EE692 Applied EM- FDTD Method
One-Dimensional Transmission Lines Notes- Lecture 2

FDTD Modeling of Lumped Parallel/Series Loads

The FDTD model for the one-dimensional (1D) lossless transmission line [1] will now be extended to include single passive lumped elements (e.g., capacitors, inductors, and resistors) placed in parallel or series with the 1D transmission line [2]. Later, the model will be extended to parallel or series RLC loads.

Parallel Inductor

To begin, a circuit model for an incremental section (Δz) of lossless transmission line with a lumped element inductor L_p in parallel is shown in Figure 1. In Fig. 1, l and c are the inductance per unit length and the capacitance per unit length.

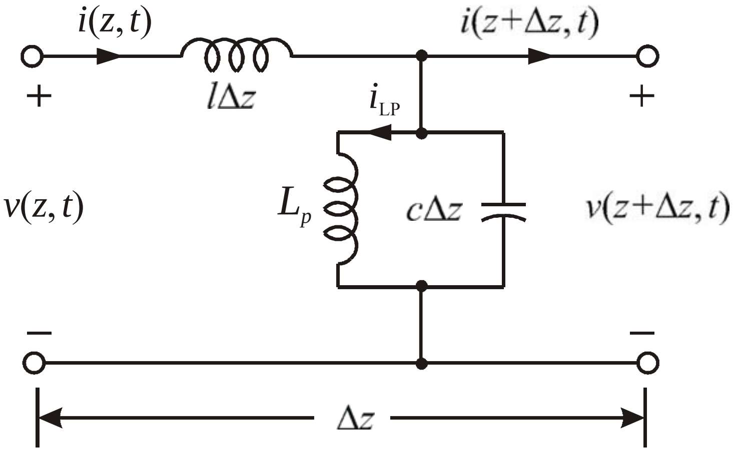


Figure 1 Incremental section of a lossless 1D transmission line with a lumped element inductor L_p in parallel.

Applying Kirchoff's Current Law (KCL) to the top right node of Fig. 1 yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z\,\frac{\partial v(z+\Delta z,t)}{\partial t} - i_{\mathrm{LP}}(z+\Delta z,t) = 0$$

where the current through the parallel inductor i_{LP} is defined by

$$i_{\mathrm{LP}}(z+\Delta z,t) = \frac{1}{L_p}\int_{t_0}^{t} v(z+\Delta z,t)\,\partial t + i_{\mathrm{LP}}(z+\Delta z,t_0)\ .$$

Applying Kirchoff's Voltage Law (KVL) clockwise around the outside loop yields

$$v(z+\Delta z,t)-v(z,t)+l\Delta z\frac{\partial i(z,t)}{\partial t}=0\ .$$

Rearranging these equations, letting Δz → 0, and applying the definition of the derivative, yields

$$-\frac{\partial i(z,t)}{\partial z}=c\frac{\partial v(z,t)}{\partial t}+\frac{i_{LP}(z,t)}{\Delta z},$$

$$i_{\mathrm{LP}}(z,t)=\frac{1}{L_p}\int_{t_0}^{t}v(z,t)\partial t+i_{\mathrm{LP}}(z,t_0),$$

and

$$-\frac{\partial v(z,t)}{\partial z}=l\frac{\partial i(z,t)}{\partial t}$$

respectively. This step of finding analytic differential equations is optional.

A discretized incremental section of a 1D lossless transmission line with a lumped element inductor L_p in parallel is shown in Figure 2.

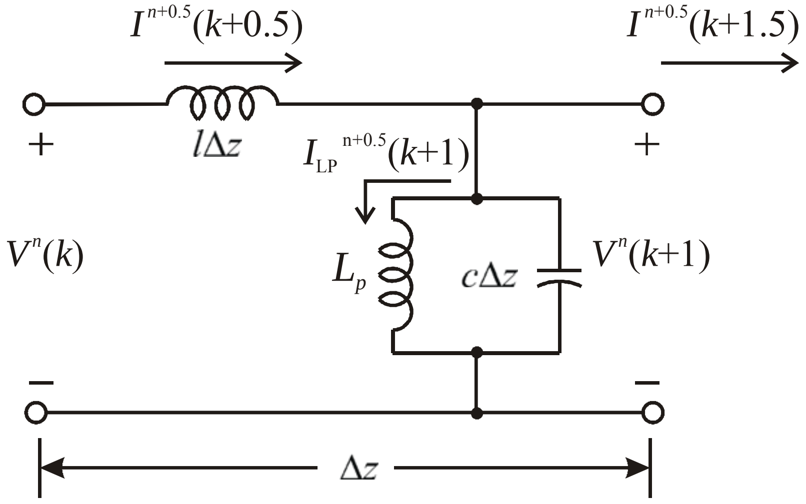


Figure 2 Discretized incremental section of a lossless transmission line with a discrete lumped inductor in parallel.

Note that the parallel inductor L_P is placed at the same spatial coordinates, i.e., $z=(k+1)\Delta z$, as the voltage node $V^n(k+1)$ upon which the current through this parallel inductor depends. However, this current is put at the same time step as the current nodes due to the Euler's method integral approximation (discussed later) and for clarity as to the order of the update equations.

Discretize the KCL and parallel inductor current equations about position $z=(k+1)\Delta z$ and time $t=(n+0.5)\Delta t$, to get

$$I^{n+0.5}(k+0.5)-I^{n+0.5}(k+1.5)-c\Delta z\left[\frac{V^{n+1}(k+1)-V^{n}(k+1)}{\Delta t}\right]-{I_{LP}}^{n+0.5}(k+1)=0$$

and

$${I_{LP}}^{n+0.5}(k+1)=\frac{1}{L_p}\left[V^{n}(k+1)\Delta t\right]+{I_{LP}}^{n-0.5}(k+1)\ .$$

Note that the integral is for the current through the parallel inductor L_P is approximated using Euler's method, i.e., add up the area of rectangles with a height equal to the value of the voltage a half step back in time and width of Δt. For example, when using Euler's method, the general integral

$$F(t)=\int_0^t f(t)\partial t=\int_{t_0}^t f(t)\partial t+\int_0^{t_0} f(t)\partial t=\int_{t_0}^t f(t)\partial t+F(t_0)$$

is approximated at $t=(n+0.5)\Delta t$ as

$$F^{n+0.5}\approx\sum_{i=0}^{n} f^{i}\Delta t=f^{n}\Delta t+\sum_{i=0}^{n-1} f^{i}\Delta t=f^{n}\Delta t+F^{n-0.5}$$

when standard FDTD notation is used as shown in Figure 3. The primary reason for using this numerical integration technique is stability, i.e., no future values of the variable being integrated are needed.

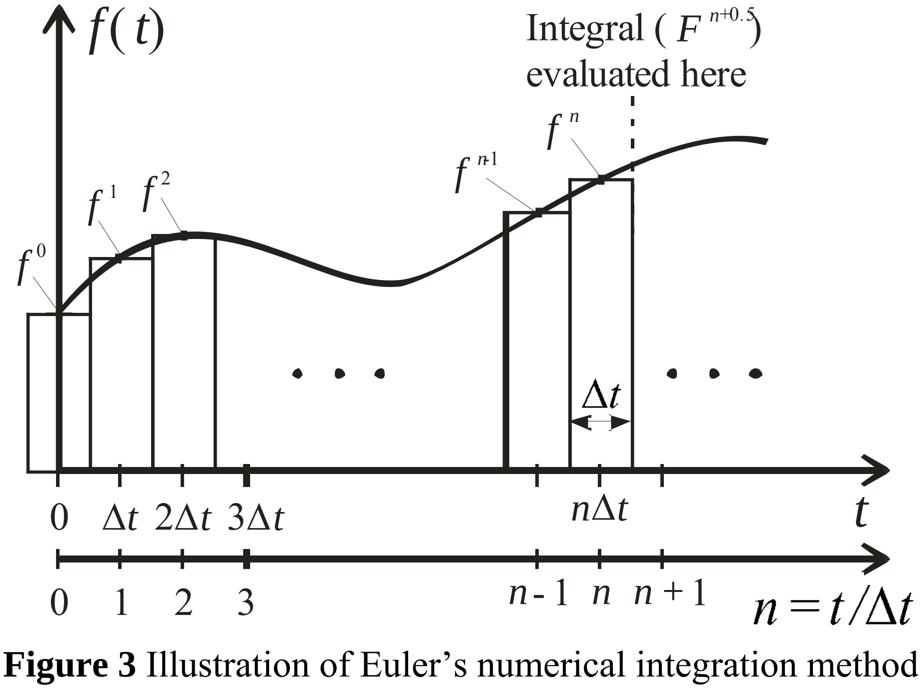


Figure 3 Illustration of Euler's numerical integration method

The KVL equation is identical to the 1D lossless transmission line case. Discretizing about position $z=(k+0.5)\Delta z$ and time $t=n\Delta t$, using the same derivative approximations, grid, and notation used for the lossless 1D transmission line, yields

$$V^n(k+1)-V^n(k)+l\Delta z\left[\frac{I^{n+0.5}(k+0.5)-I^{n-0.5}(k+0.5)}{\Delta t}\right]=0\ .$$

Re-arranging and solving each of the three discretized equations for the variable most advanced in time yields the update equations

$$\boxed{I^{n+0.5}(k+0.5)=I^{n-0.5}(k+0.5)-\frac{1}{Z_C}\left(\frac{v_p\Delta t}{\Delta z}\right)\left[V^n(k+1)-V^n(k)\right]}\ ,$$

$$\boxed{I_{\mathrm{LP}}^{n+0.5}(k+1)=I_{\mathrm{LP}}^{n-0.5}(k+1)+\frac{\Delta t}{L_p}V^n(k+1)}\ ,$$

and

$$\boxed{\begin{aligned}V^{n+1}(k+1)=V^n(k+1)-Z_C\left(\frac{v_p\Delta t}{\Delta z}\right)\left[I^{n+0.5}(k+1.5)-I^{n+0.5}(k+0.5)\right]\\ -Z_C\left(\frac{v_p\Delta t}{\Delta z}\right)I_{\mathrm{LP}}^{n+0.5}(k+1)\end{aligned}}\ .$$

Again, define the characteristic impedance $Z_C=\sqrt{\frac{l}{c}}$, phase velocity $v_p=\frac{1}{\sqrt{lc}}$, and Courant stability factor $S=\frac{v_p\Delta t}{\Delta z}$. This implies $l=\frac{Z_C}{v_p}$ and $c=\frac{1}{v_p Z_C}$. Note that the update equation for the current is unchanged from that for a 1D lossless transmission line. However, there is now an intermediate **auxiliary** equation to update the current through the parallel inductor L_P, and the voltage update equation has an additional term (when compared to that for a 1D lossless transmission line) that accounts for this current.

Series Inductor

Next, a circuit model for an incremental section (Δz) of lossless transmission line with a lumped element inductor L_S in series is shown in Figure 4a. Figure 4b shows an incremental section of this lossless transmission line after making the FDTD discretizations.

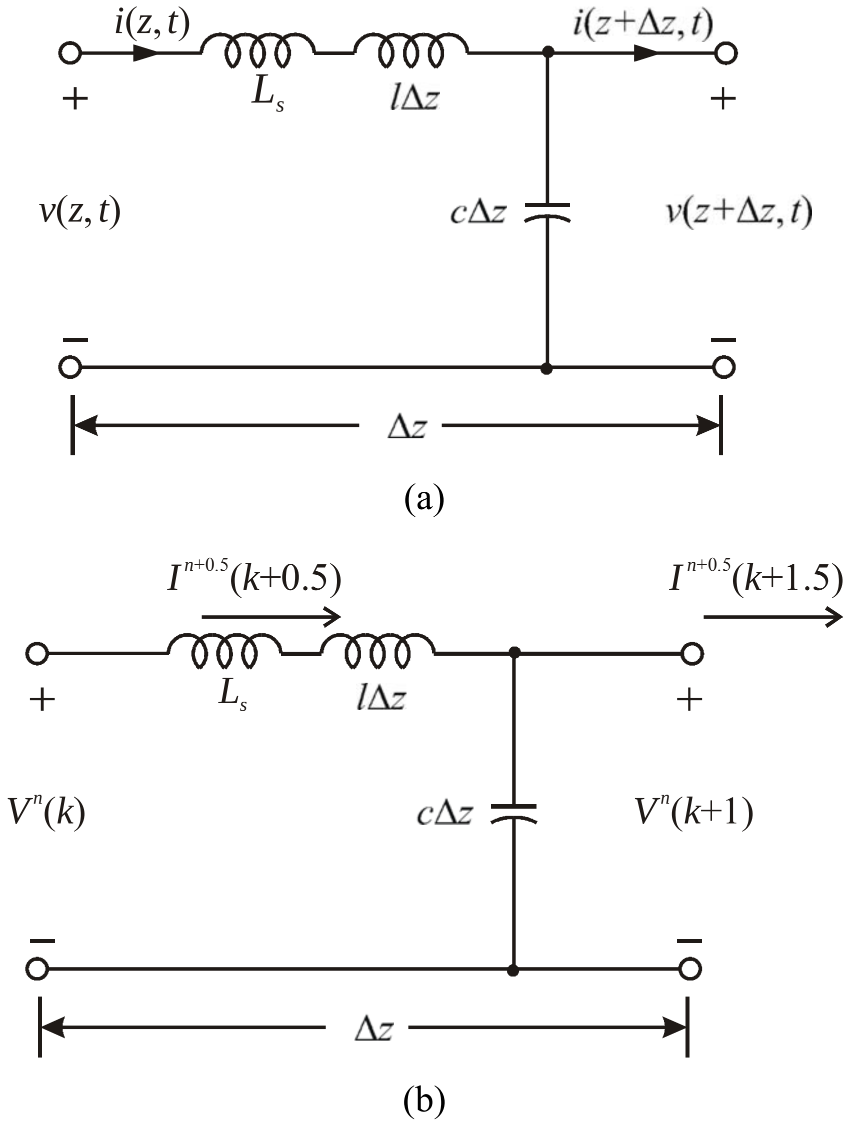


Figure 4 (a) Incremental and (b) discretized incremental section of a lossless 1D transmission line with lumped element inductor L_S in series.

Applying KCL to the top right node of Fig. 4a yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} = 0 \ .$$

Applying KVL clockwise around the outside loop of Fig. 4a yields

$$v(z+\Delta z,t) - v(z,t) + L_S \frac{\partial i(z,t)}{\partial t} + l\Delta z \frac{\partial i(z,t)}{\partial t} = 0 \ .$$

To save time, the intermediate step of letting Δz→0 and finding the derivatives with respect to *z* will be skipped. Instead, we will directly discretize the KCL equation about position $z = (k+1)\Delta z$ and time $t = (n+0.5)\Delta t$ to get

$$I^{n+0.5}(k+0.5) - I^{n+0.5}(k+1.5) - c\Delta z \left[\frac{V^{n+1}(k+1) - V^{n}(k+1)}{\Delta t} \right] = 0 \ ,$$

and discretize the KVL equation about position $z = (k+0.5)\Delta z$ and time $t = n\Delta t$, to get

$$V^{n}(k+1) - V^{n}(k) + \left(L_S + l\Delta z\right) \left[\frac{I^{n+0.5}(k+0.5) - I^{n-0.5}(k+0.5)}{\Delta t} \right] = 0 \ .$$

These equations, when re-arranged and simplified, yield the update equations

$$\boxed{I^{n+0.5}(k+0.5) = I^{n-0.5}(k+0.5) - \frac{1}{Z_C'} \left(\frac{v_p \Delta t}{\Delta z} \right) \left[V^{n}(k+1) - V^{n}(k) \right]}$$

and

$$\boxed{V^{n+1}(k+1) = V^{n}(k+1) - Z_C \left(\frac{v_p \Delta t}{\Delta z} \right) \left[I^{n+0.5}(k+1.5) - I^{n+0.5}(k+0.5) \right]} \ .$$

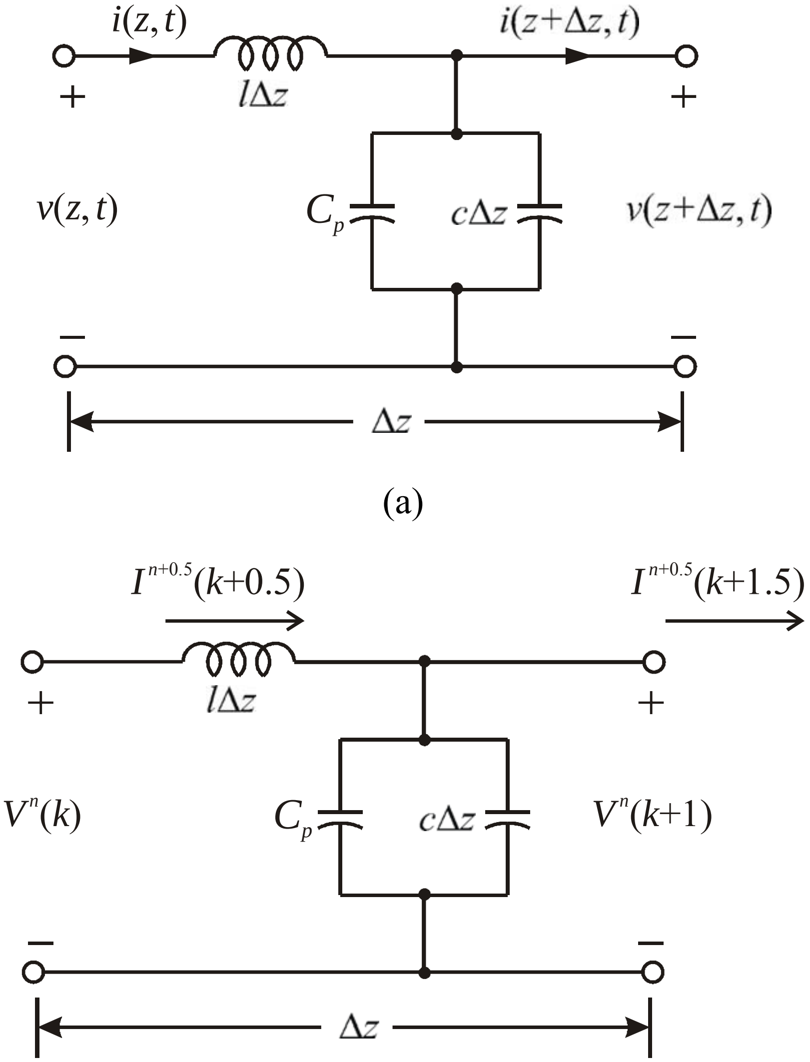
Note that in the update equation for the current, the effect of the series inductor L_S is to modify/replace the characteristic impedance Z_C with

$$\boxed{Z_C' = Z_C + \frac{L_S v_p}{\Delta z}}$$

while the voltage update equation is unchanged from that for a 1D lossless transmission line. Note, the series inductor L_S is located at $z = (k+0.5)\Delta z$ in the FDTD grid.

Parallel Capacitor

Figure 5a shows a circuit model for an incremental section (Δz) of lossless transmission line with a lumped element capacitor C_P in parallel. Figure 5b shows an incremental section of this lossless transmission line after making the FDTD discretizations.



(a)

(b)

Figure 5 (a) Incremental and (b) discretized incremental section of a lossless 1D transmission line with a lumped element capacitor C_P in parallel.

Applying KCL to the top right node of Fig. 5a yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - C_p \frac{\partial v(z+\Delta z,t)}{\partial t} = 0 \ .$$

Applying KVL clockwise around the outside loop of Fig. 5a yields

$$v(z+\Delta z,t) - v(z,t) + l\Delta z \frac{\partial i(z,t)}{\partial t} = 0 \ .$$

To save time, the intermediate step of letting Δz→0 and finding the derivatives with respect to *z* will be skipped. Instead, we will directly discretize the KCL equation about position $z = (k+1)\Delta z$ and time $t = (n+0.5)\Delta t$ to get

$$I^{n+0.5}(k+0.5) - I^{n+0.5}(k+1.5) - \left(c\Delta z + C_p\right)\left[\frac{V^{n+1}(k+1) - V^{n}(k+1)}{\Delta t}\right] = 0 \ ,$$

and discretize the KVL equation about position $z = (k+0.5)\Delta z$ and time $t = n\Delta t$, to get

$$V^{n}(k+1) - V^{n}(k) + l\Delta z\left[\frac{I^{n+0.5}(k+0.5) - I^{n-0.5}(k+0.5)}{\Delta t}\right] = 0 \ .$$

These equations, when re-arranged and simplified, yield the update equations

$$\boxed{I^{n+0.5}(k+0.5) = I^{n-0.5}(k+0.5) - \frac{1}{Z_C}\left(\frac{v_p \Delta t}{\Delta z}\right)\left[V^{n}(k+1) - V^{n}(k)\right]}$$

and

$$\boxed{V^{n+1}(k+1) = V^{n}(k+1) - Z_C''\left(\frac{v_p \Delta t}{\Delta z}\right)\left[I^{n+0.5}(k+1.5) - I^{n+0.5}(k+0.5)\right]} \ .$$

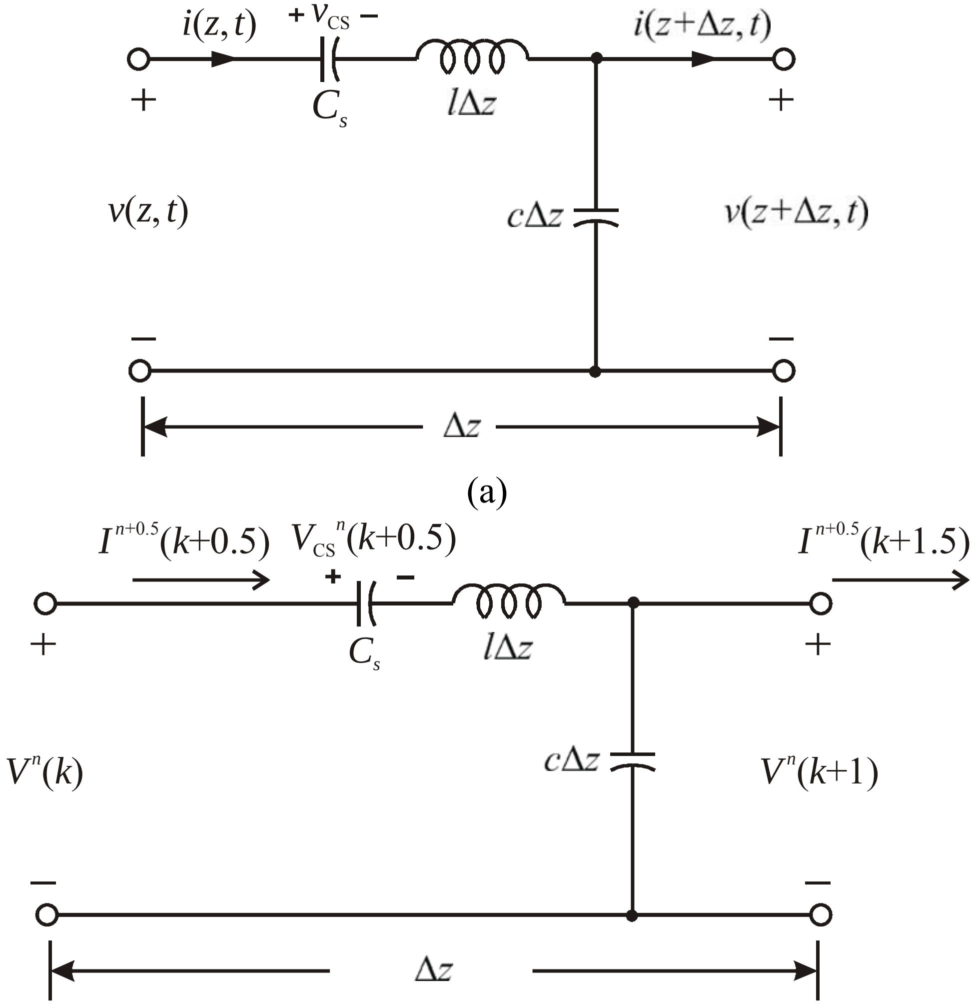
Note that in the update equation for the voltage, the effect of the discrete parallel capacitor C_P is to modify/replace the characteristic impedance Z_C with

$$\boxed{Z_C'' = \frac{Z_C}{1 + C_p v_p Z_C / \Delta z}}$$

while the current update equation is unchanged from that for a 1D lossless transmission line. Note, the parallel capacitor C_P is located at $z = (k+1)\Delta z$ in the FDTD grid.

Series Capacitor

Figure 6a shows a circuit model for an incremental section (Δz) of lossless transmission line with a lumped element capacitor C_S in series. Figure 6b shows this incremental section after making FDTD discretizations.



(a)

(b)

Figure 6 (a) Incremental and (b) discretized incremental section of lossless 1D transmission line with lumped element capacitor C_S in series.

Applying KCL to the top right node of Fig. 6a yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z\,\frac{\partial v(z+\Delta z,t)}{\partial t} = 0\ .$$

Applying KVL clockwise around the outside loop of Fig. 6a yields

$$v(z+\Delta z,t) - v(z,t) + v_{CS}(z,t) + l\Delta z\,\frac{\partial i(z,t)}{\partial t} = 0$$

where the voltage across the series capacitor v_{CS} is defined by the integral equation

$$v_{\mathrm{CS}}(z,t) = \frac{1}{C_S}\int_{t_0}^{t} i(z,t)\partial t + v_{\mathrm{CS}}(z,t_0)\,.$$

To save time, the intermediate step of letting Δ*z*→0 and finding the derivatives with respect to *z* will be skipped. Instead, we will directly discretize the KCL equation about position $z = (k+1)\Delta z$ and time $t = (n+0.5)\Delta t$ to get

$$I^{n+0.5}(k+0.5) - I^{n+0.5}(k+1.5) - c\Delta z\left[\frac{V^{n+1}(k+1) - V^{n}(k+1)}{\Delta t}\right] = 0,$$

and discretize the KVL and voltage across the series capacitor equations about position $z = (k+0.5)\Delta z$ and time $t = n\Delta t$, to get

$$V^{n}(k+1) - V^{n}(k) + V_{CS}^{n}(k+0.5) + l\Delta z\left[\frac{I^{n+0.5}(k+0.5) - I^{n-0.5}(k+0.5)}{\Delta t}\right] = 0$$

and

$$V_{CS}^{n}(k+0.5) = \frac{1}{C_S}\left[I^{n-0.5}(k+0.5)\Delta t\right] + V_{CS}^{n-1}(k+0.5)\,.$$

These equations, when re-arranged and simplified, yield the update equations

$$\boxed{V_{CS}^{n}(k+0.5) = V_{CS}^{n-1}(k+0.5) + \left(\frac{\Delta t}{C_S}\right) I^{n+0.5}(k+0.5)}\,,$$

$$\boxed{\begin{gathered} I^{n+0.5}(k+0.5) = I^{n-0.5}(k+0.5) - \frac{1}{Z_C}\left(\frac{v_p\Delta t}{\Delta z}\right)\left[V^{n}(k+1) - V^{n}(k)\right] \\ -\frac{1}{Z_C}\left(\frac{v_p\Delta t}{\Delta z}\right) V_{CS}{}^{n}(k+0.5) \end{gathered}}\,,$$

and

$$\boxed{V^{n+1}(k+1) = V^{n}(k+1) - Z_C\left(\frac{v_p\Delta t}{\Delta z}\right)\left[I^{n+0.5}(k+1.5) - I^{n+0.5}(k+0.5)\right]}\,.$$

Note that there is now an auxiliary equation to update the voltage across the series capacitor C_S. Also, the update equation for the current has an additional term (when compared to that for a 1D lossless transmission line) that accounts for C_S while the voltage update equation is unchanged from that for a 1D lossless transmission line. Note C_S is located at $z = (k+0.5)\Delta z$ in the FDTD grid.

Parallel Resistor

Figure 7a shows a circuit model for an incremental section (Δz) of lossless transmission line with a lumped element resistor R_P in parallel. Figure 7b shows the incremental section after making FDTD discretizations.

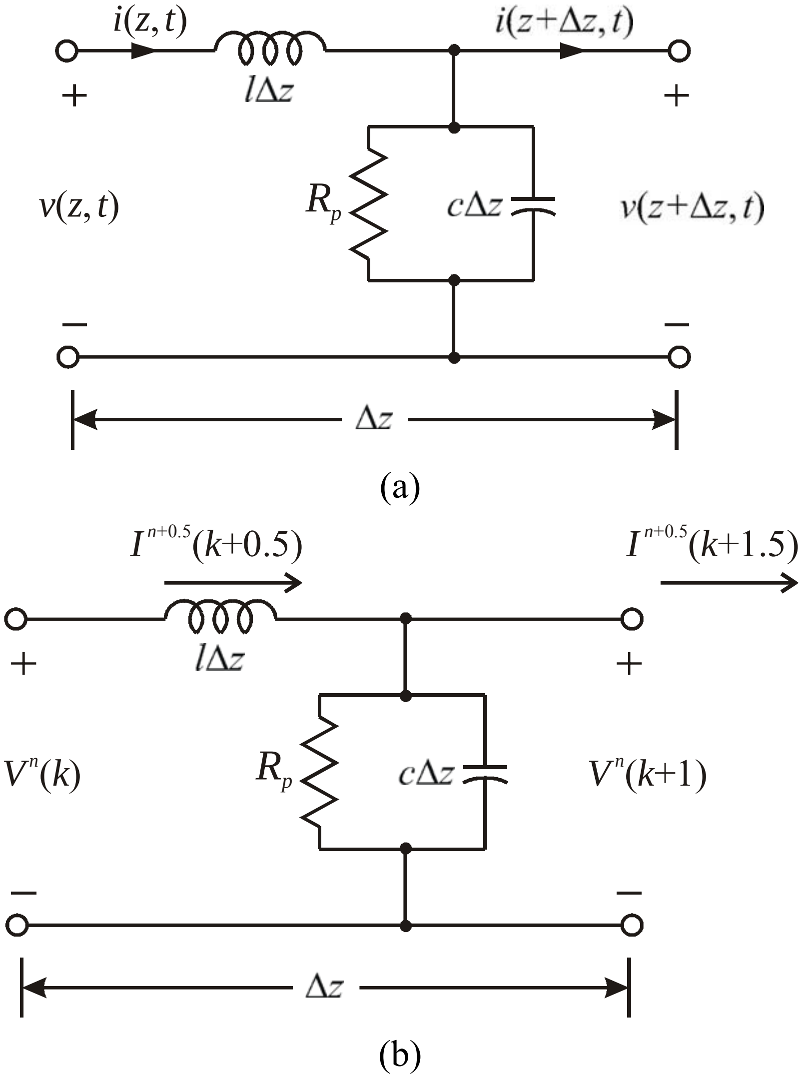


Figure 7 (a) Incremental and (b) discretized incremental section of a lossless 1D transmission line with a lumped element resistor R_P in parallel.

Applying KCL to the top right node of Fig. 7a yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - \frac{v(z+\Delta z,t)}{R_P} = 0 \ .$$

Applying KVL clockwise around the outside loop of Fig. 7b yields

$$v(z+\Delta z,t) - v(z,t) + l\Delta z \frac{\partial i(z,t)}{\partial t} = 0 \ .$$

To save time, the intermediate step of letting $\Delta z \rightarrow 0$ and finding the derivatives with respect to z will be skipped. Instead, we will directly discretize the KCL equation about position $z=(k+1)\Delta z$ and time $t=(n+0.5)\Delta t$ to get

$$I^{n+0.5}(k+0.5)-I^{n+0.5}(k+1.5)-c\Delta z\left[\frac{V^{n+1}(k+1)-V^{n}(k+1)}{\Delta t}\right]-\frac{V^{n+0.5}(k+1)}{R_p}=0\,.$$

A problem arises with the term dealing with the current through R_P (last term). Specifically, the voltage across the parallel resistor R_P is needed at time $t=(n+0.5)\Delta t$, a point not included in the temporal grid. To approximate this voltage, the simple average

$$V^{n+0.5}(k+1)\approx\frac{V^{n+1}(k+1)+V^{n}(k+1)}{2}$$

is used yielding

$$\begin{aligned}&I^{n+0.5}(k+0.5)-I^{n+0.5}(k+1.5)-c\Delta z\left[\frac{V^{n+1}(k+1)-V^{n}(k+1)}{\Delta t}\right]\\&-\frac{V^{n+1}(k+1)+V^{n}(k+1)}{2R_p}=0\end{aligned}\,.$$

The KVL equation is discretized about position $z=(k+0.5)\Delta z$ and time $t=n\Delta t$, to get

$$V^{n}(k+1)-V^{n}(k)+l\Delta z\left[\frac{I^{n+0.5}(k+0.5)-I^{n-0.5}(k+0.5)}{\Delta t}\right]=0\,.$$

These equations, when re-arranged and simplified, yield the update equations

$$\boxed{I^{n+0.5}(k+0.5)=I^{n-0.5}(k+0.5)-\frac{1}{Z_C}\left(\frac{v_p\Delta t}{\Delta z}\right)\left[V^{n}(k+1)-V^{n}(k)\right]}$$

and

$$\boxed{\begin{aligned}V^{n+1}(k+1)=&\left[\frac{1-\dfrac{Z_C}{2R_P}\left(\dfrac{v_p\Delta t}{\Delta z}\right)}{1+\dfrac{Z_C}{2R_P}\left(\dfrac{v_p\Delta t}{\Delta z}\right)}\right]V^{n}(k+1)-\left(\frac{Z_C}{1+\dfrac{Z_C}{2R_P}\left(\dfrac{v_p\Delta t}{\Delta z}\right)}\right)\\&\times\left(\frac{v_p\Delta t}{\Delta z}\right)\left[I^{n+0.5}(k+1.5)-I^{n+0.5}(k+0.5)\right]\end{aligned}}\,.$$

Note that the current update equation is unchanged from that for a 1D lossless transmission line. In the update equation for the voltage, note that there is now a damping coefficient (i.e., value is between 0 and 1) acting on the old value of the voltage. Also, in this equation, the contribution from the current terms is reduced (Z_C is effectively reduced). Note R_P is located at $z = (k+1)\Delta z$ in the FDTD grid.

Series Resistor

Lastly, Figure 8a shows a circuit model for an incremental section (Δz) of lossless transmission line with a lumped element resistor R_S in series. Figure 8b shows the incremental section after making FDTD discretizations.

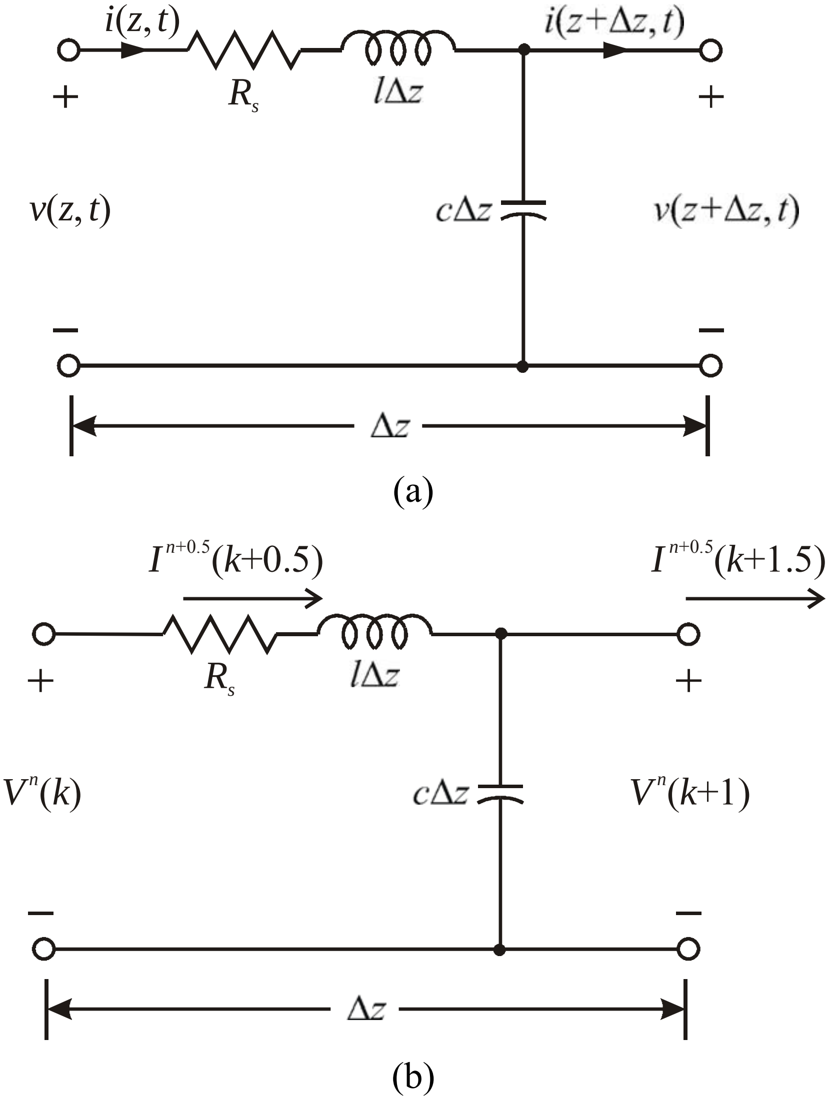


Figure 8 (a) Incremental and (b) discretized incremental section of a lossless 1D transmission line with a lumped element resistor R_S in series.

Applying KCL to the top right node of Fig. 8a yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} = 0.$$

Applying KVL clockwise around the outside loop of Fig. 8a yields

$$v(z+\Delta z,t) - v(z,t) + R_S i(z,t) + l\Delta z \frac{\partial i(z,t)}{\partial t} = 0.$$

To save time, the intermediate step of letting Δz→0 and finding the derivatives with respect to *z* will be skipped. Instead, we will directly discretize the KCL equation about position $z = (k+1)\Delta z$ and time $t = (n+0.5)\Delta t$ to get

$$I^{n+0.5}(k+0.5) - I^{n+0.5}(k+1.5) - c\Delta z \left[\frac{V^{n+1}(k+1) - V^{n}(k+1)}{\Delta t} \right] = 0,$$

and discretize the KVL equation about position $z = (k+0.5)\Delta z$ and time $t = n\Delta t$, to get

$$V^{n}(k+1) - V^{n}(k) + R_S I^{n}(k+0.5) + l\Delta z \left[\frac{I^{n+0.5}(k+0.5) - I^{n-0.5}(k+0.5)}{\Delta t} \right] = 0.$$

A problem arises with the term dealing with the voltage drop across the series resistor R_S. Specifically, the current through R_S is needed at time $t = n\Delta t$, a point not included in the temporal grid. To approximate this current, the simple average

$$I^{n}(k+0.5) \approx \frac{I^{n+0.5}(k+0.5) + I^{n-0.5}(k+0.5)}{2}$$

is used yielding

$$\begin{aligned} V^{n}(k+1) - V^{n}(k) + R_S \left[\frac{I^{n+0.5}(k+0.5) + I^{n-0.5}(k+0.5)}{2} \right] \\ + l\Delta z \left[\frac{I^{n+0.5}(k+0.5) - I^{n-0.5}(k+0.5)}{\Delta t} \right] = 0 \end{aligned}.$$

These equations, when re-arranged and simplified, yield the update equations

$$\boxed{\begin{aligned} I^{n+0.5}(k+0.5) = {} & \left[\frac{1-\dfrac{R_S}{2Z_C}\left(\dfrac{v_p\Delta t}{\Delta z}\right)}{1+\dfrac{R_S}{2Z_C}\left(\dfrac{v_p\Delta t}{\Delta z}\right)}\right] I^{n-0.5}(k+0.5) - \left(\frac{1}{1+\dfrac{R_S}{2Z_C}\left(\dfrac{v_p\Delta t}{\Delta z}\right)}\right)\frac{1}{Z_C} \\ & \times\left(\frac{v_p\Delta t}{\Delta z}\right)\left[V^n(k+1)-V^n(k)\right] \end{aligned}}$$

and

$$\boxed{V^{n+1}(k+1) = V^n(k+1) - Z_C\left(\frac{v_p\Delta t}{\Delta z}\right)\left[I^{n+0.5}(k+1.5) - I^{n+0.5}(k+0.5)\right]}\,.$$

In the update equation for the current, note that there is now a damping coefficient (i.e., value is between 0 and 1) acting on the old value of the current. Also, in this equation, the contribution from the voltage terms is reduced (Z_C is effectively increased). Note that the voltage update equation is unchanged from that for a 1D lossless transmission line. Note, R_S is located at $z = (k+0.5)\Delta z$ in the FDTD grid.

Verification of FDTD Update Equations for Lumped Elements

To demonstrate the validity of the derived FDTD update equations, a voltage pulse is launched on a long section of 1D lossless transmission line containing each type of the lumped elements/loads (see Figs. 9ab). The voltage reflected from each type of load is then examined. The FDTD results are compared with analytically derived results. For the analytic results, the 1D lossless transmission line is terminated in its characteristic impedance Z_C after the loads (see Fig. 9cd).

The time-domain reflected voltage $V_{\text{ref}}(t)$ is analytically derived by taking the inverse Fourier transform of the product of the spectrum of the incident voltage and the reflection coefficient looking in at the loads. The reflection coefficient is defined as

$$\Gamma_{\text{in}} = \frac{Z_{\text{in}} - Z_C}{Z_{\text{in}} + Z_C}$$

where $Z_{\text{in}} = Z_{\text{load}} Z_C / (Z_{\text{load}} + Z_C)$ for the parallel load and $Z_{\text{in}} = Z_{\text{load}} + Z_C$ for the series load.

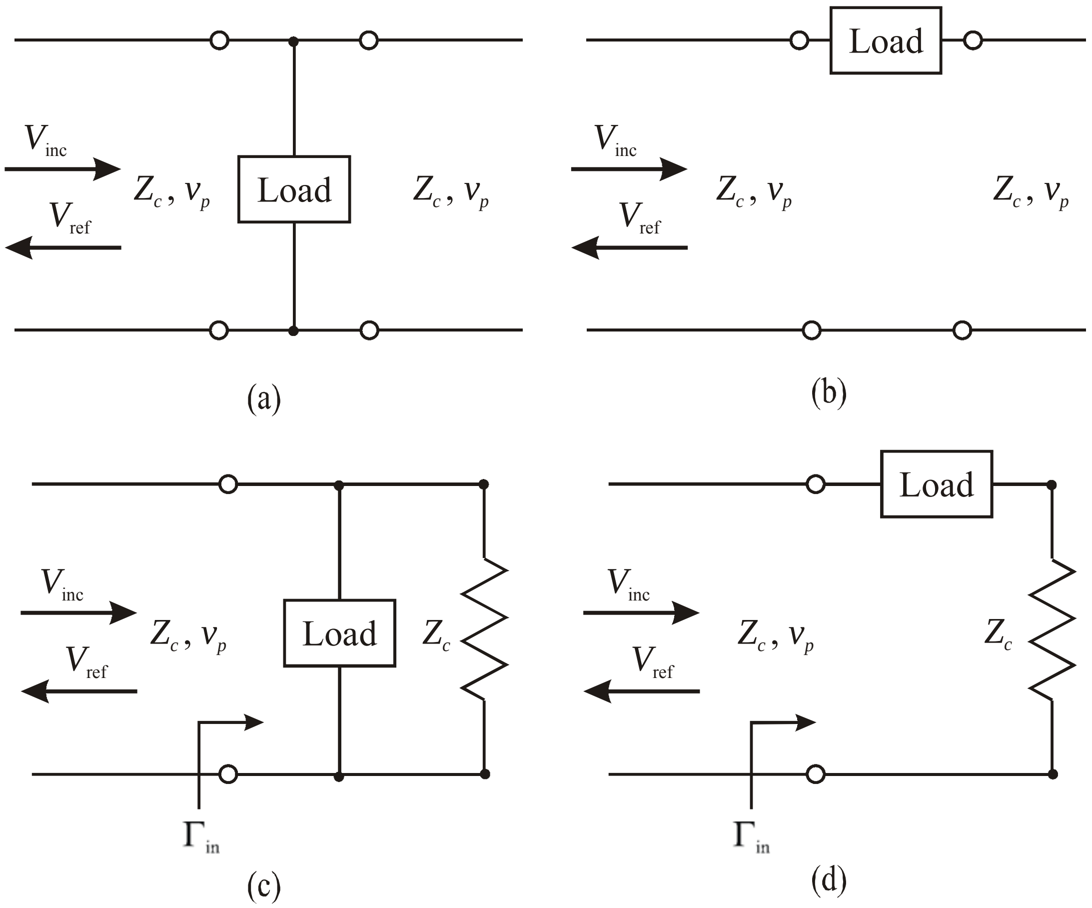


Figure 9 Transmission line circuits modeled using FDTD method for loads in (a) parallel and (b) series, and equivalent transmission line circuits used analytic results for loads in (c) parallel and (d) series.

Two incident voltage pulses were considered; a unit-amplitude, Gaussian pulse

$$\mathcal{V}_G(t) = e^{-0.5(t-\tau_d)^2/\tau_p^2},$$

and a unit-amplitude, differentiated Gaussian pulse

$$\mathcal{V}_D(t) = \frac{-\sqrt{e}\ (t-\tau_d)}{\tau_p}\, e^{-0.5(t-\tau_d)^2/\tau_p^2}$$

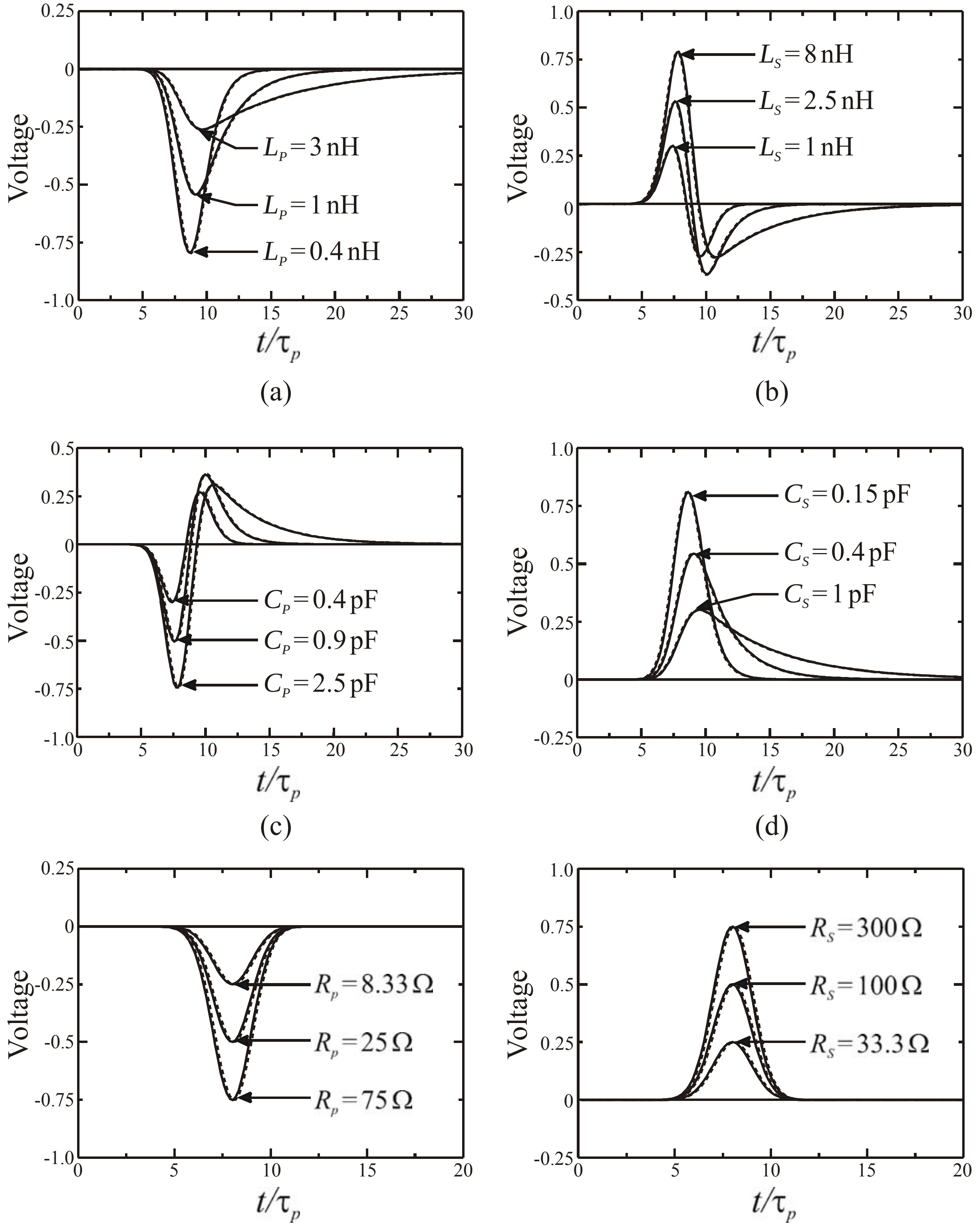
where τ_p is the characteristic time (controls width of pulses) and τ_d is a time delay. These pulses were selected because they are broadband and have well known mathematical properties. The derivations of the analytic reflected voltages for the Gaussian and differentiated Gaussian pulses, while straightforward, are somewhat long and involved, details can be seen in [2].

Figures 10 and 11 show examples of the voltages reflected from each type of load when Gaussian and differentiated Gaussian pulses are incident, respectively. Three element values were selected for each type of load to give reflected pulses with a maximum magnitudes or peak-to-peak swings of about 0.25, 0.5, and 0.75 with the Gaussian excitation. In the FDTD models, the 1D lossless transmission line ($Z_C = 50\ \Omega$, $v_p = 2.998 \times 10^8$ m/s) is driven by the one-way voltage source injector and terminated at each end by the absorbing boundary condition that will be discussed in later lecture(s) [1]. A spatial interval $\Delta z = 0.76$ mm, characteristic time $\tau_p = 16.732$ ps, and Courant stability factor of $S = v_p \Delta t / \Delta z = 0.5$ were used. For this value of Δz, the spectrum of the Gaussian pulse is down 9.34 dB from its peak value when $\Delta z = \lambda/20$ and the spectrum of the differentiated Gaussian pulse is down 4.0 dB. As shown, there is excellent agreement between the analytic and FDTD results.

References

[1] J.G. Maloney, K.L. Shlager, and G.S. Smith, “A Simple FDTD Model for Transient Excitation of Antennas by Transmission Lines,” *IEEE Trans. Ant. Propag.*, vol. 42, no. 2, pp. 289-292, Feb. 1994.

[2] T. P. Montoya and G. S. Smith, “Modeling Transmission Line Circuit Elements in the FDTD Method,” *Microwave and Optical Technology Letters*, vol. 21, no. 7, pp.105-114, April 20, 1999.



(a) (b)

(c) (d)

(e) (f)

Figure 10 Reflected voltage from a transmission line with (a) parallel and (b) series inductors, (c) parallel and (d) series capacitors, and (e) parallel and (f) series resistors inserted. Analytic (solid lines) and FDTD (dash lines) results are shown for **Gaussian** pulse excitation.

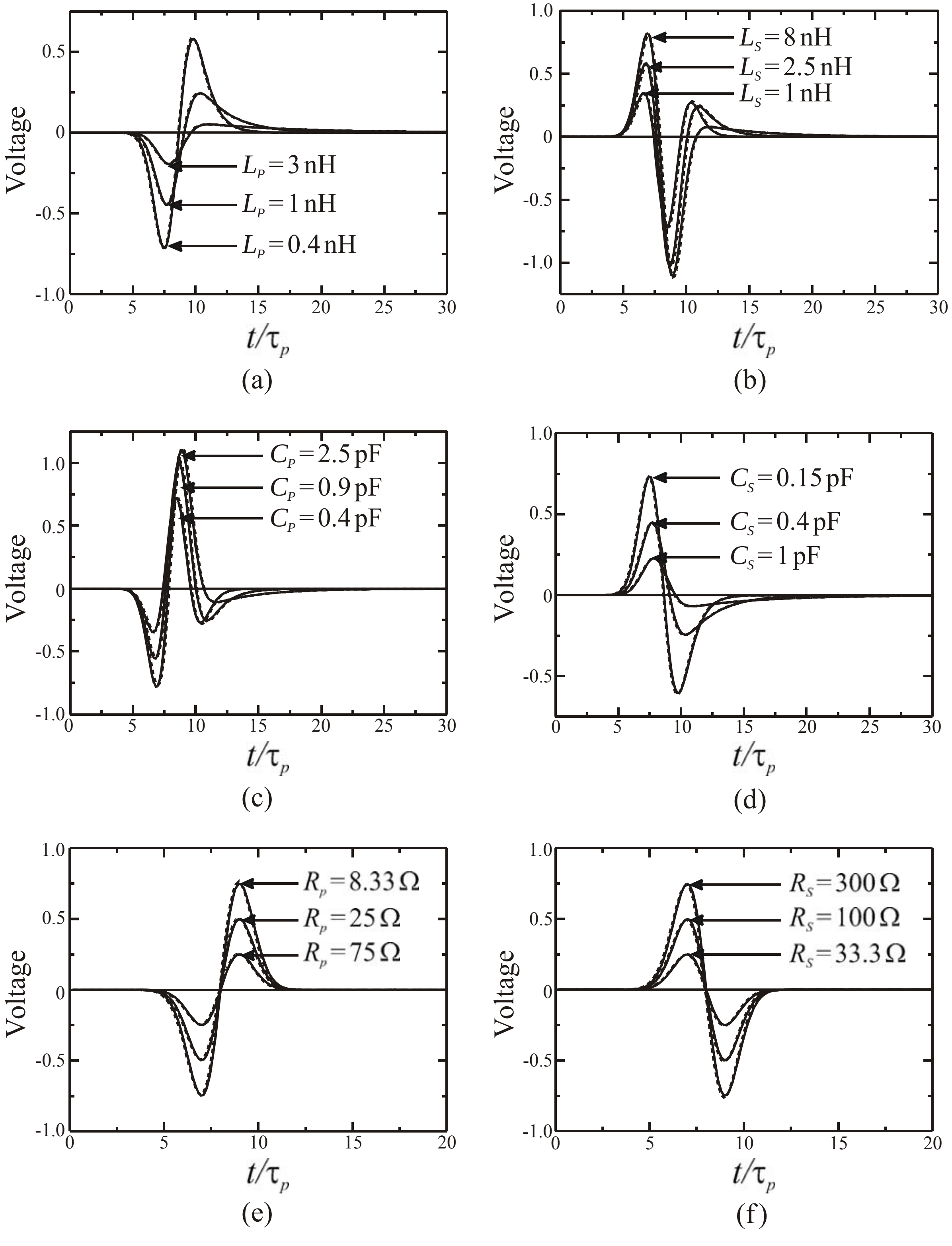


Figure 11 Reflected voltage from a transmission line with (a) parallel and (b) series inductors, (c) parallel and (d) series capacitors, and (e) parallel and (f) series resistors inserted. Analytic (solid lines) and FDTD (dash lines) results are shown for **differentiated Gaussian** pulse excitation.