EE692 Applied EM- FDTD Method
One-Dimensional Transmission Lines Notes- Lecture 1

Application of FDTD method to 1D Lossless Transmission Lines

To illustrate, we will examine a section of a one-dimensional (1D) lossless transmission line [1]. As found in many introductory EM texts, a circuit model for an incremental section (Δz) of lossless transmission line is shown in Figure 1. In it, *l* and *c* are the inductance per unit length and the capacitance per unit length, respectively.

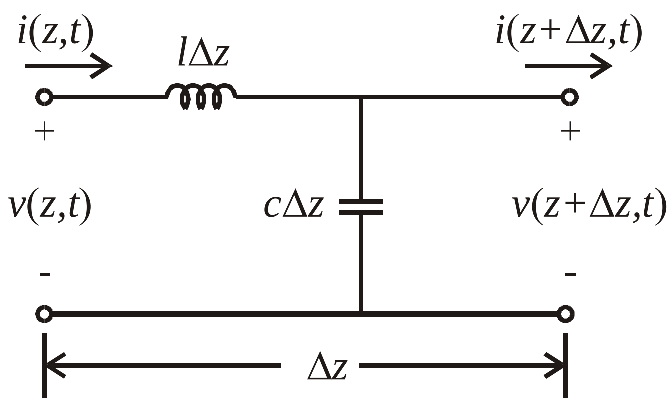

Figure 1 Incremental section of lossless transmission line

Applying Kirchoff's Current Law (KCL) to the top right node of Fig. 1 yields

$$i(z,t) - i(z+\Delta z,t) - c\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} = 0.$$

Next, applying Kirchoff's Voltage Law (KVL) clockwise around the outside loop yields

$$v(z+\Delta z,t) - v(z,t) + l\Delta z \frac{\partial i(z,t)}{\partial t} = 0 \ .$$

Rearranging these equations yields

$$-\left[\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z}\right] = c\frac{\partial v(z+\Delta z,t)}{\partial t}$$

and

$$-\left[\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z}\right] = l\frac{\partial i(z,t)}{\partial t} \ .$$

Letting $\Delta z \rightarrow 0$ and using the definition of the derivative

$$\frac{\partial f(u)}{\partial u} = \lim_{\Delta u \rightarrow 0} \frac{f(u+\Delta u) - f(u)}{\Delta u}$$

leads to the Telegrapher's equations

$$-\frac{\partial i(z,t)}{\partial z} = c\frac{\partial v(z,t)}{\partial t}$$

and

$$-\frac{\partial v(z,t)}{\partial z} = l\frac{\partial i(z,t)}{\partial t} \quad .$$

To apply the FDTD method, first spatially discretize/grid the current and voltage on a section of 1D lossless transmission line as shown in Figure 2. Note how the voltage and current nodes are spatially interleaved.

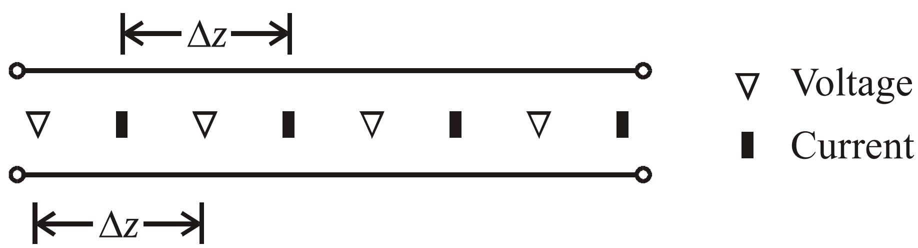

Figure 2 Current and voltage grid on lossless 1D transmission line

For the time derivatives in the Telegrapher's equations, discretize and interleave the current and voltage nodes in time, forming the mesh shown in Figure 3.

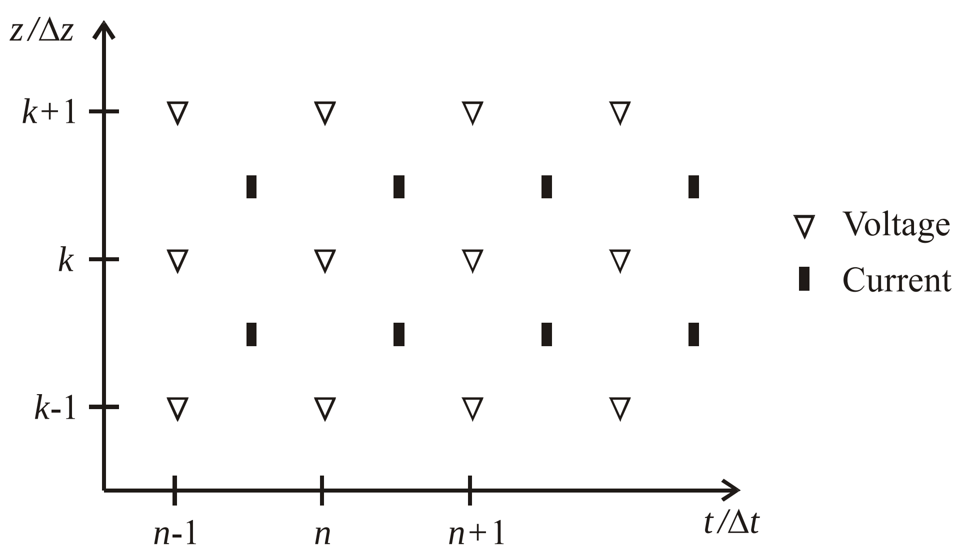

Figure 3 Current and voltage mesh for 1D lossless transmission line

The choice to place the voltage nodes at integer values of the spatial (k) and temporal (n) indices and the current nodes halfway in between is arbitrary. For convenience, adopt a standard FDTD shorthand notation where

$$i\left(z=(k+0.5)\Delta z,\ t=(n+0.5)\Delta t\right)=I^{n+0.5}(k+0.5)$$

and

$$v(z=k\Delta z, t=n\Delta t)=V^n(k)\ .$$

Based on these discretization choices, the incremental section of a 1D lossless transmission line now appears as shown in Figure 4.

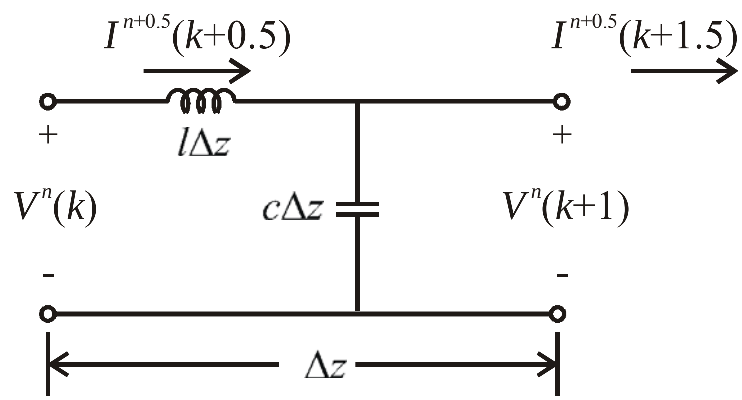

Figure 4 Discretized incremental section of lossless transmission line

To begin, discretize the Telegrapher equation $-\frac{\partial v(z,t)}{\partial z}=l\frac{\partial i(z,t)}{\partial t}$ about location $z=(k+0.5)\Delta z$ and time $t=n\Delta t$ using a second-order accurate central-difference approximation to the derivatives with respect to space and time. Assuming the spatial Δz and temporal Δt step sizes are small, the spatial and temporal derivatives can be discretized using the second-order accurate central-difference approximation

$$\left.\frac{\partial f(u)}{\partial u}\right|_{u+\frac{\Delta u}{2}} \approx \frac{f(u+\Delta u)-f(u)}{\Delta u}.$$

yielding

$$-\left[\frac{V^n(k+1)-V^n(k)}{\Delta z}\right]=l\left[\frac{I^{n+0.5}(k+0.5)-I^{n-0.5}(k+0.5)}{\Delta t}\right]\ .$$

Next, define the characteristic impedance $Z_C = \sqrt{\dfrac{l}{c}}$ and phase velocity $v_p = \dfrac{1}{\sqrt{lc}}$ for the lossless transmission line. This implies $l = \dfrac{Z_C}{v_p}$ and $c = \dfrac{1}{v_p Z_C}$. Re-arranging the discretized equation, the update equation for the current is found to be

$$\boxed{I^{n+0.5}(k+0.5) = I^{n-0.5}(k+0.5) - \frac{1}{Z_C}\left(\frac{v_p \Delta t}{\Delta z}\right)\left[V^n(k+1) - V^n(k)\right]}\,.$$

A similar procedure applied to the Telegrapher equation $-\dfrac{\partial i(z,t)}{\partial z} = c\dfrac{\partial v(z,t)}{\partial t}$ at location $z = (k+1)\Delta z$ for time $t = (n+0.5)\Delta t$ gives

$$-\left[\frac{I^{n+0.5}(k+1.5) - I^{n+0.5}(k+0.5)}{\Delta z}\right] = c\left[\frac{V^{n+1}(k+1) - V^n(k+1)}{\Delta t}\right].$$

This can be re-arranged to give an update equation for the voltage

$$\boxed{V^{n+1}(k+1) = V^n(k+1) - Z_C\left(\frac{v_p \Delta t}{\Delta z}\right)\left[I^{n+0.5}(k+1.5) - I^{n+0.5}(k+0.5)\right]}\,.$$

In both update equations, the Courant stability factor $S = \dfrac{v_p \Delta t}{\Delta z}$ appears. Note how it is similar to that defined by equation (2.28a) of the text.

Further, the Telegrapher's equations can be used to find 1D scalar wave equations for the current and voltage in the form of equation (2.1) in the text. First, take second derivatives with respect to *t* and *z*, respectively, to get

$$-\frac{\partial^2 i(z,t)}{\partial z \partial t} = c\frac{\partial^2 v(z,t)}{\partial t^2} \quad \text{and} \quad -\frac{\partial^2 i(z,t)}{\partial z^2} = c\frac{\partial^2 v(z,t)}{\partial z \partial t}$$

as well as

$$-\frac{\partial^2 v(z,t)}{\partial z \partial t} = l\frac{\partial^2 i(z,t)}{\partial t^2} \quad \text{and} \quad -\frac{\partial^2 v(z,t)}{\partial z^2} = l\frac{\partial^2 i(z,t)}{\partial z \partial t}\,.$$

Cross-substituting, re-arranging, and using $v_p = \frac{1}{\sqrt{lc}}$, then yields

$$\frac{\partial^2 i(z,t)}{\partial t^2} = v_p^2 \frac{\partial^2 i(z,t)}{\partial z^2} \text{ and } \frac{\partial^2 v(z,t)}{\partial t^2} = v_p^2 \frac{\partial^2 v(z,t)}{\partial z^2} .$$

These equations have the same form as the one-dimensional scalar wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

discussed in the text.

Modeling useful transmission line circuits will require voltage sources, parallel/series loads, and terminations. These are presented in following lecture(s).

Reference

[1] J.G. Maloney, K.L. Shlager, and G.S. Smith, "A Simple FDTD Model for Transient Excitation of Antennas by Transmission Lines," *IEEE Trans. Ant. Propag.*, vol. 42, no. 2, pp. 289-292, Feb. 1994.