.5)$$

$$\hat{E}_x^{n+1}(i, j+0.5, k+0.5) = C_a(m)\hat{E}_x^n(i, j+0.5, k+0.5)$$

$$+\hat{C}_b(m)\begin{bmatrix} \left(H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i, j, k+0.5)\right) \\ -\left(H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i, j+0.5, k)\right) \\ -J_{\text{source},x}^{n+0.5}(i, j+0.5, k+0.5)\Delta \end{bmatrix}$$

(3.39a)

$$m = \text{MEDIA}_{E_y}(i-0.5, j+1, k+0.5)$$

$$\hat{E}_y^{n+1}(i-0.5, j+1, k+0.5) = C_a(m)\hat{E}_y^n(i-0.5, j+1, k+0.5)$$

$$+\hat{C}_b(m)\begin{bmatrix} \left(H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j+1, k)\right) \\ -\left(H_z^{n+0.5}(i, j+1, k+0.5) - H_z^{n+0.5}(i-1, j+1, k+0.5)\right) \\ -J_{\text{source},y}^{n+0.5}(i-0.5, j+1, k+0.5)\Delta \end{bmatrix}$$

(3.39b)

## 3.6.6 continued

$$
\begin{aligned}
&m = \text{MEDIA}_{E_z}(i-0.5, j+0.5, k+1) \\
&\hat{E}_z^{n+1}(i-0.5, j+0.5, k+1) = C_a(m)\hat{E}_z^{n}(i-0.5, j+0.5, k+1) \\
&+\hat{C}_b(m)\begin{bmatrix} \left(H_y^{n+0.5}(i, j+0.5, k+1) - H_y^{n+0.5}(i-1, j+0.5, k+1)\right) \\ -\left(H_x^{n+0.5}(i-0.5, j+1, k+1) - H_x^{n+0.5}(i-0.5, j, k+1)\right) \\ -J_{\text{source},z}^{n+0.5}(i-0.5, j+0.5, k+1)\,\Delta \end{bmatrix}
\end{aligned}
\tag{3.39c}
$$

for the electric field components, and

$$
\begin{aligned}
&H_x^{n+0.5}(i-0.5, j+1, k+1) = H_x^{n-0.5}(i-0.5, j+1, k+1) \\
&+\begin{pmatrix} \hat{E}_y^{n}(i-0.5, j+1, k+1.5) \\ -\hat{E}_y^{n}(i-0.5, j+1, k+0.5) \end{pmatrix} - \begin{pmatrix} \hat{E}_z^{n}(i-0.5, j+1.5, k+1) \\ -\hat{E}_z^{n}(i-0.5, j+0.5, k+1) \end{pmatrix} \\
&-\hat{M}_{\text{source},x}^{n}(i-0.5, j+1, k+1)
\end{aligned}
\tag{3.40a}
$$

$$
\begin{aligned}
&H_y^{n+0.5}(i, j+0.5, k+1) = H_y^{n-0.5}(i, j+0.5, k+1) \\
&+\begin{pmatrix} \hat{E}_z^{n}(i+0.5, j+0.5, k+1) \\ -\hat{E}_z^{n}(i-0.5, j+0.5, k+1) \end{pmatrix} - \begin{pmatrix} \hat{E}_x^{n}(i, j+0.5, k+1.5) \\ -\hat{E}_x^{n}(i, j+0.5, k+0.5) \end{pmatrix} \\
&-\hat{M}_{\text{source},y}^{n}(i, j+0.5, k+1)
\end{aligned}
\tag{3.40b}
$$

and

$$
\begin{aligned}
&H_z^{n+0.5}(i, j+1, k+0.5) = H_z^{n-0.5}(i, j+1, k+0.5) \\
&+\begin{pmatrix} \hat{E}_x^{n}(i, j+1.5, k+0.5) \\ -\hat{E}_x^{n}(i, j+0.5, k+0.5) \end{pmatrix} - \begin{pmatrix} \hat{E}_y^{n}(i+0.5, j+1, k+0.5) \\ -\hat{E}_y^{n}(i-0.5, j+1, k+0.5) \end{pmatrix} \\
&-\hat{M}_{\text{source},z}^{n}(i, j+1, k+0.5)
\end{aligned}
\tag{3.40c}
$$

for the magnetic field components.

### 3.6.7 Reduction to the Two-Dimensional $TM_z$ and $TE_z$ Modes

- As discussed in section 3.3, some problems do not vary with respect to one dimension. Therefore, they reduce to two dimensions, and the six scalar component equations decouple in to two groups of three equations called TE and TM modes.
- For example, say the problem is infinite in the *z*-direction with no variations in material or structure cross-section. Then, the six scalar component equations decouple into two groups of three differential equations called the $TM_z$ (3.13) and $TE_z$ modes (3.14).
- In the following sections, the applicable FDTD update equations for $TM_z$ and $TE_z$ modes are given. The text gives a version of the update equations for the $TM_z$ and $TE_z$ modes for a finite number of material regions in (3.41) and (3.42) respectively.

**Two-Dimensional $TM_z$ Mode**

- In the $TM_z$ mode, the applicable field components are $\mathcal{H}_x$, $\mathcal{H}_y$, and $\mathcal{E}_z$. Note that the magnetic field components are transverse to the *z*-direction (i.e., they are located in the *x*-*y* plane).
- The applicable 2-D spatial grid/lattice (see Figure 3) is on an *x*-*y* plane. To see how this is a special case of the 3D grid/lattice, examine the top face of the 3D Yee unit cell shown in Figure 1(b).
- Again, the electric field component (i.e., $\mathcal{E}_z$) is placed at times $t = n\Delta t$ while the magnetic field components (i.e., $\mathcal{H}_x$ and $\mathcal{H}_y$) are placed at times $t = (n+0.5)\Delta t$ as shown in Figure 4.

## 3.6.7 continued

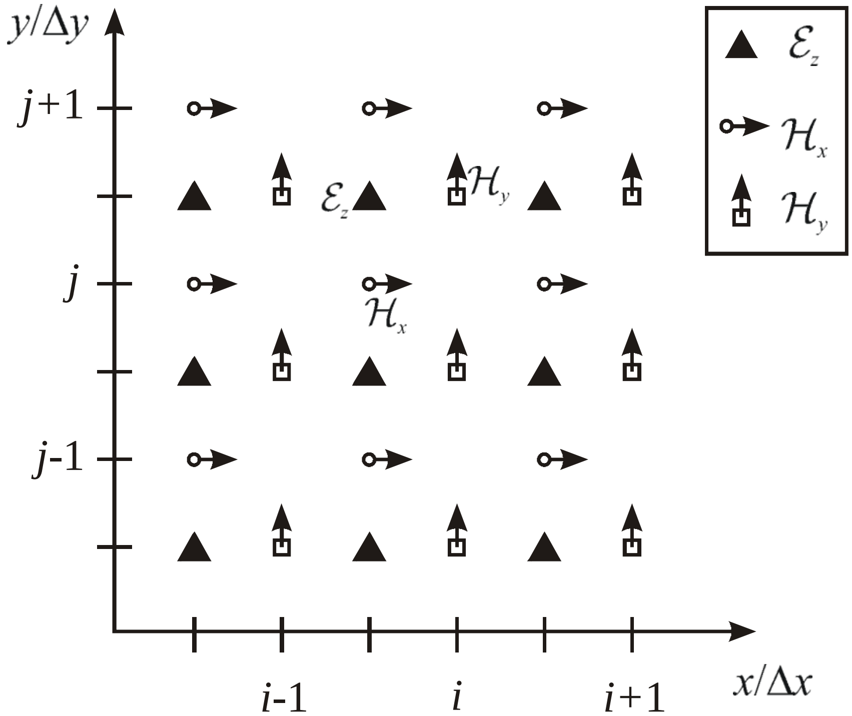



Figure 3 2D spatial lattice for $TM_z$ mode

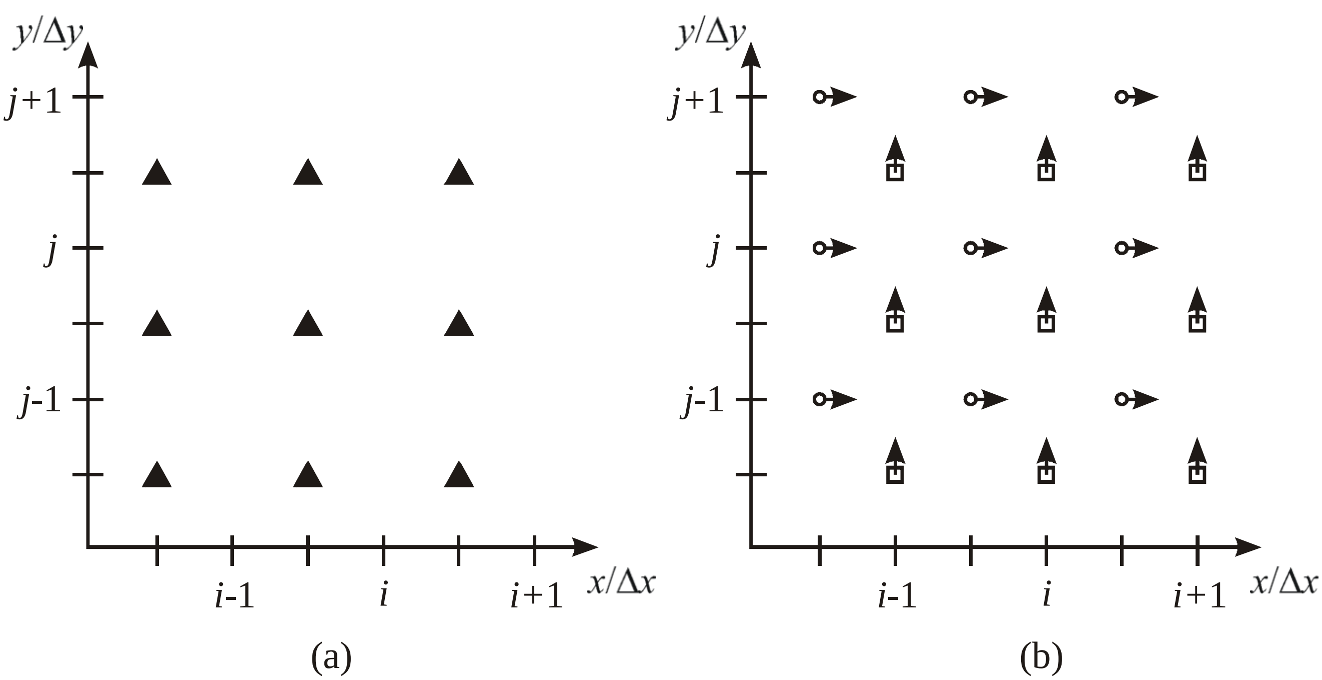



Figure 4 2D spatial lattice for $TM_z$ mode at times (a) $n\Delta t$ (only $\mathcal{E}_z$ component) and (b) $(n+0.5)\Delta t$ ($\mathcal{H}_x$ and $\mathcal{H}_y$ components).

## 3.6.7 continued

- Discretizing (3.13) for the $\mathcal{H}_x$, $\mathcal{H}_y$, and $\mathcal{E}_z$ locations indicated in Figure 3 leads to the TM$_z$ mode update equations

$$H_x^{n+0.5}(i-0.5,j)=\left(\frac{1-\dfrac{\sigma^*_{(i-0.5,j)}\Delta t}{2\mu_{(i-0.5,j)}}}{1+\dfrac{\sigma^*_{(i-0.5,j+1)}\Delta t}{2\mu_{(i-0.5,j)}}}\right)H_x^{n-0.5}(i-0.5,j)-\left(\frac{\dfrac{\Delta t}{\mu_{(i-0.5,j)}}}{1+\dfrac{\sigma^*_{(i-0.5,j)}\Delta t}{2\mu_{(i-0.5,j)}}}\right)$$

$$\times\left[\frac{E_z^n(i-0.5,j+0.5)-E_z^n(i-0.5,j-0.5)}{\Delta z}+M^n_{\text{source},x}(i-0.5,j)\right]$$

$$H_y^{n+0.5}(i,j+0.5)=\left(\frac{1-\dfrac{\sigma^*_{(i,j+0.5)}\Delta t}{2\mu_{(i,j+0.5)}}}{1+\dfrac{\sigma^*_{(i,j+0.5)}\Delta t}{2\mu_{(i,j+0.5)}}}\right)H_y^{n-0.5}(i,j+0.5)+\left(\frac{\dfrac{\Delta t}{\mu_{(i,j+0.5)}}}{1+\dfrac{\sigma^*_{(i,j+0.5)}\Delta t}{2\mu_{(i,j+0.5)}}}\right)$$

$$\times\left[\frac{E_z^n(i+0.5,j+0.5)-E_z^n(i-0.5,j+0.5)}{\Delta x}-M^n_{\text{source},y}(i,j+0.5)\right]$$

$$E_z^{n+1}(i-0.5,j+0.5)=\left(\frac{1-\dfrac{\sigma_{(i-0.5,j+0.5)}\Delta t}{2\varepsilon_{(i-0.5,j+0.5)}}}{1+\dfrac{\sigma_{(i-0.5,j+0.5)}\Delta t}{2\varepsilon_{(i-0.5,j+0.5)}}}\right)E_z^n(i-0.5,j+0.5)$$

$$+\left(\frac{\dfrac{\Delta t}{\varepsilon_{(i-0.5,j+0.5)}}}{1+\dfrac{\sigma_{(i-0.5,j+0.5)}\Delta t}{2\varepsilon_{(i-0.5,j+0.5)}}}\right)\left[\begin{array}{l}\dfrac{H_y^{n+0.5}(i,j+0.5)-H_y^{n+0.5}(i-1,j+0.5)}{\Delta x}\\[2ex]-\dfrac{H_x^{n+0.5}(i-0.5,j+1)-H_x^{n+0.5}(i-0.5,j)}{\Delta y}\\[2ex]-J^{n+0.5}_{\text{source},z}(i-0.5,j+0.5)\end{array}\right]$$

## 3.6.7 continued

## Two-Dimensional TE$_z$ Mode

- In the TE$_z$ mode, the applicable field components are $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{H}_z$. Note that the electric field components are transverse to the *z*-direction (i.e., they are located in the *x*-*y* plane).
- The applicable 2D spatial grid/lattice (see Figure 5) is on an *x*-*y* plane. To see how this is a special case of the 3D grid/lattice, examine the top face of the 3D Yee unit cell shown in Figure 1(a).
- Again, the electric field components (i.e., $\mathcal{E}_x$ and $\mathcal{E}_y$) are placed at times $t = n\Delta t$ while the magnetic field component (i.e. $\mathcal{H}_z$) is placed at times $t = (n+0.5)\Delta t$ as shown in Figure 6.

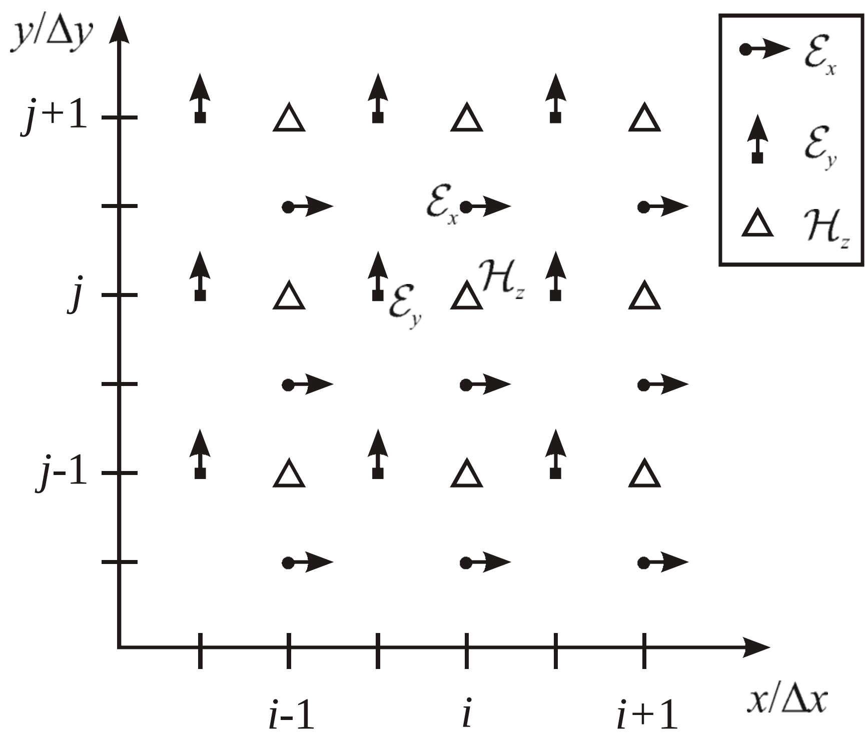



Figure 5 2D spatial lattice for TE$_z$ mode

## 3.6.7 continued

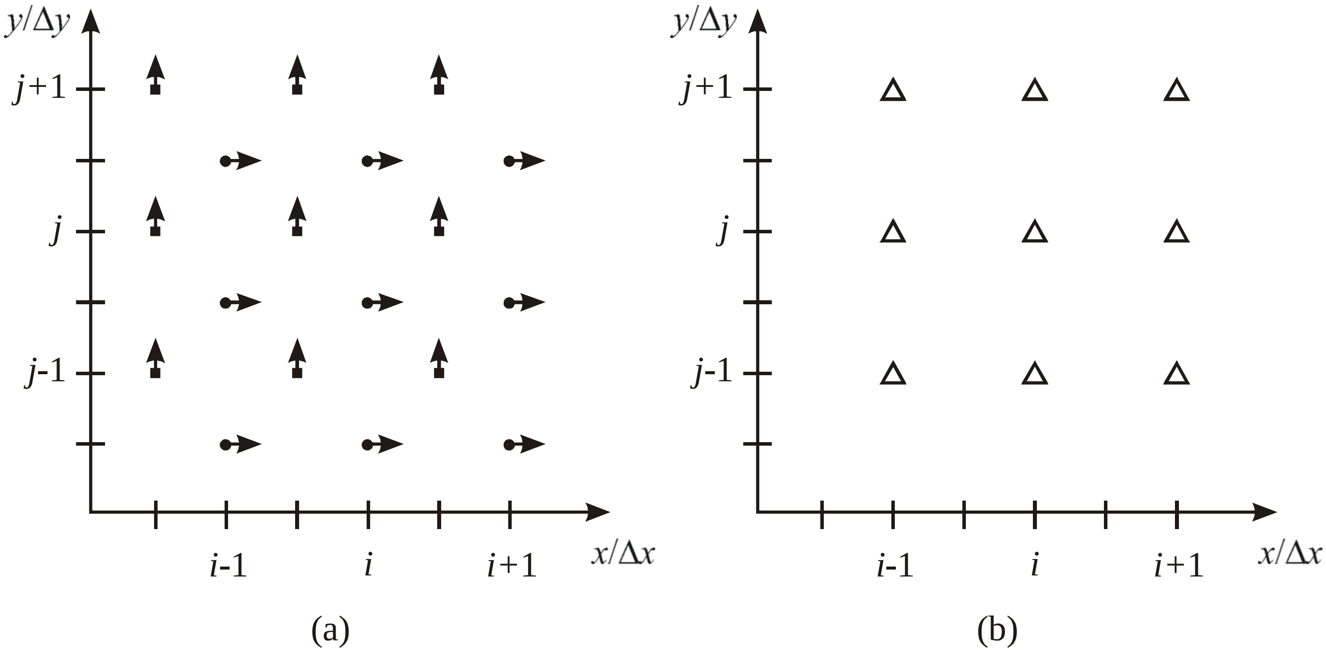



Figure 6 2D spatial lattice for $\text{TE}_z$ mode at times (a) $n\Delta t$ ($\mathcal{E}_x$ and $\mathcal{E}_y$ components) and (b) $(n+0.5)\Delta t$ (only $\mathcal{H}_z$ component).

- Discretizing (3.14) for the $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{H}_z$ locations indicated in Figure 5 leads to the $\text{TE}_z$ mode update equations

$$E_x^{n+1}(i, j+0.5) = \left( \frac{1 - \dfrac{\sigma_{(i,j+0.5)}\Delta t}{2\varepsilon_{(i,j+0.5)}}}{1 + \dfrac{\sigma_{(i,j+0.5)}\Delta t}{2\varepsilon_{(i,j+0.5)}}} \right) E_x^n(i, j+0.5) + \left( \frac{\dfrac{\Delta t}{\varepsilon_{(i,j+0.5)}}}{1 + \dfrac{\sigma_{(i,j+0.5)}\Delta t}{2\varepsilon_{(i,j+0.5)}}} \right)$$
$$\times \left[ \frac{H_z^{n+0.5}(i, j+1) - H_z^{n+0.5}(i, j)}{\Delta y} - J_{\text{source},x}^{n+0.5}(i, j+0.5) \right]$$

## 3.6.7 continued

$$E_y^{n+1}(i-0.5,j) = \left( \frac{1 - \frac{\sigma_{(i-0.5,j)}\Delta t}{2\varepsilon_{(i-0.5,j)}}}{1 + \frac{\sigma_{(i-0.5,j)}\Delta t}{2\varepsilon_{(i-0.5,j)}}} \right) E_y^n(i-0.5,j) - \left( \frac{\frac{\Delta t}{\varepsilon_{(i-0.5,j)}}}{1 + \frac{\sigma_{(i-0.5,j)}\Delta t}{2\varepsilon_{(i-0.5,j)}}} \right) \times \left[ \frac{H_z^{n+0.5}(i,j) - H_z^{n+0.5}(i-1,j)}{\Delta x} + J_{\text{source},y}^{n+0.5}(i-0.5,j) \right]$$

$$H_z^{n+0.5}(i,j) = \left( \frac{1 - \frac{\sigma^*_{(i,j)}\Delta t}{2\mu_{(i,j)}}}{1 + \frac{\sigma^*_{(i,j)}\Delta t}{2\mu_{(i,j)}}} \right) H_z^{n-0.5}(i,j) + \left( \frac{\frac{\Delta t}{\mu_{(i,j)}}}{1 + \frac{\sigma^*_{(i,j)}\Delta t}{2\mu_{(i,j)}}} \right) \left[ \frac{E_x^n(i,j+0.5) - E_x^n(i,j-0.5)}{\Delta y} - \frac{E_y^n(i+0.5,j) - E_y^n(i-0.5,j)}{\Delta x} - M_{\text{source},z}^n(i,j) \right]$$

## 3.6.8 Interpretation as Faraday's and Ampere's Laws in Integral Form

- Usually, the FDTD update equations are derived directly from the differential or point form of Ampere's and Faraday's Laws. While this works well in free space or homogeneous material regions, it is insufficient to deal with model features (e.g., thin slots in metal sheets, thin resistive sheets, thin wires, …) smaller than the spatial step sizes.
- A method for dealing with these fine features, where the underlying physics are understood, is to use the integral form of Ampere's and Faraday's Laws over the contours and surface areas of the FDTD grid/lattice. This topic is discussed in more detail in Chapter 10.
- To illustrate this method, we'll show how the FDTD update equation for $\mathcal{E}_z$ can be derived by applying the integral form of Ampere's Law to the contour and surface shown in Figure 7 where the fields are in a free space region with no electric current sources and we are at time $t=(n+0.5)\Delta t$.

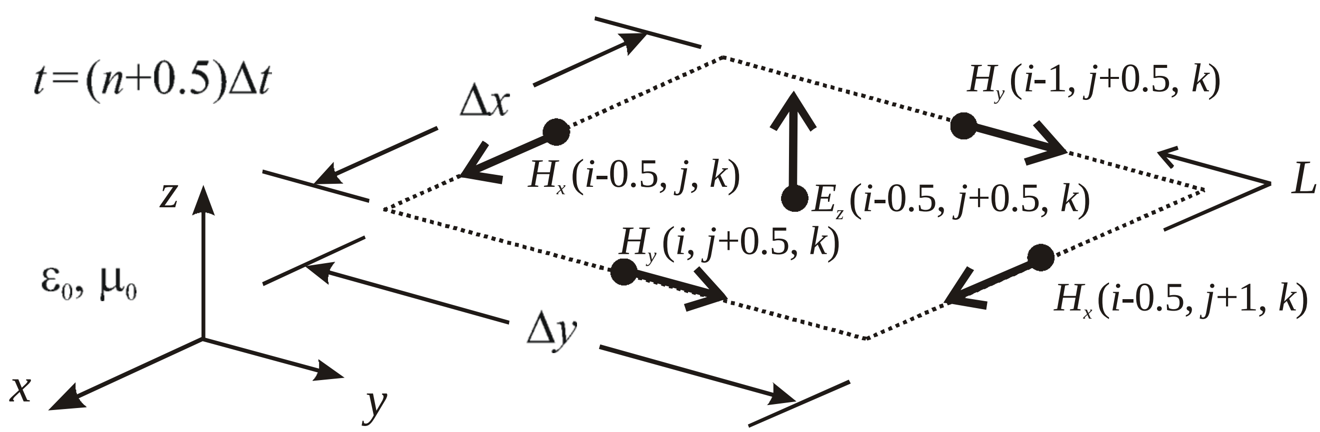



Figure 7 $\mathcal{E}_z$ FDTD update contour and surface

## 3.6.8 continued

- In this case, the general integral form of Ampere's Law (3.2) reduces to

$$\varepsilon_0 \frac{\partial}{\partial t} \iint_S \bar{\mathcal{E}} \bullet d\bar{s} = \oint_L \bar{\mathcal{H}} \bullet d\bar{l}$$

  where the surface $S$ is the rectangular area outlined by the contour $L$ (dashed line) in Figure 7 and $d\bar{s} = d\bar{s}_z = \hat{a}_z\, dxdy$ (applied right-hand rule to $L$).

- To evaluate the line integral on the RHS, assume that the $\mathcal{H}_x$ and $\mathcal{H}_y$ values are constant over the length of each of the four sides of $S$. This yields

$$\oint_L \bar{\mathcal{H}} \bullet d\bar{l} = H_x^{n+0.5}(i-0.5, j, k)\Delta x + H_y^{n+0.5}(i, j+0.5, k)\Delta y$$
$$-H_x^{n+0.5}(i, j+1, k)\Delta x - H_y^{n+0.5}(i-1, j+0.5, k)\Delta y$$

- To evaluate the surface integral, assume that $\mathcal{E}_z$ is constant (i.e., can be pulled outside the integral) over the entire surface $S$. This yields

$$\varepsilon_0 \frac{\partial}{\partial t} \iint_S \bar{\mathcal{E}} \bullet d\bar{s} = \varepsilon_0 \frac{\partial E_z(i-0.5, j+0.5, k)}{\partial t} \iint_S dxdy$$
$$= \varepsilon_0 \frac{\partial E_z(i-0.5, j+0.5, k)}{\partial t} \Delta x \Delta y$$

  Then, use the standard FDTD approximation on the time-derivative to get

$$\varepsilon_0 \frac{\partial}{\partial t} \iint_S \bar{\mathcal{E}} \bullet d\bar{s} = \varepsilon_0 \frac{\partial E_z(i-0.5, j+0.5, k)}{\partial t} \Delta x \Delta y$$
$$= \varepsilon_0 \frac{E_z^{n+1}(i-0.5, j+0.5, k) - E_z^{n}(i-0.5, j+0.5, k)}{\Delta t} \Delta x \Delta y$$

## 3.6.8 continued

- Equating the results for the two sides of the integral form of Ampere's Law yields

$$\varepsilon_0 \frac{E_z^{n+1}(i-0.5, j+0.5, k) - E_z^n(i-0.5, j+0.5, k)}{\Delta t}\Delta x \Delta y =$$
$$H_x^{n+0.5}(i-0.5, j, k)\Delta x + H_y^{n+0.5}(i, j+0.5, k)\Delta y$$
$$-H_x^{n+0.5}(i, j+1, k)\Delta x - H_y^{n+0.5}(i-1, j+0.5, k)\Delta y$$

- We can then solve for the update equation

$$E_z^{n+1}(i-0.5, j+0.5, k) = E_z^n(i-0.5, j+0.5, k)$$
$$+\left(\frac{\Delta t}{\varepsilon_0 \Delta x}\right)\left(H_y^{n+0.5}(i, j+0.5, k) - H_y^{n+0.5}(i-1, j+0.5, k)\right)$$
$$-\left(\frac{\Delta t}{\varepsilon_0 \Delta y}\right)\left(H_x^{n+0.5}(i, j+1, k) - H_x^{n+0.5}(i-0.5, j, k)\right)$$

  which is what we get from (3.29c) under the same conditions.

## 3.6.9 Divergence-Free Nature

- This section shows how applying the integral form of Gauss' Law (3.3) to the six sides of a Yee unit cell (see Figure 1(b)) and then substituting in the appropriate update equations for the relevant electric field components satisfies the condition that the net electric flux be zero. Further work on this will be reserved for the homework.