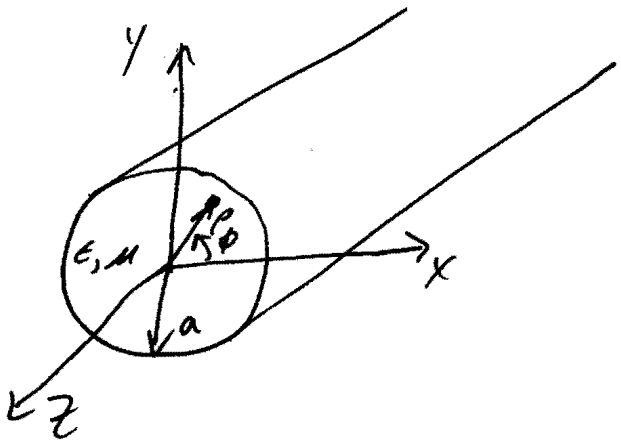
Chapter 9 Circular Cross-Section Waveguides and Cavities

9.2 Circular Waveguide



$0 \le \rho < \infty$

$0 \le \phi < 2\pi$

$-\infty < z < \infty$

Why?

→ Good for feeding conical horn antennas

→ Easy to manufacture

→ Good for applications requiring moving rotary joints

→ some modes are very low loss ⇒ good for longer distances @ high frequencies

Disadvantages

→ occupies more space than comparable rectangular waveguides

→ modes not separated by as great a BW.
E.g., Rectangular 2:1 vs Circular 1.3:1 for dominant mode maximum bandwidth

9.2.1 Transverse Electric (TE^z) Modes

Here, let $\bar{A}=0$ & $\bar{F}=\hat{a}_z F_z(\rho,\phi,z)$. From chapter 3, $\bar{F}$ must satisfy the wave equation

$$\nabla^2 F_z + \beta^2 F_z = 0$$

$$\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F_z}{\partial \phi^2} + \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z = 0$$

For our circular waveguide, we can only have a propagating wave in the $\pm z$-directions. Therefore, we choose a combination of (3-67b), (3-68b), & (3-69a) as our assumed solution

$$F_z = \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \times \left[A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}\right] \qquad (9\text{-}4)$$

with the constraint/dispersion equation

$$\beta_\rho^2 + \beta_z^2 = \beta^2 \qquad (9\text{-}4a).$$

Some boundary & physical conditions that we can use to find our unknown coefficients are:

$$E_\phi = 0 \quad @ \ \rho = a \qquad (9\text{-}5a)$$

9.2.1 cont.

For conservation of energy, $\bar{E}$ & $\bar{H}$ must be finite w/in the waveguide (9-5b)

Since the fields, $\bar{E}$ & $\bar{H}$, can not have two values at the same point, they must repeat every 2π wrt ϕ. (9-5c)

$\Rightarrow$ Equation (9-5b) implies that $\underline{B_1 = 0}$ since $Y_m(\beta_\rho \rho) \to \infty$ as $\rho \to 0$

$\Rightarrow$ Equation (9-5c) implies that $\underline{m = 0, 1, 2, \ldots}$ since then the $\cos(m\phi)$ & $\sin(m\phi)$ terms will repeat every 2π wrt ϕ

$\Rightarrow$ If we further only consider waves propagating in the $+z$-direction $\Rightarrow \underline{B_3 = 0}$, we arrive at

$$F_z^+ = A_{mn} J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z} \quad (9\text{-}8)$$

To apply (9-5a), we need to find E_ϕ which from (6-80) is $E_\phi^+ = \frac{1}{\epsilon}\frac{\partial F_z^+}{\partial \rho}$

9.2.1 cont.

$$E_\phi^+ = \beta_\rho \frac{A_{mn}}{\epsilon} J_m'(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$$

Letting $\rho = a$, we get

$$E_\phi^+(\rho=a) = 0 = \beta_\rho \frac{A_{mn}}{\epsilon} J_m'(\beta_\rho a)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$$

which, for a non-trivial solution, means

$$J_m'(\beta_\rho a) = 0$$

$\Rightarrow \beta_\rho a = \chi_{mn}' \equiv n^{th}$ zero $(n = 1, 2, \ldots)$ of the derivative of the Bessel function of the first kind of order m $(m = 0, 1, 2, \ldots) \Rightarrow$ See Table 9-1.

$$\beta_\rho = \frac{\chi_{mn}'}{a} \quad (9\text{-}10)$$

TABLE 9-1 Zeroes χ_{mn}' of derivative $J_m'(\chi_{mn}') = 0$ $(n = 1, 2, 3, \ldots)$ of the Bessel function $J_m(x)$

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$	$m=11$
$n=1$	3.8318	1.8412	3.0542	4.2012	5.3175	6.4155	7.5013	8.5777	9.6474	10.7114	11.7708	12.8264
$n=2$	7.0156	5.3315	6.7062	8.0153	9.2824	10.5199	11.7349	12.9324	14.1155	15.2867	16.4479	17.6003
$n=3$	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872	15.2682	16.5294	17.7740	19.0046	20.2230	21.4309
$n=4$	13.3237	11.7060	13.1704	14.5859	15.9641	17.3129	18.6375	19.9419	21.2291	22.5014	23.7607	25.0085
$n=5$	16.4706	14.8636	16.3475	17.7888	19.1960	20.5755	21.9317	23.2681	24.5872	25.8913	27.1820	28.4609

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9.2.1 cont.

Examining Table 9-1, the first three zeroes occur for $(m=1, n=1)$ w/ $x'_{11} = 1.8412$, $(m=2, n=1)$ w/ $x'_{21} = 3.0542$, and $(m=0, n=1)$ w/ $x'_{01} = 3.8318$. Substituting this into the dispersion equation leads to

$$(\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2 - \beta_p^2} = \sqrt{\beta^2 - \left(\frac{x'_{mn}}{a}\right)^2} & \beta > \beta_p \quad (9\text{-}11a) \\ 0 & \beta = \beta_p \quad (9\text{-}11b) \\ -j\sqrt{\beta_p^2 - \beta^2} = -j\sqrt{\left(\frac{x'_{mn}}{a}\right)^2 - \beta^2} & \beta < \beta_p \quad (9\text{-}11c) \end{cases}$$

This leads to the cutoff frequency $(\beta = \beta_p)$

$$\beta_c = \omega_c\sqrt{\mu\epsilon} = 2\pi f_c\sqrt{\mu\epsilon} = \beta_p = \frac{x'_{mn}}{a}$$

$$\underline{f_{c,mn} = \frac{x'_{mn}}{2\pi a\sqrt{\mu\epsilon}}} \qquad (9\text{-}12a)$$

Now, we have several alternative ways to express

$$\beta_{z,mn} = \begin{cases} \beta\sqrt{1-\left(\beta_p/\beta\right)^2} = \beta\sqrt{1-\left(\beta_c/\beta\right)^2} & f > f_c \\ = \beta\sqrt{1-\left(\frac{x'_{mn}}{\beta a}\right)^2} = \beta\sqrt{1-\left(f_c/f\right)^2} & \\ 0 & f = f_c \\ -j\beta\sqrt{\left(\frac{\beta_p}{\beta}\right)^2 - 1} = -j\beta\sqrt{\left(\frac{\beta_c}{\beta}\right)^2 - 1} & f < f_c \\ = -j\beta\sqrt{\left(x'_{mn}/\beta a\right)^2 - 1} = -j\beta\sqrt{\left(f_c/f\right)^2 - 1} & \end{cases}$$

9.2.1 cont.

By definition, the guide wavelength is

$$\lambda_{g,mn} = \frac{2\pi}{\beta_{z,mn}} \qquad (9\text{-}14)$$

$$= \begin{cases} \dfrac{2\pi}{\beta\sqrt{1-(f_c/f)^2}} = \dfrac{\lambda}{\sqrt{1-(f_c/f)^2}} & f > f_c \quad (9\text{-}14a) \\ \infty & f = f_c \quad (9\text{-}14b) \end{cases}$$

λ ← unbounded

$\Rightarrow$ No wavelength for $f < f_c$ (evanescent)

In order of cutoff frequencies, the order in which TE_{mn}^z modes occur is:

$$TE_{11}^z,\ TE_{21}^z,\ TE_{01}^z,\ \ldots$$

$\Rightarrow$ Unlike rectangular waveguides, the order of the TE_{mn}^z modes is <u>fixed</u>.

$\Rightarrow$ BW between modes is also <u>fixed</u>. E.g.

$$\frac{f_{c,21}}{f_{c,11}} = \frac{3.0542}{1.8412} = 1.6588:1$$

← smaller than typical rectangular waveguide (2:1)

From (6-80), the field components are

$$E_\rho = \frac{-1}{\epsilon\rho}\frac{\partial F_z}{\partial\phi} \qquad H_\rho = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial\rho\,\partial z}$$

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial\rho} \qquad H_\phi = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial\phi\,\partial z}$$

$$E_z = 0 \ (TE^z!) \qquad H_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z\right)$$

9.2.1 cont. (9-15a) to (9-15g)

$$E_\rho^+ = -A_{mn}\frac{m}{\epsilon\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$E_\phi^+ = A_{mn}\frac{\beta_\rho}{\epsilon}J_m'(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$E_z^+ = 0$$

$$H_\rho^+ = -A_{mn}\frac{\beta_\rho\beta_z}{\omega\mu\epsilon}J_m'(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$H_\phi^+ = -A_{mn}\frac{m\beta_z}{\omega\mu\epsilon}\frac{1}{\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$H_z^+ = -jA_{mn}\frac{\beta_\rho^2}{\omega\mu\epsilon}J_m(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

where $J_m'(\beta_\rho\rho) = \dfrac{\partial J_m(\beta_\rho\rho)}{\partial(\beta_\rho\rho)}$

The TE_{mn}^{+z} wave impedances are

$$(Z_w^{+z})^{TE_{mn}} = \frac{E_\rho^+}{H_\phi^+} = \frac{-E_\phi^+}{H_\rho^+} = \frac{\omega\mu}{(\beta_z)_{mn}} \qquad (9\text{-}16)$$

$$= \begin{cases} \dfrac{\omega\mu}{\beta\sqrt{1-(f_c/f)^2}} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}} & f > f_c \quad (9\text{-}16a) \\ \infty & f = f_c \quad (9\text{-}16b) \\ \dfrac{+j\omega\mu}{\beta\sqrt{(f_c/f)^2-1}} = \dfrac{+j\eta}{\sqrt{(f_c/f)^2-1}} & f < f_c \quad (9\text{-}16c) \end{cases}$$

9.2.1 cont.

$\Rightarrow$ For $f > f_c$, $(Z_w^{+z})^{TE}_{mn} > \eta = \sqrt{\mu/\epsilon}$ ← inside waveguide

$\Rightarrow$ For $f < f_c$, the wave impedance is <u>inductive</u>

$\Rightarrow$ waveguide behaves like an inductor & stores energy in magnetic field.

ex. For a 3/4" copper pipe, determine the cutoff frequencies for the lowest 3 TE^z_{mn} modes. Then, calculate some parameters for the TE^z_{11} mode at a frequency of 11 GHz. Air-filled.

TE^z_{11} $\quad x'_{mn} = 1.8412 \Rightarrow f_{c,11} = \dfrac{1.8412\,(2.9979\times10^8)}{2\pi(0.75/2)(0.0254)}$

$$\underline{\underline{f_{c,11} = 9.223 \text{ GHz}}}$$

TE^z_{21} $\quad x'_{mn} = 3.0542 \qquad f_{c,21} = \dfrac{3.0542\,(2.9979\times10^8)}{2\pi(0.375)0.0254}$

$$\underline{\underline{f_{c,21} = 15.299 \text{ GHz}}}$$

TE^z_{01} $\quad x'_{mn} = 3.8318 \qquad f_{c,01} = \dfrac{3.8318\,(2.9979\times10^8)}{2\pi(0.375)0.0254}$

$$\underline{\underline{f_{c,01} = 19.194 \text{ GHz}}}$$

9.2.1 cont.

ex. cont. TE_{11}^{z} mode

@ $f = 11\,GHz$ $\quad \beta = \omega\sqrt{\mu_0\varepsilon_0} = \dfrac{(2\pi)11\times10^9}{2.9979\times10^8}$

$$\beta = 230.545 \text{ rad/m}$$

(9-10) $\beta_\rho = \dfrac{x_{11}'}{a} = \dfrac{1.8412}{0.375(0.0254)} = 193.302 \dfrac{\text{rad}}{\text{m}}$

(9-13a) $\beta_{z,11} = \beta\sqrt{1-(f_c/f)^2} = 230.545\sqrt{1-(9.223/11)^2}$

$$\beta_{z,11} = 125.64 \frac{\text{rad}}{\text{m}}$$

(9-14) $\lambda_{g,11} = \dfrac{2\pi}{\beta_{z,11}} = 0.0500\,m = 5.00\,cm$

For comparison, $\lambda = c/f = \dfrac{2.9979\times10^8}{11\times10^9} = 2.725\,cm$

(9-16a) $(Z_w^{+z})_{11}^{TE} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}} \xrightarrow{air} \dfrac{376.7303}{\sqrt{1-(9.223/11)^2}}$

$$(Z_w^{+z})_{11}^{TE} = 691.288\,\Omega$$

9.2.2 Transverse Magnetic (TM^z) Modes

Here, we will let $\bar{F}=0$ and assume the vector magnetic potential is $\bar{A} = \hat{a}_z A_z(\rho,\phi,z)$.

From Chapter 3, $\bar{A}$ must satisfy the wave equation $\nabla^2 A_z + \beta^2 A_z = 0$ (9-18)

$$\frac{\partial^2 A_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z = 0$$

Again, for our circular waveguide, we can only have propagation in the $\pm z$-directions.

Therefore, from Chapter 3, we choose a combination of (3-67b), (3-68b), & (3-69a) as our assumed general solution

$$A_z = \underbrace{\left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]}_{\text{standing}}\underbrace{\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]}_{\text{periodic}}$$

$$\times\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \leftarrow \text{Traveling} \quad (9\text{-}19)$$

w/ dispersion equation

$$\beta_\rho^2 + \beta_z^2 = \beta^2 \quad (9\text{-}19a)$$

9.2.2 cont.

For the TM^z modes, the boundary conditions and physical constraints for the circular waveguide are:

$$E_\phi = 0 \quad @\ \rho = a \qquad (9\text{-}20a)$$

$$E_z = 0 \quad @\ \rho = a \qquad (9\text{-}20b)$$

Conservation of energy $\Rightarrow$ $\bar{E}$ & $\bar{H}$ must be <u>finite</u>. (9-20c)

Fields can not be multivalued @ same location $\Rightarrow$ $\bar{E}$ & $\bar{H}$ must repeat every 2π for ϕ. (9-20d)

$\Rightarrow$ Since $Y_m(\beta_\rho \rho) \to \infty$ as $\rho \to 0$, we set $\underline{B_1 = 0}$ to satisfy (9-20c).

$\Rightarrow$ To ensure that $\cos(m\phi)$ & $\sin(m\phi)$ repeat every 2π $\Rightarrow$ $\underline{m = 0, 1, 2, \ldots}$

$\Rightarrow$ If we consider only waves propagating in the $+z$-direction $\Rightarrow$ $\underline{B_3 = 0}$

9.2.2 cont.

At this point, we have reduced A_z to

$$A_z^+ = B_{mn} J_m(\beta_\rho \rho)\{C_2 \cos(m\phi) + D_2 \sin(m\phi)\} e^{-j\beta_z z} \quad (9\text{-}22)$$

To find β_ρ (i.e., eigenvalues), we can apply either (9-20a) or (9-20b). Using (6-70) on the A_z of (9-22), we can compute

$$E_z^+ = -j \frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2 A_z^+}{\partial z^2} + \beta^2 A_z^+\right)$$

$$= -j B_{mn} \frac{\beta_\rho^2}{\omega\mu\epsilon} J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z} \quad (9\text{-}23)$$

Applying (9-20b)

$$E_z^+(\rho = a) = 0 = -j B_m \frac{\beta_\rho^2}{\omega\mu\epsilon} J_m(\beta_\rho a)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$$

for a non-trivial solution

$$\Rightarrow J_m(\beta_\rho a) = 0$$

$$\Rightarrow \beta_\rho a = x_{mn} \equiv$$ n^{th} zero $(n = 1, 2, \ldots)$ of the Bessel function of the first kind J_m of order m where $m = 0, 1, 2, \ldots$

$$\Downarrow$$

$$\beta_\rho = \frac{x_{mn}}{a} \quad (9\text{-}25)$$

TABLE 9-2 Zeroes χ_{mn} of $J_m(\chi_{mn}) = 0$ ($n = 1, 2, 3, \ldots$) of Bessel function $J_m(x)$

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$	$m=11$
$n=1$	2.4049	3.8318	5.1357	6.3802	7.5884	8.7715	9.9361	11.0864	12.2251	13.3543	14.4755	15.5898
$n=2$	5.5201	7.0156	8.4173	9.7610	11.0647	12.3386	13.5893	14.8213	16.0378	17.2412	18.4335	19.6160
$n=3$	8.6537	10.1735	11.6199	13.0152	14.3726	15.7002	17.0038	18.2876	19.5545	20.8071	22.0470	23.2759
$n=4$	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801	20.3208	21.6415	22.9452	24.2339	25.5095	26.7733
$n=5$	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178	23.5861	24.9349	26.2668	27.5838	28.8874	30.1791

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For the TM^z modes, the lowest cutoff frequencies occur for TM^z_{01} ($\chi_{01} = 2.4049$), TM^z_{11} ($\chi_{11} = 3.8318$), and TM^z_{21} ($\chi_{21} = 5.1357$)

Now, the guided wavelength λ_g is

$$\lambda_g = \frac{2\pi}{\beta_\rho}$$

and

$$\beta_{z,mn} = \begin{cases} \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{\beta^2 - \left(\frac{\chi_{mn}}{a}\right)^2} & \beta > \beta_\rho \quad (9\text{-}26a) \\ 0 & \beta = \beta_\rho \quad (9\text{-}26b) \\ -j\sqrt{\beta_\rho^2 - \beta^2} = -j\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 - \beta^2} & \beta < \beta_\rho \quad (9\text{-}26c) \end{cases}$$

9.2.2 cont.

Using $\beta_c = \omega_c\sqrt{\mu\epsilon} = \beta_\rho = \frac{\chi_{mn}}{a}$, the cutoff frequencies for TM^z modes are

$$f_{c,mn} = \frac{\chi_{mn}}{2\pi a\sqrt{\mu\epsilon}} \qquad (9\text{-}27)$$

The phase constant in the z-direction can be found from the dispersion equation as

$$\beta_{z,mn} = \begin{cases} \sqrt{\beta^2-\beta_\rho^2} = \beta\sqrt{1-\left(\beta_c/\beta\right)^2} & f > f_c \quad (9\text{-}28a) \\ = \beta\sqrt{1-(f_c/f)^2} & \\ 0 & f = f_c \quad (9\text{-}28b) \\ -j\sqrt{\beta_\rho^2-\beta^2} = -j\beta\sqrt{(\beta_c/\beta)^2-1} & f < f_c \quad (9\text{-}28c) \\ = -j\beta\sqrt{(f_c/f)^2-1} & \end{cases}$$

The guided wavelength $\lambda_z = \lambda_g = \frac{2\pi}{\beta_z}$ is then

$$\lambda_{g,mn} = \begin{cases} \dfrac{2\pi}{\beta\sqrt{1-(f_c/f)^2}} = \dfrac{\lambda}{\sqrt{1-(f_c/f)^2}} & f > f_c \\ \infty & f = f_c \end{cases}$$

For $f < f_c$, 'waves' are evanescent (non-propagating).

9.2.2 cont.

So, overall what is the order of TE^z + TM^z modes?

Lowest

TE_{11} $(\chi_{11}' = 1.8412)$

TM_{01} $(\chi_{01} = 2.4049)$

TE_{21} $(\chi_{21}' = 3.0542)$

TE_{01} $(\chi_{01}' = 3.8318)$

&

TM_{11} $(\chi_{11} = 3.8318)$

$\vdots$

larger

Figure 9-2 illustrates the field configurations for some of the lowest order modes.

$\Rightarrow$ Note: TE_{0n} + TM_{1n} modes have same cut-off frequencies, called degenerate modes. This results from $J_0'(\beta_\rho a) = -J_1(\beta_\rho a)$

$$\Rightarrow \underline{\underline{\chi_{0n}' = \chi_{1n}}} \quad n = 1, 2, 3, \ldots$$

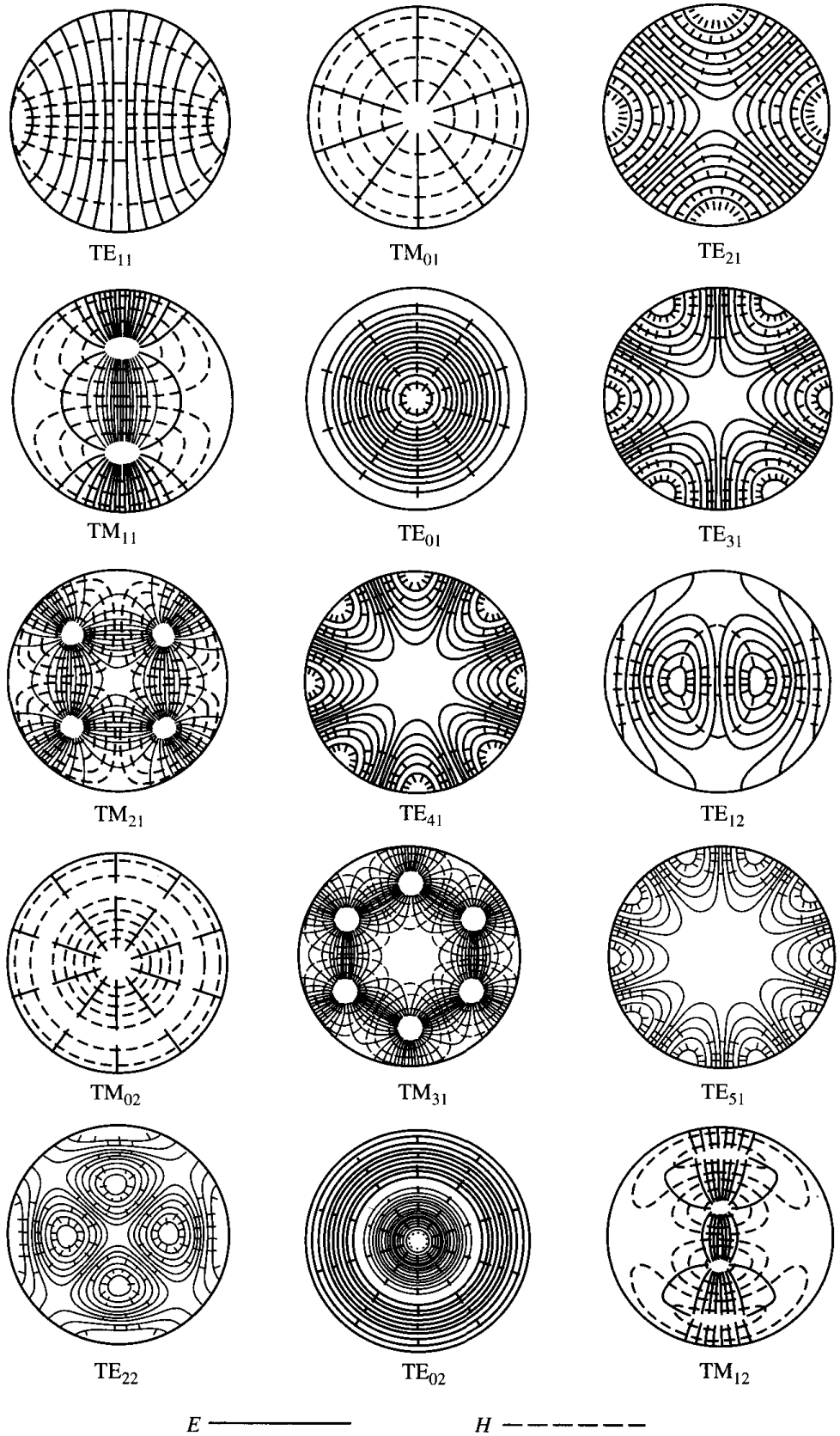

Figure 9-2 Field configurations of the first 30 TE^z and/or TM^z modes in a circular waveguide. (Source: C. S. Lee, S. W. Lee, and S. L. Chuang, "Plot of modal field distribution in rectangular and circular waveguides," *IEEE Trans. Microwave Theory Tech.*, © 1966, IEEE.).

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9.2.2 cont. (9-31a) – (9-31g)

For the TM^z modes, the $\bar{E}$ + $\bar{H}$ field components can be found using A_z^+ (9-22) and (6-70)

$$E_\rho^+ = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2 A_z^+}{\partial\rho\,\partial z} = -B_{mn}\frac{\beta_\rho\beta_z}{\omega\mu\epsilon}J_m'(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$E_\phi^+ = \frac{-j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 A_z^+}{\partial\phi\,\partial z} = -B_{mn}\frac{m\beta_z}{\omega\mu\epsilon\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$E_z^+ = \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2 A_z^+}{\partial z^2}+\beta^2 A_z^+\right)$$

$$= -j\,B_{mn}\frac{\beta_\rho^2}{\omega\mu\epsilon}J_m(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$H_\rho^+ = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z^+}{\partial\phi} = B_{mn}\frac{m}{\mu}\frac{1}{\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$H_\phi^+ = \frac{-1}{\mu}\frac{\partial A_z^+}{\partial\rho} = -B_{mn}\frac{\beta_\rho}{\mu}J_m'(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$H_z^+ = 0 \quad (TM^z \text{ mode})$$

Note: $J_m'(\beta_\rho\rho) = \dfrac{\partial J_m(\beta_\rho\rho)}{\partial(\beta_\rho\rho)}$

With the field components computed, it is now possible to determine the wave impedance $(Z_w^{+z})_{mn}^{TM} = \dfrac{E_\rho^+}{H_\phi^+} = \dfrac{-E_\phi^+}{H_\rho^+} = \dfrac{\beta_{z,mn}}{\omega\epsilon}$

9.2.2 cont.

$$(Z_w^{+z})^{TM}_{mn} = \begin{cases} \dfrac{\beta\sqrt{1-(f_c/f)^2}}{\omega\epsilon} = \eta\sqrt{1-(f_c/f)^2} & f > f_c \\ 0 & f = f_c \\ \dfrac{-j\beta\sqrt{(f_c/f)^2-1}}{\omega\epsilon} = -j\eta\sqrt{(f_c/f)^2-1} & f < f_c \end{cases}$$

⇒ wave impedance <u>less</u> than η above cutoff

⇒ 'short' @ cutoff

⇒ looks capacitive below cutoff, stores energy in electric field

How can we excite specific modes?

Once again, it is necessary to 'match' the pattern, electric or magnetic, field that is desired. <u>Figure 9-3</u>, supplemented by excerpts from Figure 9-2, shows examples of how to excite the lowest two modes → TE_{11}^z and TM_{01}^z.

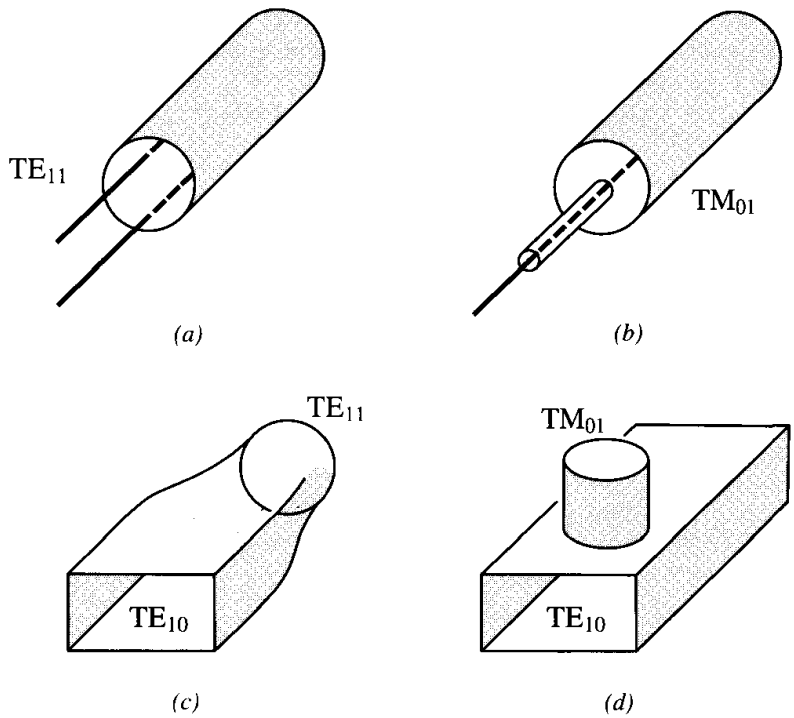


Figure 9-3 Excitation of TE_{mn} and TM_{mn} modes in a circular waveguide. (*a*) TE_{11} mode. (*b*) TM_{01} mode. (*c*) TE_{10} (rectangular)-TE_{11} (circular). (*d*) TE_{10} (rectangular)-TM_{01} (circular).

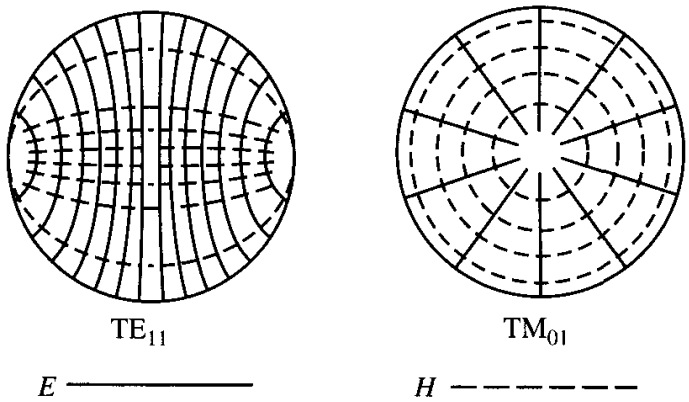


Figure 9-2 excerpt

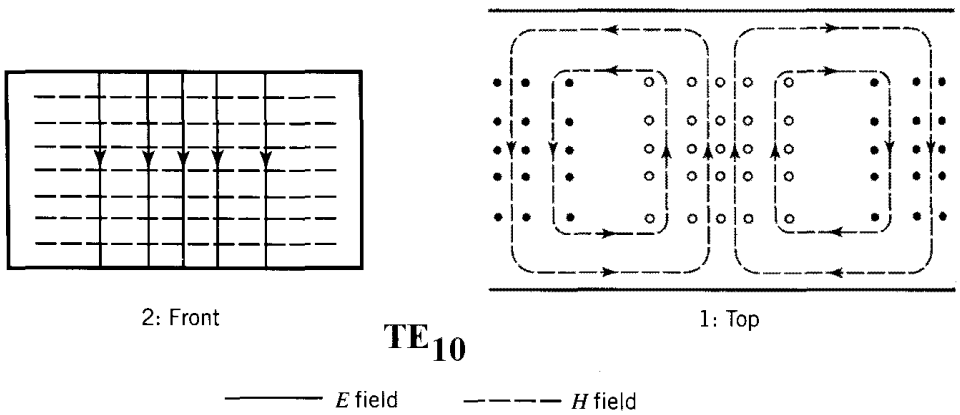


Figure 8-7 excerpt

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

9.2.2 cont.

ex. For a 3/4" copper pipe, determine the cutoff frequencies for the lowest TM^z_{mn} modes. Then, calculate a few parameters at 14 GHz for the TM^z_{01} mode. Air-filled.

$\boxed{TM^z_{01}}$ $\chi_{01} = 2.4049 \Rightarrow f_{c,01} = \dfrac{\chi_{10}}{2\pi a\sqrt{\mu\epsilon}}$ (9-27)

$$f_{c,01} = \frac{2.4049(2.9979\times10^8)}{2\pi(0.375)0.0254} \Rightarrow \underline{\underline{f_{c,10} = 12.047\text{ GHz}}}$$

$\boxed{TM^z_{11}}$ $\chi_{11} = 3.8318$

$$f_{c,11} = \frac{3.8318(2.9979\times10^8)}{2\pi(0.375)0.0254} \Rightarrow \underline{\underline{f_{c,11} = 19.194\text{ GHz}}}$$

$\Rightarrow$ Same as f_c for TE^z_{01} mode!

<u>For TM^z_{01} mode @ 14 GHz</u>

$$\beta = \omega\sqrt{\mu_0\epsilon_0} = \frac{2\pi(14\times10^9)}{2.9979\times10^8} \Rightarrow \underline{\underline{\beta = 293.421\ \tfrac{\text{rad}}{\text{m}}}}$$

(9-25) $\beta_\rho = \dfrac{\chi_{01}}{a} = \dfrac{2.4049}{0.375(0.0254)} \Rightarrow \underline{\underline{\beta_\rho = 252.483\ \tfrac{\text{rad}}{\text{m}}}}$

(9-26a) $\beta_{z,01} = \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{293.4^2 - 252.5^2} \Rightarrow \underline{\underline{\beta_{z,01} = 149.49\ \tfrac{\text{rad}}{\text{m}}}}$

(9-29a) $\lambda_{g,01} = \dfrac{2\pi}{\beta_{z,01}} = \dfrac{2\pi}{149.49} \Rightarrow \underline{\underline{\lambda_{g,01} = 4.203\text{ cm}}}$

(9-32a) $(Z_w^{+z})^{TM}_{01} = \eta\sqrt{1-(f_c/f)^2} = 376.7303\sqrt{1-(12.05/14)^2}$

$$\Rightarrow \underline{\underline{(Z_w^{+z})^{TM}_{01} = 191.94\ \Omega}}$$

9.2.3 Attenuation

Unlike rectangular waveguides where there is a clear choice for a preferred mode of operation, TE_{10}, circular waveguides are often oversized and used for higher modes. The reason, as we will see, is due to the attenuation characteristics for circular waveguides.

As previously examined in Section 8.2.5 for rectangular waveguides, the attenuation coefficient (conduction) is defined as

$$\alpha_{c,mn} = \frac{P_c/\ell}{2P_{mn}}$$

← Power lost to conduction losses per unit length

← Power flowing in +z direction for mode

Conduction attenuation coefficient TE^z_{mn} modes

$$(\alpha_c)^{TE}_{mn} = \frac{R_s}{a\eta\sqrt{1-(f_c/f)^2}}\left[\left(\frac{f_c}{f}\right)^2 + \frac{m^2}{\chi'^2_{mn}-m^2}\right] \quad \left(\frac{Np}{m}\right) \qquad (9\text{-}34a)$$

$R_s \rightarrow$ walls + $\eta \rightarrow$ interior

9.2.3 cont.

Conduction attenuation coefficient TM^z_{mn} modes

$$(\alpha_c)^{TM}_{mn} = \frac{R_s}{a\eta} \frac{1}{\sqrt{1-(f_c/f)^2}} \quad (Np/m) \quad (9\text{-}34b)$$

Figure 9-4 shows plots of the attenuation coefficient for six selected modes (first five plus TE_{02}) wrt frequency normalized by $f^{TE}_{c,11}$ (lowest/dominant mode) for copper circular waveguides of radius $a = 1.5$ cm and 3 cm. Free space is assumed w/in the waveguides, i.e., no dielectric losses.

* Note that the TE_{01} mode has very low losses. Drawback it is the fourth possible mode, preceded by TE_{11}, TM_{01}, + TE_{21} and tied w/ TM_{11}.
* Note TE_{0n} modes have α_c monotonically decreasing w/ increasing frequency!

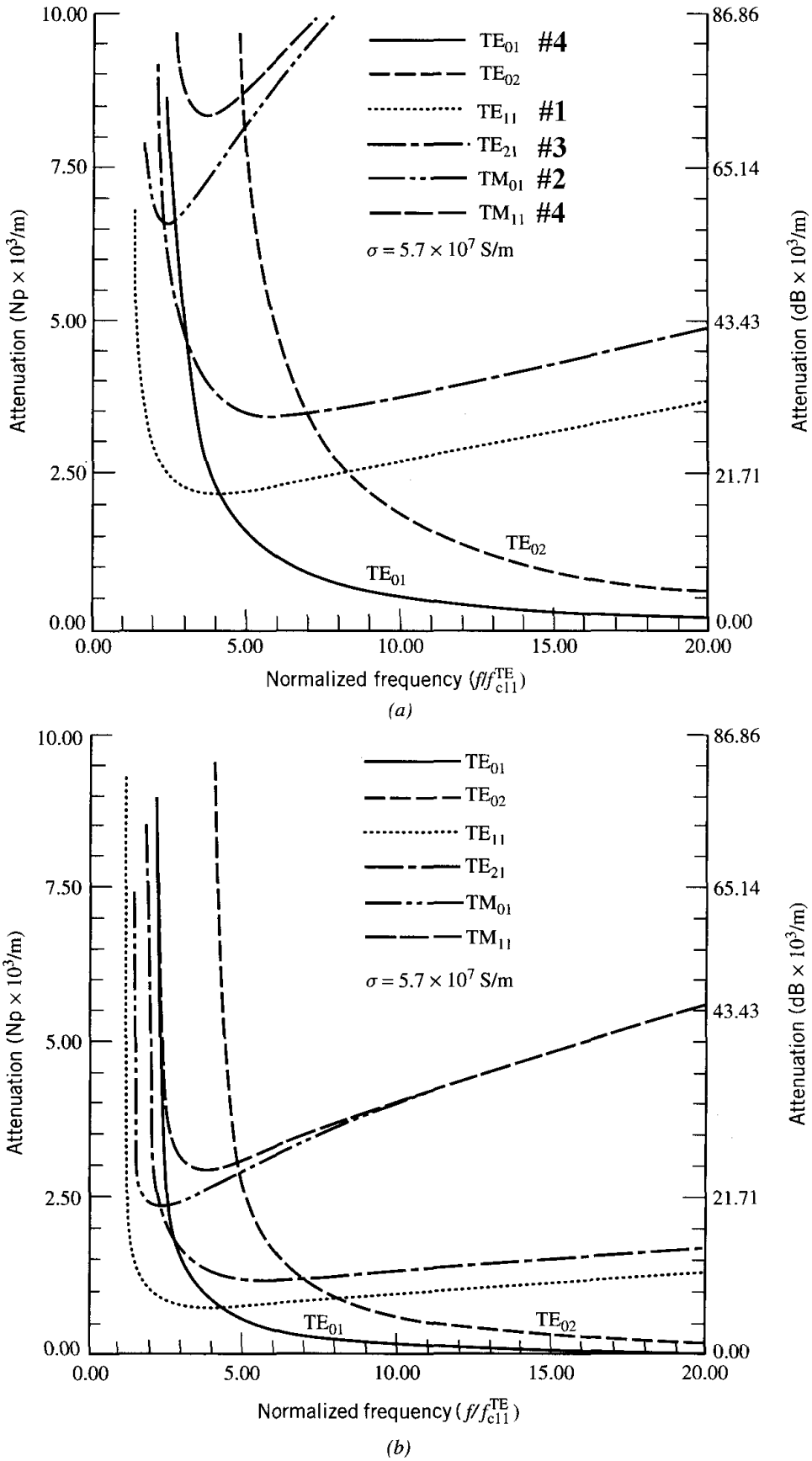


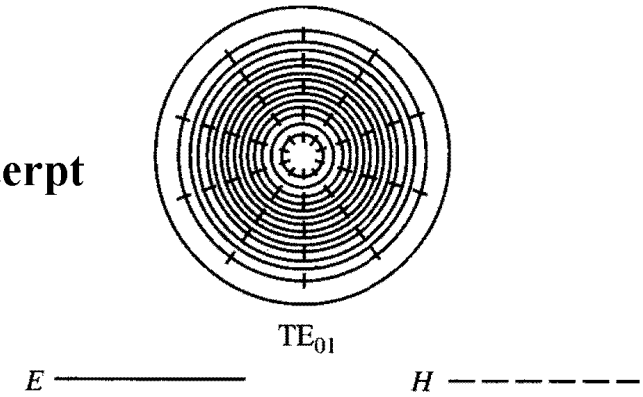
Figure 9-4 Attenuation for TE^z_{mn} and TM^z_{mn} modes in a circular waveguide. (a) $a = 1.5$ cm. (b) $a = 3$ cm

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

9.2.2 cont.

Due to the very low losses, the TE_{01} mode is attractive for high frequency longer distance transmissions. Why? $|\bar{E}| \to 0$ near $\rho = a$!

Figure 9-2 excerpt



Drawbacks → waveguide must be oversized (compared to one using TE_{11}^{z})

→ 4 other modes are possible, meaning power loss/transfer to other modes is a definite problem, particularly if the waveguide bends or has imperfections

→ exciting the TE_{01} mode <u>only</u> can be challenging

⇒ Active research area(s) to try and ameliorate or minimize these issues

Example- For an air-filled 3/4" copper ($\sigma = 5.7*10^7$ S/m) pipe, determine the cutoff frequencies and conduction attenuation coefficients for the lowest five modes at 32 GHz.

$a := 0.5 \cdot 0.75 \cdot 0.0254$ $\quad a = 9.525 \times 10^{-3}$ m $\quad \sigma cu := 5.7 \cdot 10^{7}$ S/m

$\mu 0 := 4 \cdot \pi \cdot 10^{-7}$ H/m $\quad \varepsilon 0 := 8.8541878 \cdot 10^{-12}$ F/m

$\eta 0 := \sqrt{\frac{\mu 0}{\varepsilon 0}}$ $\quad \eta 0 = 376.7303$ Ω $\quad f := 32 \cdot 10^{9}$ Hz

use (8-61) $\quad Rs := \sqrt{\frac{2 \cdot \pi \cdot f \cdot \mu 0}{2 \cdot \sigma cu}}$ $\quad \boxed{Rs = 0.047078}$ Ω

1) TE_{11} (dominant mode) $\quad \chi 11p := 1.8412 \quad m := 1$

(9-12a) $\quad fc_TE11 := \frac{\chi 11p}{2 \cdot \pi \cdot a \cdot \sqrt{\mu 0 \cdot \varepsilon 0}}$ $\quad \boxed{fc_TE11 = 9.2231 \times 10^{9}}$ Hz

(9-34a) $\quad \alpha c_TE11 := \frac{Rs}{a \cdot \eta 0 \cdot \sqrt{1 - \left(\frac{fc_TE11}{f}\right)^2}} \cdot \left[\left(\frac{fc_TE11}{f}\right)^2 + \frac{m^2}{\chi 11p^2 - m^2}\right]$

$\boxed{\alpha c_TE11 = 6.871 \times 10^{-3}}$ Np/m

2) TM_{01} (second) $\quad \chi 01 := 2.4049$

(9-27) $\quad fc_TM01 := \frac{\chi 01}{2 \cdot \pi \cdot a \cdot \sqrt{\mu 0 \cdot \varepsilon 0}}$ $\quad \boxed{fc_TM01 = 1.2047 \times 10^{10}}$ Hz

(9-34a) $\quad \alpha c_TM01 := \frac{Rs}{a \cdot \eta 0 \cdot \sqrt{1 - \left(\frac{fc_TM01}{f}\right)^2}}$

$\boxed{\alpha c_TM01 = 1.416 \times 10^{-2}}$ Np/m

3) TE_{21} (third) $\chi 21p := 3.0542$ $m := 2$

(9-12a) $$fc_TE21 := \frac{\chi 21p}{2\cdot\pi\cdot a\cdot\sqrt{\mu 0\cdot\varepsilon 0}}$$ $fc_TE21 = 1.5299\times 10^{10}$ Hz

(9-34a) $$\alpha c_TE21 := \frac{Rs}{a\cdot\eta 0\cdot\sqrt{1-\left(\frac{fc_TE21}{f}\right)^2}}\cdot\left[\left(\frac{fc_TE21}{f}\right)^2+\frac{m^2}{\chi 21p^2-m^2}\right]$$

$\alpha c_TE21 = 1.463\times 10^{-2}$ Np/m

4a) TM_{11} (fourth w/ TE_{01}) $\chi 11 := 3.8318$

(9-27) $$fc_TM11 := \frac{\chi 11}{2\cdot\pi\cdot a\cdot\sqrt{\mu 0\cdot\varepsilon 0}}$$ $fc_TM11 = 1.9195\times 10^{10}$ Hz

(9-34a) $$\alpha c_TM11 := \frac{Rs}{a\cdot\eta 0\cdot\sqrt{1-\left(\frac{fc_TM11}{f}\right)^2}}$$ $\alpha c_TM11 = 1.64\times 10^{-2}$ Np/m

4b) TE_{01} (fourth w/ TM_{11}) $\chi 01p := 3.8318$ $m := 0$

(9-12a) $$fc_TE01 := \frac{\chi 01p}{2\cdot\pi\cdot a\cdot\sqrt{\mu 0\cdot\varepsilon 0}}$$ $fc_TE01 = 1.9195\times 10^{10}$ Hz

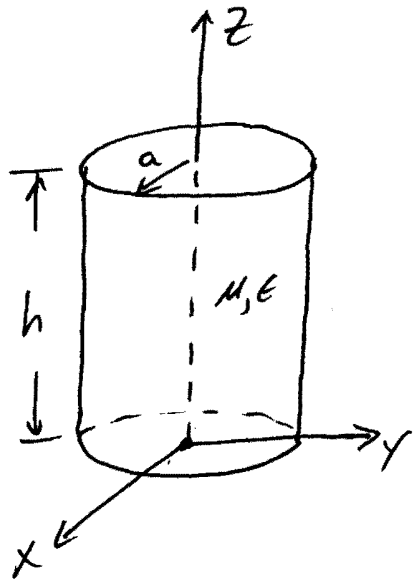
(9-34a) $$\alpha c_TE01 := \frac{Rs}{a\cdot\eta 0\cdot\sqrt{1-\left(\frac{fc_TE01}{f}\right)^2}}\cdot\left[\left(\frac{fc_TE01}{f}\right)^2+\frac{m^2}{\chi 01p^2-m^2}\right]$$

$\alpha c_TE01 = 5.9\times 10^{-3}$ Np/m

$$\frac{f}{fc_TE11} = 3.47$$ **Roughly 3.5x the lowest cutoff frequency.**

At 32 GHz, the TE_{01} mode definitely has the lowest attenuation.

9.3 Circular Cavity



Here, we have closed off the ends of a length h of circular waveguide, radius a, with flat metal plates. Inside the cavity is some material w/ electrical properties (μ, ϵ) or free space.

* Now, wave propagation is no longer possible in the $\pm z$-directions. Therefore, in our equations for F_z and A_z, we will replace the traveling wave $h(z)$ equation with standing wave, i.e.,

$$h(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z} \Rightarrow h(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z),$$

equations before applying boundary conditions @ $z=0$ and $z=h$.

* To actually excite/utilize circular cavities, irises/holes or probes that penetrate the walls are utilized as appropriate to match the applicable field pattern(s).

9.3.1 Transverse Electric (TE^z) modes

→ The boundary conditions and constants found for the ρ + ϕ terms are unchanged. Therefore, we start with

$$F_z = A_{mn} J_m(\beta_\rho \rho)\underbrace{\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]}_{\text{same}} \times \underbrace{\left[C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)\right]}_{\text{new}} \quad (9\text{-}35)$$

where $\beta_\rho = \dfrac{x'_{mn}}{a}$ and $m = 0, 1, 2, \ldots$

To find β_z, we apply the new boundary conditions that $\bar{E}_{tan} = 0$ @ $z = 0$ & h

$$E_\rho = 0 \quad @\ z = 0 \ \&\ z = h \qquad (9\text{-}36a)$$

$$E_\phi = 0 \quad @\ z = 0 \ \&\ z = h \qquad (9\text{-}36b)$$

Using either the E_ρ or E_ϕ condition will lead to the same answer. Per (6-80) + (9-35)

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial \rho}$$

$$= \beta_\rho \frac{A_{mn}}{\epsilon} J'_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \times \left[C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)\right]$$

where $J'_m(\beta_\rho \rho) = \dfrac{\partial(J_m(\beta_\rho \rho))}{\partial(\beta_\rho \rho)}$

9.3.1 cont.

@ $z=0$

$$E_\phi = 0 = \beta_\rho \frac{A_{mn}}{\epsilon} J_m'(\beta_\rho \rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[C_3+0\right]$$

For non-trivial sol'n $\Rightarrow \underline{C_3 = 0}$

@ $z=h$

$$E_\phi = 0 = \beta_\rho \frac{A_{mn}}{\epsilon} J_m'(\beta_\rho \rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]D_3\sin(\beta_z h)$$

For non-trivial sol'n $\Rightarrow \sin(\beta_z h) = 0$

$$\beta_z h = p\pi \qquad p = 1, 2, \ldots$$

$$\Rightarrow \underline{\beta_z = \frac{p\pi}{h}} \qquad (9\text{-}38b)$$

Using our dispersion/constraint equation, we can now find our resonant phase constants

$$\beta_\rho^2 + \beta_z^2 = \beta_r^2$$

$$\left(\frac{\xi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \omega_r^2 \mu\epsilon = \frac{\omega_r^2}{v^2} \qquad (9\text{-}39)$$

9.3.1 cont.

leading to the resonant frequencies (9-39a)

$$\boxed{(f_r)^{TE^z}_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \qquad \begin{array}{l} m = 0,1,2,\ldots \\ n = 1,2,\ldots \\ p = 1,2,\ldots \end{array}}$$

$$F_z = A_{mnp} J_m(\beta_\rho \rho)\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\sin(\beta_z z) \quad (9\text{-}40)$$

The various TE^z_{mnp} mode electric & magnetic fields may be found by applying (6-80) to (9-40).

$$E_\rho = -\frac{1}{\epsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} \qquad H_\rho = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial \rho\,\partial z}$$

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial \rho} \qquad H_\phi = \frac{-j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial \phi\,\partial z}$$

$$E_z = 0 \ (TE^z!) \qquad H_z = \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z\right)$$

Hmm! Smells like HW problem(s).

9.3.2 Transverse Magnetic (TM^z) Modes

→ Again the boundary conditions, etcetera and results found for the ρ & ϕ terms are unchanged. Therefore, we start with

$$A_z = B_{mn} J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \leftarrow \text{same}$$
$$\times\left[C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)\right] \leftarrow \text{new} \quad (9\text{-}41)$$

where $\beta_\rho = \dfrac{x_{mn}}{a}$ (9-41a) & $m = 0, 1, 2, \ldots$ (9-41b).

Per (6-70) & (9-41), for the TM^z case

$$E_\phi = \frac{-j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z}$$

$$= -jB_{mn}\frac{m\beta_z}{\omega\mu\epsilon}\frac{1}{\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi) + D_2\cos(m\phi)\right]$$
$$\times\left[-C_3\sin(\beta_z z) + D_3\cos(\beta_z z)\right]$$

@ $z = 0$

$$E_\phi = 0 = -jB_{mn}\frac{m\beta_z}{\omega\mu\epsilon}\frac{1}{\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi) + D_2\cos(m\phi)\right]$$
$$\times\left[0 + D_3(1)\right]$$

For a non-trivial sol'n $\Rightarrow$ $\underline{D_3 = 0}$

9.3.2 cont.

@ $z=h$

$$E_\phi = 0 = +j B_{mn} \frac{m\beta_z}{\omega\mu\epsilon} \frac{1}{\rho} J_m(\beta_\rho \rho)\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right] C_3 \sin(\beta_z h)$$

For a non-trivial solution $\Rightarrow \sin(\beta_z h) = 0$

$$\beta_z h = p\pi \quad p = 0, 1, 2, \ldots$$

$$\Rightarrow \underline{\beta_z = \frac{p\pi}{h}} \qquad (9\text{-}43b)$$

From the constraint/dispersion equation

$$\beta_\rho^2 + \beta_z^2 = \beta_r^2 \leftarrow \text{resonant}$$

$$\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \beta_r^2 = \omega_r^2 \sqrt{\mu\epsilon} = \frac{\omega_r^2}{v^2} \qquad (9\text{-}44)$$

and

$$\boxed{f_{r,mnp}^{TM^z} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \qquad \begin{array}{l} m = 0, 1, 2, \ldots \\ n = 1, 2, \ldots \\ p = 0, 1, 2, \ldots \end{array}} \qquad (9\text{-}45)$$

$$A_z = B_{mnp} J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \cos(\beta_z z) \qquad (9\text{-}46)$$

The various TM_{mnp}^z mode fields are found using (6-70) & (9-46)

9.3.2 cont.

$$E_\rho = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial\rho\,\partial z} \qquad H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}$$

$$E_\phi = \frac{-j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z} \qquad H_\phi = \frac{-1}{\mu}\frac{\partial A_z}{\partial\rho}$$

$$E_z = \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z\right) \qquad H_z = 0 \ (TM^z!)$$

More HW opportunities!

⇒ Obviously, the resonant frequencies are functions of a & h (or h/a).

⇒ For TE^z_{mnp} modes, the lowest resonant frequency occurs for $m=n=p=1$ since $x'_{11} = 1.8412$ is the smallest & $p \neq 0$

$$f^{TE^z}_{r,111} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{1.8412}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2} \qquad (9\text{-}47a)$$

⇒ For TM^z_{mnp} modes, the lowest resonant frequency occurs when $m=p=0$ & $n=1$

$x_{01} = 2.4049$

$$f^{TM^z}_{r,010} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{2.4049}{a}\right)^2} \qquad (9\text{-}47b)$$

⇒ Independent of h!

9.3.2 cont.

So, which of $f_{r,111}^{TE^z}$ and $f_{r,010}^{TM^z}$ is lower and when?

$$\text{Set } f_{r,111}^{TE^z} = f_{r,010}^{TM^z}$$

$$\frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{1.8412}{a}\right)^2+\left(\frac{\pi}{h}\right)^2} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{2.4049}{a}\right)^2}$$

$$\Downarrow$$

$$\frac{h}{a} = 2.03063$$

⇒ For $h/a < 2.03$, the lowest (dominant) mode is the $\underline{TM_{010}^z}$.

⇒ For $h/a > 2.03$, the lowest (dominant) mode is the $\underline{TE_{111}^z}$.

⇒ See Table 9-4 to see how the resonant frequencies for various TE_{mnp}^z & TM_{mnp}^z modes compare to either $f_{r,010}^{TM^z}$ when $h/a < 2.03$ or to $f_{r,111}^{TE^z}$ when $h/a > 2.03$.

TABLE 9-4 Resonant frequencies for the TE_{mnp} and TM_{mnp} modes of a circular cavity

$$R_{dom}^{mnp} = \frac{(f_r)_{mnp}}{(f_r)_{dom}}$$

$\frac{h}{a}$	R_{dom}^{mnp}									
0	$\frac{TM_{010}}{TM_{010}}$	$\frac{TM_{110}}{TM_{010}}$	$\frac{TM_{210}}{TM_{010}}$	$\frac{TM_{020}}{TM_{010}}$	$\frac{TM_{310}}{TM_{010}}$	$\frac{TM_{120}}{TM_{010}}$	$\frac{TM_{410}}{TM_{010}}$	$\frac{TM_{220}}{TM_{010}}$	$\frac{TM_{030}}{TM_{010}}$	$\frac{TM_{510}}{TM_{010}}$
	1.000	1.593	2.136	2.295	2.653	2.917	3.155	3.500	3.598	3.647
0.5	$\frac{TM_{010}}{TM_{010}}$	$\frac{TM_{110}}{TM_{010}}$	$\frac{TM_{210}}{TM_{010}}$	$\frac{TM_{020}}{TM_{010}}$	$\frac{TM_{310}}{TM_{010}}$	$\frac{TE_{111}}{TM_{010}}$	$\frac{TM_{011}}{TM_{010}}$	$\frac{TE_{211}}{TM_{010}}$	$\frac{TM_{120}}{TM_{010}}$	$\frac{TE_{011}}{TM_{010}}$
	1.000	1.593	2.136	2.295	2.653	2.722	2.797	2.905	2.917	3.060
1.00	$\frac{TM_{010}}{TM_{010}}$	$\frac{TE_{111}}{TM_{010}}$	$\frac{TM_{110}}{TM_{010}}$	$\frac{TM_{011}}{TM_{010}}$	$\frac{TE_{211}}{TM_{010}}$	$\frac{TM_{111}}{TM_{010}}$	$\frac{TE_{011}}{TM_{010}}$	$\frac{TM_{210}}{TM_{010}}$	$\frac{TE_{311}}{TM_{010}}$	$\frac{TM_{020}}{TM_{010}}$
	1.000	1.514	1.593	1.645	1.822	2.060	2.060	2.136	2.181	2.295
2.03	$\frac{TM_{010}}{TM_{010}, TE_{111}}$	$\frac{TE_{111}}{TM_{010}, TE_{111}}$	$\frac{TM_{011}}{TM_{010}, TE_{111}}$	$\frac{TE_{211}}{TM_{010}, TE_{111}}$	$\frac{TE_{212}}{TM_{010}, TE_{111}}$	$\frac{TM_{110}}{TM_{010}, TE_{111}}$	$\frac{TM_{012}}{TM_{010}, TE_{111}}$	$\frac{TE_{011}}{TM_{010}, TE_{111}}$	$\frac{TM_{111}}{TM_{010}, TE_{111}}$	$\frac{TE_{212}}{TM_{010}, TE_{111}}$
	1.000	1.000	1.189	1.424	1.497	1.593	1.630	1.718	1.718	1.808
3.0	$\frac{TE_{111}}{TE_{111}}$	$\frac{TM_{010}}{TE_{111}}$	$\frac{TM_{011}}{TE_{111}}$	$\frac{TE_{112}}{TE_{111}}$	$\frac{TM_{012}}{TE_{111}}$	$\frac{TE_{211}}{TE_{111}}$	$\frac{TE_{113}}{TE_{111}}$	$\frac{TE_{212}}{TE_{111}}$	$\frac{TM_{110}}{TE_{111}}$	$\frac{TM_{013}}{TE_{111}}$
	1.000	1.136	1.238	1.317	1.506	1.524	1.719	1.748	1.809	1.868
4.0	$\frac{TE_{111}}{TE_{111}}$	$\frac{TM_{010}}{TE_{111}}$	$\frac{TE_{112}}{TE_{111}}$	$\frac{TM_{011}}{TE_{111}}$	$\frac{TM_{012}}{TE_{111}}$	$\frac{TE_{113}}{TE_{111}}$	$\frac{TE_{211}}{TE_{111}}$	$\frac{TM_{013}}{TE_{111}}$	$\frac{TE_{212}}{TE_{111}}$	$\frac{TE_{114}}{TE_{111}}$
	1.000	1.202	1.209	1.264	1.435	1.494	1.575	1.682	1.717	1.819
5.0	$\frac{TE_{111}}{TE_{111}}$	$\frac{TE_{112}}{TE_{111}}$	$\frac{TM_{010}}{TE_{111}}$	$\frac{TM_{011}}{TE_{111}}$	$\frac{TE_{113}}{TE_{111}}$	$\frac{TM_{012}}{TE_{111}}$	$\frac{TM_{013}}{TE_{111}}$	$\frac{TE_{114}}{TE_{111}}$	$\frac{TE_{211}}{TE_{111}}$	$\frac{TE_{212}}{TE_{111}}$
	1.000	1.146	1.236	1.278	1.354	1.395	1.571	1.602	1.603	1.698
10.0	$\frac{TE_{111}}{TE_{111}}$	$\frac{TE_{112}}{TE_{111}}$	$\frac{TE_{113}}{TE_{111}}$	$\frac{TE_{114}}{TE_{111}}$	$\frac{TM_{010}}{TE_{111}}$	$\frac{TE_{115}}{TE_{111}}$	$\frac{TM_{011}}{TE_{111}}$	$\frac{TM_{012}}{TE_{111}}$	$\frac{TM_{013}}{TE_{111}}$	$\frac{TE_{116}}{TE_{111}}$
	1.000	1.042	1.107	1.194	1.288	1.296	1.299	1.331	1.383	1.411
∞	$\frac{TE_{11p}}{TE_{111}}$	$\frac{TM_{01p}}{TE_{111}}$	$\frac{TE_{21p}}{TE_{111}}$	$\frac{TE_{01p}}{TE_{111}}$	$\frac{TM_{11p}}{TE_{111}}$	$\frac{TE_{31p}}{TE_{111}}$	$\frac{TM_{21p}}{TE_{111}}$	$\frac{TE_{41p}}{TE_{111}}$	$\frac{TE_{12p}}{TE_{111}}$	$\frac{TM_{02p}}{TE_{111}}$
	1.000	1.306	1.659	2.081	2.081	2.282	2.790	2.888	2.896	2.998

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

9.3.2 cont.

ex. For an air-filled circular cavity of radius $a = 10\text{cm}$, find the resonant frequencies of the TE_{111} & TM_{010} modes when:

a) $h = 15\text{cm}$ $\left(h/a = \frac{15}{10} = 1.5 \right)$

$$f_{r,010}^{TM^z} = \frac{2.9979 \times 10^8}{2\pi}\left(\frac{2.4049}{0.1}\right) \Rightarrow \underline{\underline{f_{r,010}^{TM^z} = 1.14745 \text{ GHz}}}$$

(lower)

$$f_{r,111}^{TE^z} = \frac{2.9979 \times 10^8}{2\pi}\sqrt{\left(\frac{1.8412}{0.1}\right)^2 + \left(\frac{\pi}{0.15}\right)^2} \Rightarrow \underline{\underline{f_{r,111}^{TE^z} = 1.3305 \text{ GHz}}}$$

b) $h = 20.306\text{cm}$ $\left(h/a = \frac{20.306}{10} = 2.0306 \right)$

$$f_{r,010}^{TM^z} = \frac{2.9979 \times 10^8}{2\pi}\left(\frac{2.4049}{0.1}\right) \Rightarrow \underline{\underline{f_{r,010}^{TM^z} = 1.14745 \text{ GHz}}}$$

No change

$$f_{r,111}^{TE^z} = \frac{2.9979 \times 10^8}{2\pi}\sqrt{\left(\frac{1.8412}{0.1}\right)^2 + \left(\frac{\pi}{0.20306}\right)^2} \Rightarrow \underline{\underline{f_{r,111}^{TE^z} = 1.14746 \text{ GHz}}}$$

essentially equal

c) $h = 30\text{cm}$ $\left(h/a = \frac{30}{10} = 3 \right)$

$$f_{r,010}^{TM^z} = \frac{2.9979 \times 10^8}{2\pi}\left(\frac{2.4049}{0.1}\right) \Rightarrow \underline{\underline{f_{r,010}^{TM^z} = 1.14745 \text{ GHz}}}$$

No change

$$f_{r,111}^{TE^z} = \frac{2.9979 \times 10^8}{2\pi}\sqrt{\left(\frac{1.8412}{0.1}\right)^2 + \left(\frac{\pi}{0.3}\right)^2} \Rightarrow \underline{\underline{f_{r,111}^{TE^z} = 1.0106 \text{ GHz}}}$$

(lower)

9.3.3 Quality Factor Q

For the TM^z_{010} mode (dominant when $h/a < 2.03$), the magnetic potential function reduces to

$$A_z = B_{010} J_0(\beta_\rho \rho) \qquad (9\text{-}49)$$

$$\text{where } \beta_\rho = \frac{\chi_{01}}{a} = \frac{2.4049}{a} \qquad (9\text{-}49a)$$

From (6-70), the corresponding fields are

$$E_\rho = E_\phi = H_z = H_\rho = 0$$

$$E_z = -j \frac{\beta_r^2}{\omega_r \mu \epsilon} B_{010} J_0\left(\frac{\chi_{01}}{a}\rho\right) \qquad (9\text{-}50)$$

$$H_\phi = -\frac{\chi_{01}}{a} \frac{B_{010}}{\mu} J_0'\left(\frac{\chi_{01}}{a}\rho\right)$$

As previously mentioned, $W_m = W_E$. Therefore, the total energy stored in the cavity is

$$W = W_E + W_m = 2W_E = \frac{\epsilon}{2} \iiint_V |\bar{E}|^2 dv \leftarrow dv = \rho\, d\rho\, d\phi\, dz$$

$$= (1)\left(\frac{\beta_r^2}{\omega_r \mu \epsilon}\right)^2 |B_{010}|^2 \frac{\epsilon}{2} \int_{z=0}^{h} dz \int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{a} J_0^2\left(\frac{\chi_{01}}{a}\rho\right) \rho\, d\rho$$

$$= |B_{010}|^2 \pi h \epsilon \left(\frac{\beta_r^2}{\omega_r \mu \epsilon}\right)^2 \int_{\rho=0}^{a} J_0^2\left(\frac{\chi_{01}}{a}\rho\right) \rho\, d\rho$$

9.3.3 cont.

A worthy math reference (Schaum's Mathematical Handbook) gives:

$$\int_{\rho=0}^{a} \rho J_0^2\left(\frac{x_{01}}{a}\rho\right) d\rho = \frac{a^2}{2} J_1^2(x_{01})$$

leading to

$$W = |B_{010}|^2 \frac{\pi h \epsilon}{2}\left(\frac{a\beta_r^2}{\omega_r \mu \epsilon}\right)^2 J_1^2(x_{01}) \qquad (9\text{-}53)$$

Assuming no dielectric losses, the total power dissipated by the cavity occurs due to conduction losses on the walls

$$P_d = \frac{R_s}{2} \oint\!\!\oint_{A\,(\text{walls})} |H|^2 ds$$

Noting H_ϕ does NOT depend on z

$$P_d = \frac{R_s}{2}\left[\iint_{\substack{\text{side}\\ \rho=a}} |H_\phi|^2 a\, d\phi\, dz + 2\iint_{\substack{\text{bottom}\\ z=0}} |H_\phi|^2 \rho\, d\rho\, d\phi \right]$$

where

$$|H_\phi|^2 = |B_{010}|^2 \left(\frac{x_{01}}{a}\right)^2 \frac{1}{\mu^2}\left[J_0'\left(\frac{x_{01}}{a}\rho\right)\right]^2$$

$$P_d = \frac{R_s}{2}|B_{010}|^2\left(\frac{x_{01}}{a}\right)^2\frac{1}{\mu^2}\left[\int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{h} dz \left(a\left[J_0'(x_{01})\right]^2\right) + 2\int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{a} \rho\left[J_0'\left(\frac{x_{01}}{a}\rho\right)\right]^2 d\rho\right]$$

9.3.3 cont.

use $J_0'\left(\frac{x_{01}}{a}\rho\right) = -J_1\left(\frac{x_{01}}{a}\rho\right)$

to get, @ $\rho = a$, $J_0'(x_{01}) = -J_1(x_{01})$ and

$$P_d = |B_{010}|^2 \frac{\pi R_s}{\mu^2}\left(\frac{x_{01}}{a}\right)^2 \left\{ a h J_1^2(x_{01}) + \underbrace{2\int_{\rho=0}^{a} \rho J_1^2\left(\frac{x_{01}}{a}\rho\right) d\rho}_{\frac{a^2}{2} J_1^2(x_{01})} \right\}$$

$$P_d = |B_{010}|^2 \frac{\pi R_s}{\mu^2}\left(\frac{x_{01}}{a}\right)^2 a(h+a) J_1^2(x_{01}) \quad (9\text{-}56)$$

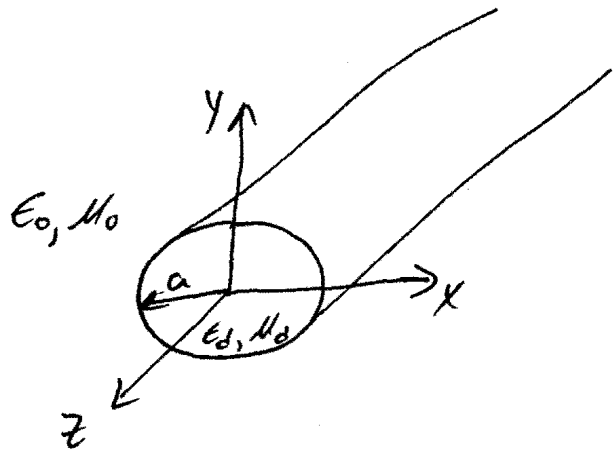
$$Q = \frac{\omega_r W}{P_d} \Rightarrow \boxed{\begin{aligned} Q_{010} &= \frac{\beta_r^4 h a^3}{2\omega_r \epsilon (h+a) R_s x_{01}^2} \\ &= \frac{x_{01}\sqrt{\mu/\epsilon}}{2(1+a/h)R_s} = \frac{1.2025\,\eta}{R_s(1+a/h)} \quad (9\text{-}57) \end{aligned}}$$

Note: TM_{010} $\beta_\rho = \frac{x_{01}}{a} = \beta_r$, $\beta_z = 0$

9.5 Dielectric Waveguides and Resonators

Big example – fiber optic cables

9.5.1 Circular Dielectric Waveguide



Typically, $\epsilon_d > \epsilon_0$ and $\mu_d = \mu_0$.

For the analysis, we will assume perfect dielectric (no losses).

⇒ Generally, the fields will be TE or TM. However, pure TE(H) or TM(E) modes only when fields are symmetric and <u>independent of ϕ</u> [i.e., not even periodic w/ ϕ as is $\cos(m\phi)$ or $\sin(m\phi)$].

⇒ Otherwise, we have combinations of TE(H) and TM(E) modes that can be non-symmetric and vary w/ ϕ

$$\Downarrow$$

<u>Hybrid Modes</u> (HEM_{mn}). Other nomenclature is HE (when TE(H) predominate) or EH (when TM(E)...)

9.5.1 cont.

Most of these modes have cutoff frequencies, determined by a/λ. However, there is one mode, HEM_{11} (AKA: HE_{11}) that does not have a cutoff frequency.

⇒ HEM_{11} or HE_{11} is the dominant mode, sometimes called the dipole mode.

⇒ Use prior work on cylindrical waveguides (i.e, metal pipes) for TE^z_{mn} & TM^z_{mn} modes and fields. TE^z_{mn} fields are listed in equations (9-15a) – (9-15g). TM^z_{mn} fields are listed in equations (9-31a) – (9-31g).

⇒ Keep terms w/ $\cos(m\phi)$ variation for E_z and $\sin(m\phi)$ variation for H_z. This leads to the HEM modes.

⇒ Assume $+z$-direction for wave propagation.

9.5.1 cont.

HEM Modes ($\rho \le a$) (9-81a) – (9-81f)

$$E_\rho^d = \frac{-j}{(\beta_\rho^d)^2}\left[m\omega\mu_d A_m \frac{1}{\rho} J_m(\beta_\rho^d \rho) + \beta_z \beta_\rho^d B_m J_m'(\beta_\rho^d \rho)\right]\cos m\phi \, e^{-j\beta_z z}$$

$$E_\phi^d = \frac{j}{(\beta_\rho^d)^2}\left[\omega\mu_d \beta_\rho^d A_m J_m'(\beta_\rho^d \rho) + m\beta_z B_m \frac{1}{\rho} J_m(\beta_\rho^d \rho)\right]\sin m\phi \, e^{-j\beta_z z}$$

$$E_z^d = B_m J_m(\beta_\rho^d \rho)\cos m\phi \, e^{-j\beta_z z}$$

$$H_\rho^d = \frac{-j}{(\beta_\rho^d)^2}\left[\beta_z \beta_\rho^d A_m J_m'(\beta_\rho^d \rho) + m\omega\epsilon_d B_m \frac{1}{\rho} J_m(\beta_\rho^d \rho)\right]\sin m\phi \, e^{-j\beta_z z}$$

$$H_\phi^d = \frac{-j}{(\beta_\rho^d)^2}\left[m\beta_z A_m \frac{1}{\rho} J_m(\beta_\rho^d \rho) + \omega\epsilon_d \beta_\rho^d B_m J_m'(\beta_\rho^d \rho)\right]\cos m\phi \, e^{-j\beta_z z}$$

$$H_z^d = \frac{-j}{(\beta_\rho^d)^2} J_m(\beta_\rho^d \rho)\sin m\phi \, e^{-j\beta_z z}$$

where $(\beta_\rho^d)^2 + \beta_z^2 = \beta_d^2 = \omega^2\mu_d\epsilon_d$ (9-81g)

$$A_m = -j\frac{(\beta_\rho^d)^2}{\omega\mu_d\epsilon_d} A_{mn} D_2 \qquad (9\text{-}81h)$$

$$B_m = -j\frac{(\beta_\rho^d)^2}{\omega\mu_d\epsilon_d} B_{mn} C_2 \qquad (9\text{-}81i)$$

$$J_m' = \frac{\partial(J_m(\beta_\rho^d \rho))}{\partial(\beta_\rho^d \rho)} \qquad (9\text{-}81j)$$

9.5.1 cont.

Notes: ⇒ 'Hybrid' modes ⇒ E_z^d & H_z^d are non-zero!

⇒ A_m & B_m are not independent, find relation by applying boundary conditions

For a dielectric waveguide, E & H outside the rod, i.e., $\rho \geq a$, should be evanescent & decay in ρ direction

$a \downarrow$ (gets smaller) ⇒ loosely bound

→ less attenuation

→ fields extend out farther from dielectric

→ β only a bit more than β_0

$a \uparrow$ (gets bigger) ⇒ tightly bound

→ more attenuation

→ fields stay closer to dielectric

→ $\beta_z \to \beta_d$

For both, $\beta_z > \beta_0$ and $v_p > v = c$

9.5.1 cont.

What about the fields outside the dielectric waveguide?

$\Rightarrow$ Use functions that decay in ρ direction, e.g.,

(1) Hankel functions of order m, first kind $H_m^{(1)}$
or, second kind $H_m^{(2)}$

with imaginary arguments

or

(2) modified Bessel functions of order m, second kind

$$K_m(\alpha) = \begin{cases} j^{m+1} \frac{\pi}{2} H_m^{(1)}(j\alpha) & (9\text{-}82a) \\ -j^{m+1} \frac{\pi}{2} H_m^{(2)}(-j\alpha) & (9\text{-}82b) \end{cases}$$

Adapt (9-81a) – (9-81j) 'ρ terms', to get field components for HEM modes outside the dielectric waveguide, i.e., $\rho \geq a$.

Note that α_ρ^o replaces β_ρ^d in earlier field expressions

9.5.1 cont.

HEM Modes ($\rho \geq a$) (9-83a) – (9-83f)

$$E_\rho^0 = \frac{j}{(\alpha_\rho^0)^2}\left[m\omega\mu_0 C_m \frac{1}{\rho} K_m(\alpha_\rho^0 \rho) + \beta_z \alpha_\rho^0 D_m K_m'(\alpha_\rho^0 \rho)\right]\cos m\phi \, e^{-j\beta_z z}$$

$$E_\phi^0 = \frac{-j}{(\alpha_\rho^0)^2}\left[\omega\mu_0 \alpha_\rho^0 C_m K_m'(\alpha_\rho^0 \rho) + m\beta_z D_m \frac{1}{\rho} K_m(\alpha_\rho^0 \rho)\right]\sin m\phi \, e^{-j\beta_z z}$$

$$E_z^0 = D_m K_m(\alpha_\rho^0 \rho)\cos m\phi \, e^{-j\beta_z z}$$

$$H_\rho^0 = \frac{j}{(\alpha_\rho^0)^2}\left[\beta_z \alpha_\rho^0 C_m K_m'(\alpha_\rho^0 \rho) + m\omega\epsilon_0 D_m \frac{1}{\rho} K_m(\alpha_\rho^0 \rho)\right]\sin m\phi \, e^{-j\beta_z z}$$

$$H_\phi^0 = \frac{j}{(\alpha_\rho^0)^2}\left[m\beta_z C_m \frac{1}{\rho} K_m(\alpha_\rho^0 \rho) + \omega\epsilon_0 \alpha_\rho^0 D_m K_m'(\alpha_\rho^0 \rho)\right]\cos m\phi \, e^{-j\beta_z z}$$

$$H_z^0 = C_m K_m(\alpha_\rho^0 \rho)\sin(m\phi) \, e^{-j\beta_z z}$$

where $(j\alpha_\rho^0)^2 + \beta_z^2 = -(\alpha_\rho^0)^2 + \beta_z^2 = \beta_0^2 = \omega^2\mu_0\epsilon_0$ (9-83g)

and $$K_m'(\alpha_\rho^0 \rho) = \frac{\partial K_m(\alpha_\rho^0 \rho)}{\partial(\alpha_\rho^0 \rho)} \qquad (9\text{-}83h)$$

$\Rightarrow$ C_m & D_m are NOT independent of each other or from A_m & B_m, relations found by applying boundary conditions

9.5.1 cont.

Apply the tangential $\bar{E}$ + $\bar{H}$ boundary conditions

@ $\rho = a$

$$E_\phi^d = E_\phi^0 \qquad (9\text{-}84a)$$

$$E_z^d = E_z^0 \qquad (9\text{-}84b)$$

$$H_\phi^d = H_\phi^0 \qquad (9\text{-}84c)$$

$$H_z^d = H_z^0 \qquad (9\text{-}84d)$$

These lead to (9-85a) - (9-85d)

$$\frac{1}{(\beta_\rho^d)^2}\left[\omega\mu_d\beta_\rho^d A_m J_m'(\beta_\rho^d a) + m\beta_z B_m \frac{1}{a} J_m(\beta_\rho^d a)\right]$$

$$= \frac{-1}{(\alpha_\rho^0)^2}\left[\omega\mu_0\alpha_\rho^0 C_m K_m'(\alpha_\rho^0 a) + m\beta_z D_m \frac{1}{a} K_m(\alpha_\rho^0 a)\right],$$

$$B_m J_m(\beta_\rho a) = D_m K_m(\alpha_\rho^0 a),$$

$$\frac{-1}{(\beta_\rho^d)^2}\left[m\beta_z A_m \frac{1}{a} J_m(\beta_\rho^d a) + \omega\epsilon_d\beta_\rho^d B_m J_m'(\beta_\rho a)\right]$$

$$= \frac{1}{(\alpha_\rho^0)^2}\left[m\beta_z C_m \frac{1}{a} K_m(\alpha_\rho^0 a) + \omega\epsilon_0\alpha_\rho^0 D_m K_m'(\alpha_\rho^0 a)\right],$$

and,

$$A_m J_m(\beta_\rho^d a) = C_m K_m(\alpha_\rho^0 a)$$

9.5.1 cont.

Multiplying our constraint/dispersion equations (9-81g)) & (9-83g) by a^2 leads to

$$(\beta_\rho^d a)^2 + (\beta_z a)^2 = (\beta_d a)^2 \quad \Rightarrow \quad \beta_z a = \sqrt{(\beta_d a)^2 - (\beta_\rho^d a)^2} \tag{9-85e}$$

$$-(\alpha_\rho^o a)^2 + (\beta_z a)^2 = (\beta_o a)^2 \tag{9-85f}$$

If we subtract (9-85f) from (9-85e), we get-

$$\begin{aligned} &(\beta_\rho^d a)^2 + (\alpha_\rho^0 a)^2 = (\beta_d a)^2 - (\beta_0 a)^2 \\ &\Rightarrow \quad \alpha_\rho^0 a = \sqrt{(\beta_d a)^2 - (\beta_0 a)^2 - (\beta_\rho^d a)^2} = \sqrt{(\beta_0 a)^2(\varepsilon_r \mu_r - 1) - (\beta_\rho^d a)^2} \end{aligned} \tag{9-86}$$

Now, we will define some new variables to shorten the subsequent equations-

(lowercase chi) $\chi = \beta_\rho^d a$ (9-87a)

(lowercase xi) $\xi = \alpha_\rho^o a$ (9-87b)

(lowercase zeta) $\varsigma = \beta_z a$ (9-87c)

$$\beta_d = \beta_0 \sqrt{\varepsilon_r \mu_r} \tag{9-87d}$$

Using these new variables we can rewrite (9-85e) & (9-86) as-

$$\varsigma = \sqrt{(\beta_0 a)^2 \varepsilon_r \mu_r - \chi^2} \tag{9-88a}$$

$$\xi = \sqrt{(\beta_0 a)^2 (\varepsilon_r \mu_r - 1) - \chi^2} \tag{9-88b}$$

Assuming the dielectric waveguide is non-magnetic, we can put the equations (9-85a) - (9-85d) in matrix form $[F][g] = 0$ where (9-89a) & (9-89b) are-

$$[F] = \begin{bmatrix} \frac{\omega \mu_0 a}{\chi} J_m'(\chi) & \frac{m\varsigma}{\chi^2} J_m(\chi) & \frac{\omega \mu_0 a}{\xi} K_m'(\xi) & \frac{m\varsigma}{\xi^2} K_m(\xi) \\ 0 & J_m(\chi) & 0 & -K_m(\xi) \\ \frac{m\varsigma}{\chi^2} J_m(\chi) & \frac{\omega \varepsilon_d a}{\chi} J_m'(\chi) & \frac{m\varsigma}{\xi^2} K_m(\xi) & \frac{\omega \varepsilon_0 a}{\xi} K_m'(\xi) \\ J_m(\chi) & 0 & -K_m(\xi) & 0 \end{bmatrix} \quad \text{and} \quad [g] = \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}.$$

9.5.1 cont.

For a non-trivial solution, the determinant of (9-89a) must be zero, i.e.,

$$\begin{aligned}|F| &= \frac{\omega^2\mu_0\varepsilon_d a^2}{\chi^2}[J_m{}'(\chi)]^2[K_m(\xi)]^2 + \frac{\omega^2\mu_0\varepsilon_0 a^2}{\chi\xi}J_m(\chi)J_m{}'(\chi)K_m(\xi)K_m{}'(\xi) \\ &-\frac{(m\varsigma)^2}{\chi^4}[J_m(\chi)]^2[K_m(\xi)]^2 - \frac{(m\varsigma)^2}{\chi^2\xi^2}[J_m(\chi)]^2[K_m(\xi)]^2 \\ &+\frac{\omega^2\mu_0\varepsilon_d a^2}{\chi\xi}J_m(\chi)J_m{}'(\chi)K_m(\xi)K_m{}'(\xi) + \frac{\omega^2\mu_0\varepsilon_0 a^2}{\xi^2}[J_m{}'(\chi)]^2[K_m{}'(\xi)]^2 \\ &-\frac{(m\varsigma)^2}{\chi^2\xi^2}[J_m(\chi)]^2[K_m(\xi)]^2 - \frac{(m\varsigma)^2}{\xi^4}[J_m(\chi)]^2[K_m(\xi)]^2 = 0\end{aligned} \tag{9-90}$$

Dividing (9-90) by $\omega^2\mu_0\varepsilon_d a^2[K_m(\xi)]^2$ and doing some algebra leads to the eigenvalue equation for the dielectric waveguide-

$$G_1(\chi)\,G_2(\chi) - G_3^2(\chi) = 0 \tag{9-91}$$

where

$$G_1(\chi) = \frac{J_m{}'(\chi)}{\chi} + \frac{K_m{}'(\xi)J_m(\chi)}{\varepsilon_r\,\xi\,K_m(\xi)} \tag{9-91ab}$$

$$G_2(\chi) = \frac{J_m{}'(\chi)}{\chi} + \frac{K_m{}'(\xi)J_m(\chi)}{\xi\,K_m(\xi)} \tag{9-91b}$$

$$G_3(\chi) = \frac{m\varsigma}{\beta_0 a\sqrt{\varepsilon_r}}J_m(\chi)\left(\frac{1}{\chi^2} + \frac{1}{\xi^2}\right) \tag{9-91c}$$

$$\varsigma = \sqrt{(\beta_0 a)^2\varepsilon_r - \chi^2} \tag{9-91d}$$

$$\xi = \sqrt{(\beta_0 a)^2(\varepsilon_r - 1) - \chi^2} \tag{9-91e}$$

To keep $\xi = \alpha_\rho^o a$ real, the value of $\chi = \beta_\rho^d a$ needs to be less than-

$$\chi_{\max} = \left(\beta_\rho^d a\right)_{\max} = \beta_0\,a\sqrt{\varepsilon_r - 1} = \omega\,a\sqrt{\mu_0\varepsilon_0(\varepsilon_r - 1)}\,. \tag{9-92}$$

If $\chi = \beta_\rho^d a > \chi_{\max}$, then $\xi = \alpha_\rho^o a$ becomes imaginary and our modified Bessel function of the second kind K_m from (9-82) becomes a Hankel function of the second kind $H_m^{(2)}$ and give **unattenuated** outward traveling waves in the ρ direction, i.e., our dielectric waveguide becomes an antenna!

9.5.1 cont.

⇒ For a given value of m, there are a finite number n of χ_{mn} (eigenvalues) where our dielectric rod acts as a waveguide. Each combo of $m, n,$ + χ_{mn} leads to a hybrid mode HEM_{mn}.

⇒ If n is odd, $HEM_{mn} \Rightarrow HE_{mn}$ mode

⇒ If n is even, $HEM_{mn} \Rightarrow EH_{mn}$ mode

⇒ Also, if $\chi > \beta_0 a \sqrt{\epsilon_r}$, then $\xi = \beta_z a$ becomes imaginary ⇒ evanescent in the z-direction (not propagating :( )

How do we get χ_{mn}? Need to solve (9-91), a transcendental equation.

For example, if $m = 0$, we get

TM$_{0n}$ modes (9-93a)

$$G_1(\chi_{0n}) = \frac{J_0'(\chi_{0n})}{\chi_{0n}} + \frac{K_0'(\xi_{0n}) J_0(\chi_{0n})}{\epsilon_r \xi_{0n} K_0(\xi_{0n})} = \frac{-J_1(\chi_{0n})}{\chi_{0n}} - \frac{K_1(\xi_{0n}) J_0(\chi_{0n})}{\epsilon_r \xi_{0n} K_0(\xi_{0n})} = 0$$

and

9.5.1 cont.

TE_{on} modes (9-93b)

$$G_2(X_{on}) = \frac{J_0'(X_{on})}{X_{on}} + \frac{K_0'(\xi_{on})J_0(X_{on})}{\xi_{on}K_0(\xi_{on})} = \frac{-J_1(X_{on})}{X_{on}} - \frac{K_1(\xi_{on})J_0(X_{on})}{\xi_{on}K_0(\xi_{on})} = 0$$

Since $G_3(X_{on}) = 0$

Otherwise, these equations need to be solved iteratively. Figures 9-9 & 9-10 show solutions to X_{mn} as a function of $\beta_0 a$ for many TE_{on}, TM_{on}, & HEM_{mn} modes when $\epsilon_r = 20$ and $\epsilon_r = 38$. Obviously X_{mn} is NOT constant for a given mode, unlike our metal circular waveguides

* Once X_{mn} is found, then (9-91d) can be used to determine $\xi = \beta_z a$.

* Figure 9-11 shows β_z/β_0 vs $2a/\lambda_0$ for the HE_{11}, TE_{01}, & TM_{01} modes for polystyrene ($\epsilon_r = 2.56$). Note, HE_{11} does NOT have a cutoff! However $\beta_z/\beta_0 \to 1$ as $a\downarrow$ which implies fields are loosely bound (bigger ϵ_r helps).

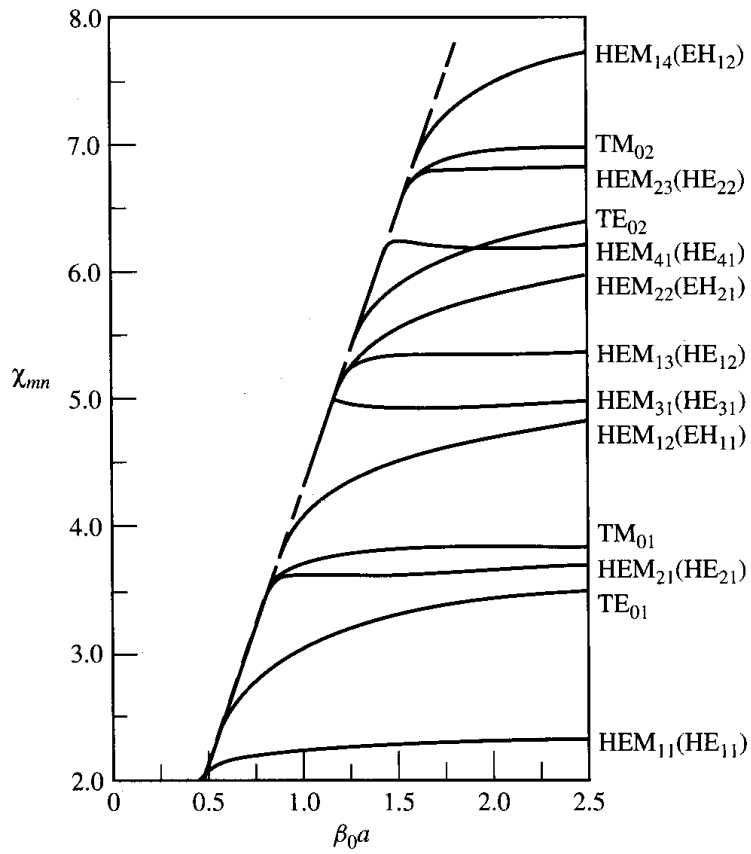


Figure 9-9 The first 13 eigenvalues of the dielectric rod waveguide ($\varepsilon_r = 20$). (Source: D. Kajfez and P. Guillon (Eds.), *Dielectric Resonators*, 1986, Artech House, Inc.).

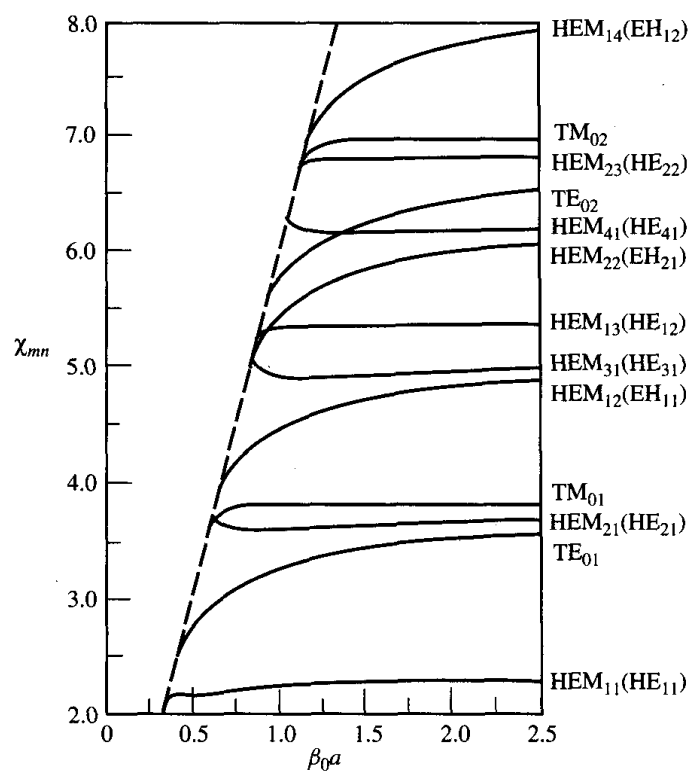


Figure 9-10 The first 13 eigenvalues of the dielectric rod waveguide ($\varepsilon_r = 38$). (Source: D. Kajfez and P. Guillon (Eds.), *Dielectric Resonators*, 1986, Artech House, Inc.).

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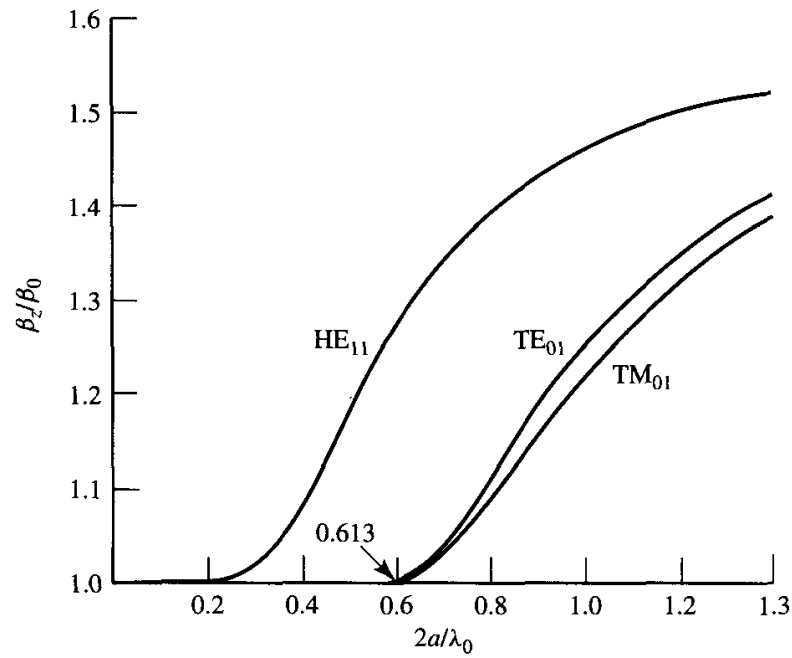

Figure 9-11 Ratio of β_z/β_0, for first three surface-wave modes on a polystyrene rod ($\varepsilon_r = 2.56$). (Source: R. E. Collin, *Field Theory of Guided Waves*, 1960, McGraw-Hill Book Co.).

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

Note that the TM_{01} mode stops propagating, i.e., $\beta_z/\beta_0 = 1$, at around the value-

$$\frac{2a}{\lambda_0} = \frac{1}{\pi}\frac{2.4049}{\sqrt{\varepsilon_r - 1}} = \frac{1}{\pi}\frac{2.4049}{\sqrt{2.56-1}} = 0.613$$

in Figure 9-11, more on this later.

Next, **<u>Figure 9-12</u>** shows some typical field patterns for the HE_{11}, TE_{01}, TM_{01}, HE_{21}, EH_{11}, and HE_{31} modes. For all plots, a ratio

$$\tau^2 = \chi^2/(\chi^2+\xi^2) = (\beta_\rho^d)^2/\left[(\beta_\rho^d)^2 + (\alpha_\rho^0 a)^2\right] = 0.1$$

was selected.

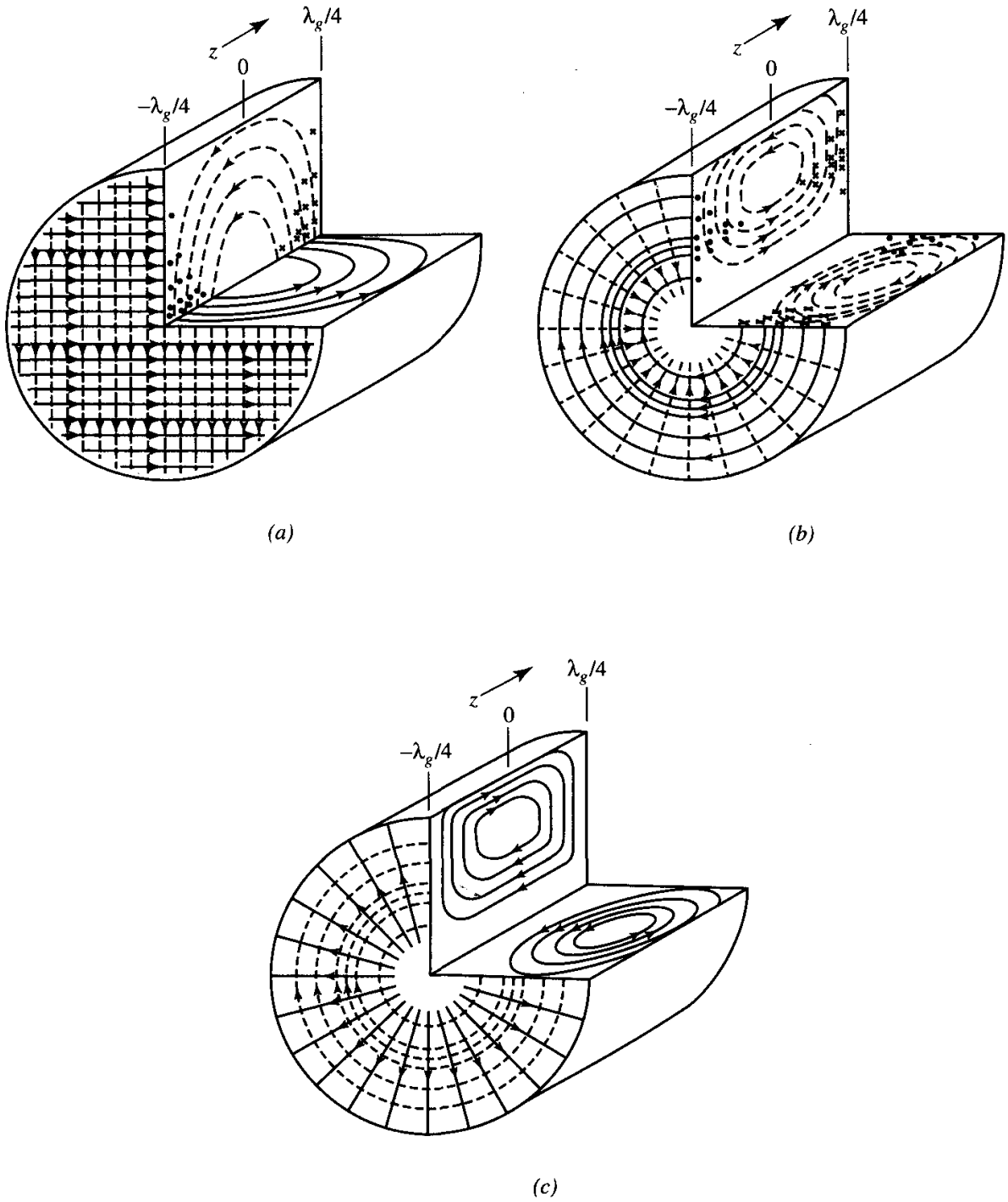

Figure 9-12 Field patterns in the central core of a dielectric rod waveguide (in all cases $\tau = 0.1$; **E**:——, **H**: ----). (Source: T. Okoshi, *Optical Fibers*, 1982, Academic Press.). (*a*) HEM_{11} (HE_{11}) mode. (*b*) TE_{01} mode. (*c*) TM_{01} mode.

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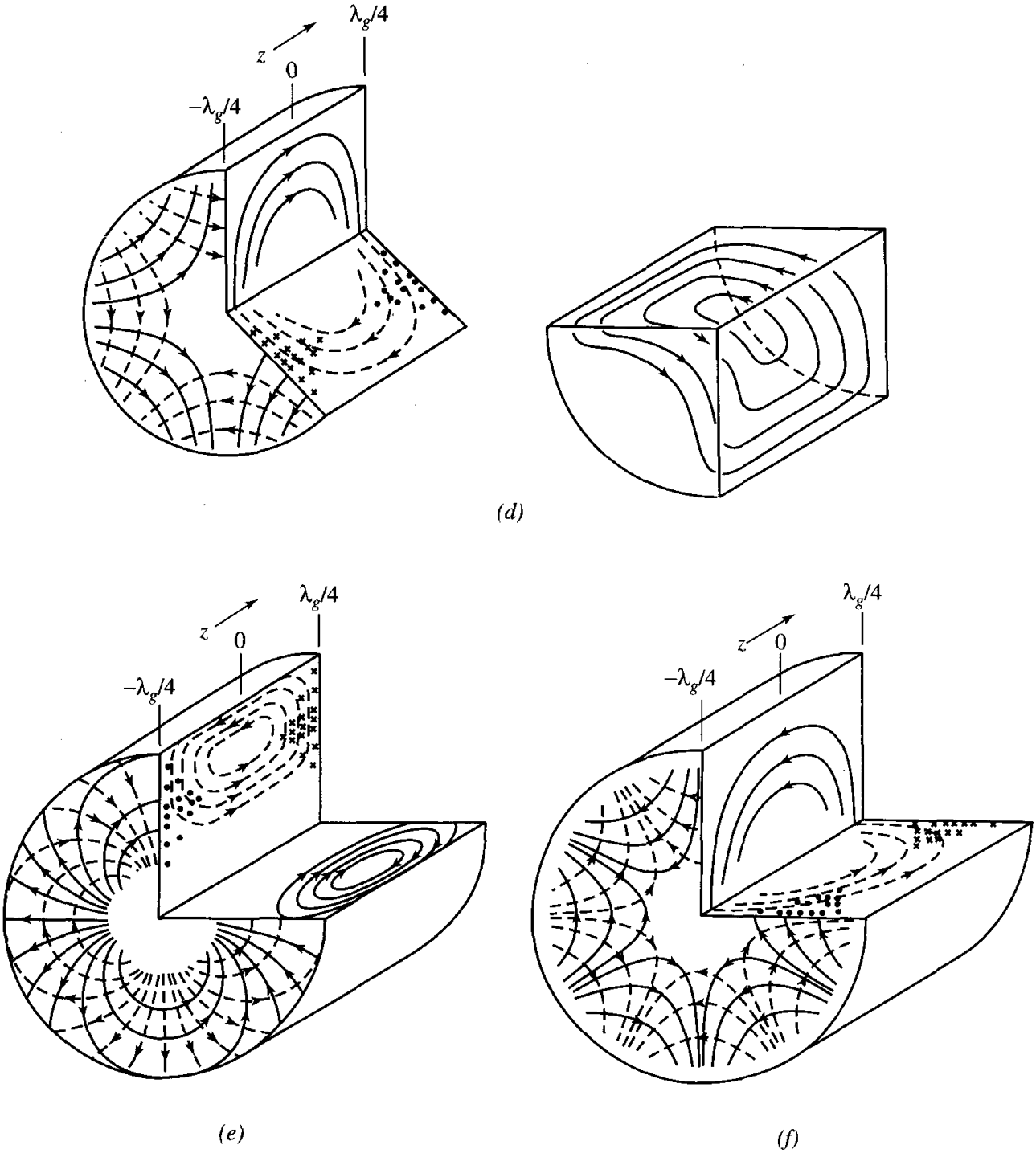

Figure 9-12 (*d*) HEM_{21} (HE_{21}) mode. (*e*) HEM_{12} (EH_{11}) mode. (*f*) HEM_{31} (HE_{31}) mode. (*Continued*)

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9.5.1 cont.

For all circular dielectric waveguide modes, the sum $\chi^2 + \xi^2 = (\beta_0 a)^2(\epsilon_r - 1) = \text{constant}$ (9-94)

To keep dominant HE_{11} mode well separated, choose

normalized core radius $\rightarrow$ $V^2 = \chi^2 + \xi^2 = (\beta_0 a)^2(\epsilon_r - 1) \geq (2.4049)^2 = 5.7835$ (9-94a)

or $$\frac{2a}{\lambda_0} \geq \frac{1}{\pi}\frac{2.4049}{\sqrt{\epsilon_r - 1}}. \quad (9\text{-}94b)$$

why? $\chi_{mn} = \chi_{01} = 2.4049$ for the TM_{01} mode. So, values of $2a/\lambda_0$ below this value ensure that the HE_{11} (dominant/dipole) mode is the <u>only</u> mode.

$\Rightarrow$ <u>Table</u> on following page gives the first 20 modes in order of ascending cutoff frequency along w/ Bessel function(s) that go to zero.

$\Rightarrow$ <u>Figure 9-13</u> on following page shows how far, normalized diameter, the e^{-1} point extends outside waveguide for HE_{11} (sweet spot).

Mode(s)	$J_m(\chi_{mn}) = 0$	χ_{mn} at cutoff	Total number of propagating modes
HE_{11} (HEM_{11})	$J_1(\chi_{10}) = 0$	$\chi_{10} = 0$	2
TE_{01}, TM_{01}, HE_{21} (HEM_{21})	$J_0(\chi_{01}) = 0$	$\chi_{01} = 2.4049$	6
HE_{12} (HEM_{13}), EH_{11} (HEM_{12}), HE_{31} (HEM_{31})	$J_1(\chi_{11}) = 0$	$\chi_{11} = 3.8318$	12
EH_{21} (HEM_{22}), HE_{41} (HEM_{41})	$J_2(\chi_{21}) = 0$	$\chi_{21} = 5.1357$	16
TE_{02}, TM_{02}, HE_{22} (HEM_{23})	$J_0(\chi_{02}) = 0$	$\chi_{02} = 5.5201$	20

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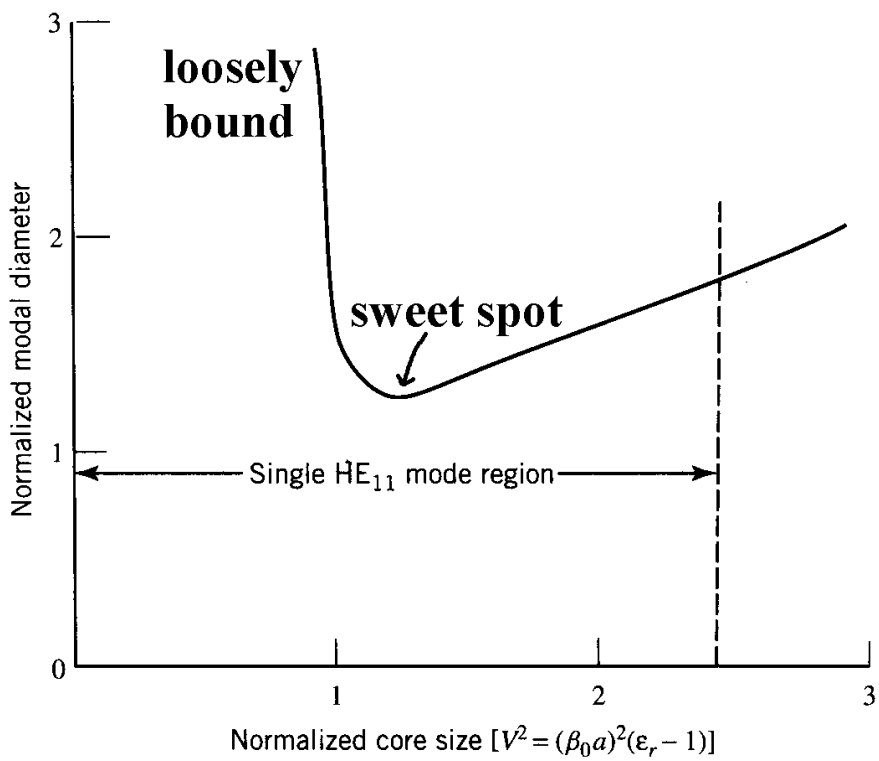


Figure 9-13 Normalized modal diameter as a function of normalized core size for dielectric rod waveguide. (Source: T. G. Giallorenzi, "Optical communications research and technology: Fiber optics," *Proc. IEEE*, © 1978, IEEE.).

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

9.5.2 Circular Dielectric Resonator

→ can have many shapes, e.g., spheres, disks, ...

→ Usually a ceramic w/ high ϵ_r and low $\tan\delta$ to achieve high Q

→ Theorized & studied by Richtmyer ~1939, but not much done with them for many years due to lack of suitable materials

→ Started gaining interest again in late 60's and into 70's w/ development of better materials, e.g., barium tetratitanate & $(Zr\text{-}Sn)TiO_4$

→ Advantages include relatively compact size versus waveguide filters and cost less when used in some oscillators (dielectric resonator oscillators / DROs) @ mm frequencies

→ Typically want $\epsilon_r > 30$ to get good energy confinement.

For a planar interface,

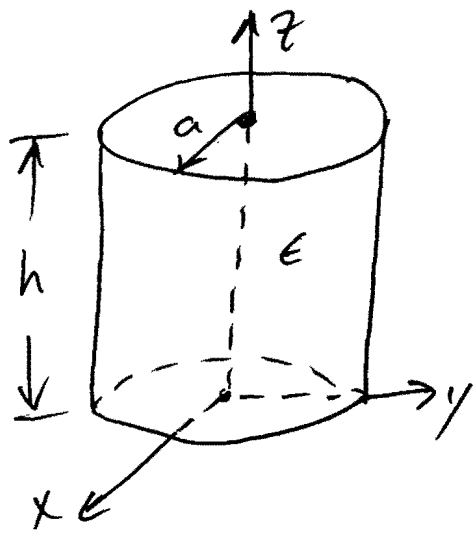
$$\Gamma = \frac{\eta_0 - \eta}{\eta_0 + \eta} = \frac{\sqrt{\epsilon_r} - 1}{\sqrt{\epsilon_r} + 1} \longrightarrow +1 \text{ as } \epsilon_r \uparrow$$

9.5.2 cont.

when ϵ_r is large & $\Gamma \to 1$, we can <u>approximate</u> the dielectric-air interface for the dielectric resonator as a Perfect Magnetic Conductor (PMC).

→ PMC property $\bar{H}_{tan} \simeq 0$ & $\bar{E}_{normal} = 0$

[PEC ⇒ $\bar{E}_{tan} = 0$ & $\bar{H}_{norm} = 0$]



← For PMC model, assume top, bottom, and side surfaces are PMCs.

w/ PMC approximation, we get the boundary conditions

@ $\rho = a$, $H_\phi = 0$ (9-96a)

@ $z = 0$, $H_\phi = 0$ & @ $z = h$, $H_\phi = 0$ (9-96b)

@ $\rho = a$, $H_z = 0$ (9-97a)

@ $z = 0$, $H_\rho = 0$ & @ $z = h$, $H_\rho = 0$ (9-97b)

9.5.2 cont.

Applying either set of boundary conditions leads to same results, will use (9-96a & b)

A. TE^z Modes

→ Start w/ (9-35) from circular cavity work in section 9.3.1. [Note: w/ PMC walls, we will not worry about $\rho > a$ fields.]

$$F_z = A_{mn} J_m(\beta_\rho^d \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \times \left[C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)\right] \qquad m = 0, 1, \ldots$$

Use (6-80), to compute H_ϕ

$$H_\phi = -j \frac{1}{\omega_r \mu_d \epsilon_d} \frac{1}{\rho} \frac{\partial^2 F_z}{\partial \phi \partial z}$$

← already assuming we are @ a resonant frequency

$$= -j A_{mn} \frac{m \beta_z}{\omega_r \mu_d \epsilon_d} \frac{1}{\rho} J_m(\beta_\rho^d \rho)\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right] \times \left[-C_3 \sin(\beta_z z) + D_3 \cos(\beta_z z)\right]$$

Apply (9-96a)

$$H_\phi(\rho = a) = 0 = -j A_{mn} \frac{m \beta_z}{\omega_r \mu_d \epsilon_d} \frac{1}{a} J_m(\beta_\rho^d a)\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right] \times \left[-C_3 \sin(\beta_z z) + D_3 \cos(\beta_z z)\right]$$

For non-trivial sol'n $\Rightarrow J_m(\beta_\rho^d a) = 0 \Rightarrow \beta_\rho^d a = X_{mn}$

9.5.2 cont.

A. cont.

So, we find that $\underline{\beta_\rho^d = \frac{\chi_{mn}}{a}}$ (9-100)

where χ_{mn} are the zeros of J_m (see Table 9-2).

Next, apply (9-96b)

$$H_\phi(z=0) = 0 = -j A_{mn} \frac{m\beta_z}{\omega_r \mu_d \epsilon_d} \frac{1}{\rho} J_m(\beta_\rho^d \rho)\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right] \times \left[0 + D_3(1)\right]$$

For non-trivial sol'n $\Rightarrow$ $\underline{D_3 = 0}$ (9-100a)

$$H_\phi(z=h) = 0 = -j A_{mn} \frac{m\beta_z}{\omega_r \mu_d \epsilon_d} \frac{1}{\rho} J_m(\beta_\rho^d \rho)\left[-C_2 \sin m\phi + D_2 \cos m\phi\right] \times \left[-C_3 \sin(\beta_z h)\right]$$

For non-trivial sol'n $\Rightarrow$ $\sin(\beta_z h) = 0$

$$\Rightarrow \beta_z h = p\pi$$

$$\underline{\beta_z = \frac{p\pi}{h} \quad p = 0, 1, 2, \ldots} \quad (9\text{-}100b)$$

The dispersion/constraint equation (9-4a)

becomes $\beta_\rho^{d2} + \beta_z^2 = \left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \omega_r^2 \mu_d \epsilon_d$

$$\Rightarrow \beta_r = \omega_r \sqrt{\mu_d \epsilon_d} = \sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \quad (9\text{-}101)$$

$$\boxed{(f_r)_{mnp}^{TE^z} = \frac{1}{2\pi\sqrt{\mu_d \epsilon_d}} \sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \quad \begin{array}{l} m = 0, 1, \ldots \\ n = 1, 2, \ldots \\ p = 0, 1, 2, \ldots \end{array}}$$

9.5.2 cont.

A. cont.

The dominant TE^z_{mnp} mode is TE^z_{010}

w/ $$f_{r,010}^{TE^z} = \frac{x_{01}}{2\pi a\sqrt{\mu_d \epsilon_d}} = \frac{2.4049}{2\pi a\sqrt{\mu_d \epsilon_d}} \quad (9\text{-}101a)$$

(dominant overall if $h/a < 2.03$ per PMC model)

B. TM^z Modes

Start w/ (9.41)

$$A_z = B_{mn} J_m(\beta_\rho^d \rho)\left[C_2\cos(m\phi) + D_2 \sin(m\phi)\right]\left[C_3\cos(\beta_z z) + D_3 \sin(\beta_z z)\right]$$

$m = 0, 1, 2, \ldots$

Per (6-70)

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = \frac{-B_{mn}}{\mu}\beta_\rho^d J_m'(\beta_\rho^d \rho)\left[C_2\cos m\phi + D_2 \sin m\phi\right] \times \left[C_3\cos(\beta_z z) + D_3 \sin(\beta_z z)\right]$$

Apply (9-96a)

$$H_\phi(\rho = a) = 0 = \frac{-B_{mn}}{\mu}\beta_\rho^d J_m'(\beta_\rho^d a)\left[C_2\cos m\phi + D_2 \sin m\phi\right] \times \left[C_3\cos(\beta_z z) + D_3 \sin(\beta_z z)\right]$$

$$\Rightarrow J_m'(\beta_\rho^d a) = 0 \Rightarrow \beta_\rho^d a = x_{mn}' \text{ (see Table 9-1)}$$

$$\underline{\underline{\beta_\rho^d = \frac{x_{mn}'}{a}}}$$

9.5.2 cont.

B. cont.

Apply (9-96b)

$$H_\phi(z=0) = 0 = \frac{-B_{mn}}{\mu}\beta_\rho^d J_m(\beta_\rho^d \rho)\left[C_2\cos m\phi + D_2 \sin m\phi\right] \times \left[C_3(1) + 0\right]$$

$$\Rightarrow \underline{C_3 = 0}$$

$$H_\phi(z=h) = 0 = \frac{-B_{mn}}{\mu}\beta_\rho^d J_m(\beta_\rho^d \rho)\left[C_2\cos m\phi + D_2 \sin m\phi\right] D_3 \sin(\beta_z h)$$

$$\Rightarrow \sin(\beta_z h) = 0 \Rightarrow \beta_z h = p\pi \quad p = 1, 2, 3, \ldots$$

$$\underline{\underline{\beta_z = \frac{p\pi}{h}}}$$

$$(\beta_\rho^d)^2 + \beta_z^2 = \left(\frac{x_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \beta_r^2 = \omega_r^2 \mu_d \epsilon_d$$

$$\Rightarrow \beta_r = \omega_r\sqrt{\mu_d \epsilon_r} = \sqrt{\left(\frac{x_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

$$(9\text{-}102) \quad \boxed{f_{r,mnp}^{TM^z} = \frac{1}{2\pi\sqrt{\mu_d \epsilon_d}}\sqrt{\left(\frac{x_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \quad \begin{array}{l} m = 0, 1, 2, \ldots \\ n = 1, 2, \ldots \\ p = 1, 2, \ldots \end{array}}$$

Dominant TM_{mnp}^z mode is TM_{111}^z mode w/

$$f_{r,111}^{TM^z} = \frac{1}{2\pi\sqrt{\mu_d \epsilon_d}}\sqrt{\left(\frac{x_{11}'}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2} \qquad x_{11}' = 1.8412$$

(dominant overall when $h/a > 2.03$ per PMC model)

9.5.2 cont.

Comparing the cylindrical cavity frequencies and cylindrical dielectric resonator frequencies (PMC model) shows that the TE^z_{mnp} + TM^z_{mnp} for each trade places.

⇒ Key difference: ϵ_d is usually large while ϵ for cavity is usually ϵ_0.

C. $TE_{01\delta}$ Mode

* In practice, most circular dielectric resonators encounter/are used in the geometry shown in Figure 9-15.

* For the $TE_{01\delta}$ mode, δ is a noninteger and $\delta < 1$. This implies the field(s) undergo less than $\lambda/2$ phase shift in $0 \le z \le h$ w/in the dielectric resonator

* Numerical modeling is needed to get truly accurate results when dielectric resonator is on a circuit board w/in a metal box w/ other circuit elements.

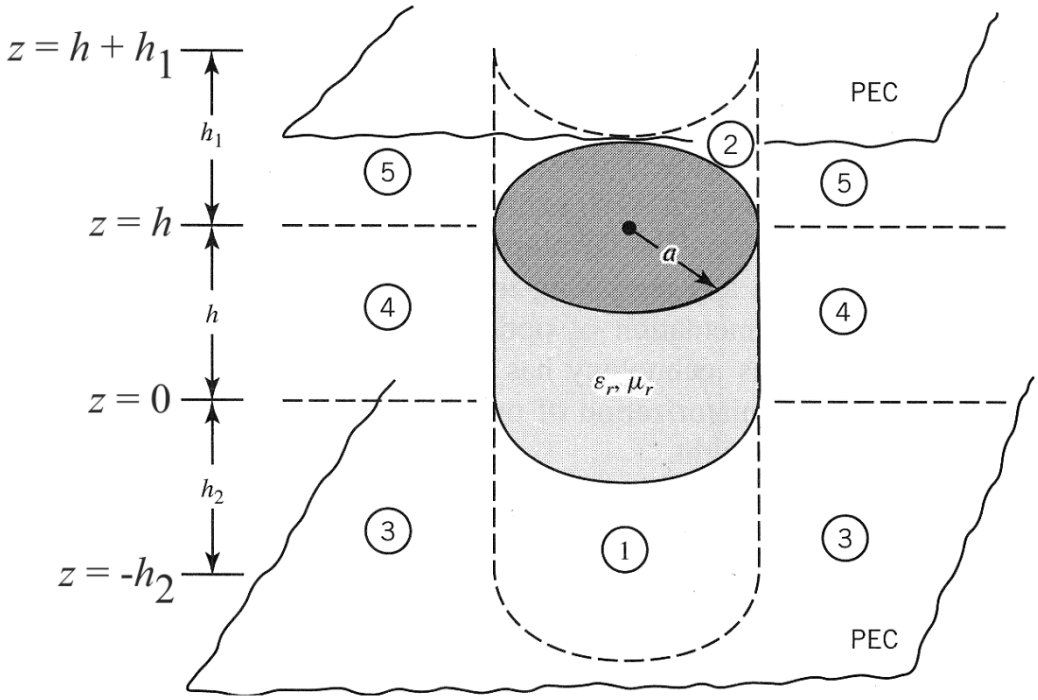


Figure 9-15 Modeling of circular dielectric rod resonator. (Source: D. Kajfez and P. Guillon (Eds.), *Dielectric Resonators*, 1986, Artech House, Inc.).

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

In Figure 9-15-

Bottom of dielectric cylinder is at $z = 0$ & $\rho \le a$

Top of dielectric cylinder is at $z = h$ & $\rho \le a$

Region 1- volume of space between bottom of dielectric cylinder and PEC plane at $z = -h_2$

Region 2- volume of space between top of dielectric cylinder and PEC plane at $z = h + h_1$

Region 3- volume of space $-h_2 < z < 0$ NOT below dielectric cylinder

Region 4- volume of space $0 < z < h$ & $\rho > a$, i.e., outside & adjacent to dielectric cylinder

Region 5- volume of space $h < z < h + h_1$ NOT above dielectric cylinder

Typically, the medium in regions 1 – 5 is air/free space (ε_0, μ_0).

The distances h_1 and h_2 above and below the dielectric cylinder are anywhere from 0 to ∞, i.e., $0 \le h_1 \le \infty$ and $0 \le h_2 \le \infty$.

9.5.2 cont.

C_1 cont.

For the situation shown in Figure 9-15 -

$$\beta_z h \simeq \frac{\psi_1}{2} + \frac{\psi_2}{2} + q\pi \qquad q = 0, 1, 2, \ldots \qquad (9\text{-}103)$$

where

$$\frac{\psi_1}{2} = \tan^{-1}\left[\frac{\alpha_1}{\beta_z}\coth(\alpha_1 h_1)\right] \qquad (9\text{-}103a)$$

$$\frac{\psi_2}{2} = \tan^{-1}\left[\frac{\alpha_2}{\beta_z}\coth(\alpha_2 h_2)\right] \qquad (9\text{-}103b)$$

$$\alpha_1 = \sqrt{\left(\frac{\chi_{01}}{a}\right)^2 + \beta_0^2 \epsilon_{r1}} \leftarrow \text{Region 1} \qquad (9\text{-}103c)$$

$$\alpha_2 = \sqrt{\left(\frac{\chi_{01}}{a}\right)^2 + \beta_0^2 \epsilon_{r2}} \leftarrow \text{Region 2} \qquad (9\text{-}103d)$$

* For the $TE_{01\delta}$ mode, select $q = 0$ in (9-103).

$$\delta = \frac{1}{\pi}\left(\frac{\psi_1}{2} + \frac{\psi_2}{2}\right) \qquad (9\text{-}104)$$

* From Table 9-2, $\chi_{01} = 2.4049$.
* $\beta_0 = \omega\sqrt{\mu_0 \epsilon_0}$
* From Kajfez + Guillon, Dielectric Resonators, 1986, Artech House

$$f_{r,01\delta}^{TE}(\text{GHz}) = \frac{34}{a\sqrt{\epsilon_r}}\left(\frac{a}{h} + 3.45\right) \quad \text{for } h_1 = h_2 \to \infty$$

$$0.5 < \frac{a}{h} < 2$$

$$30 < \epsilon_r < 50$$

Example- Compute some resonant frequencies for a circular dielectric resonator with dielectric constant 38, radius 5.25 mm, and height 4.6 mm.

$a := 0.00525$ m $\quad h := 0.0046$ m $\quad \frac{h}{a} = 0.876 \quad \varepsilon r := 38$

$\mu 0 := 4 \cdot \pi \cdot 10^{-7}$ H/m $\quad \varepsilon 0 := 8.8541878 \cdot 10^{-12}$ F/m

TE_{010} mode (dominant) $\quad \chi 01 := 2.4049$

(9-101a) $$fc_TE010 := \frac{\chi 01}{2 \cdot \pi \cdot a \cdot \sqrt{\mu 0 \cdot \varepsilon r \cdot \varepsilon 0}}$$

$$\boxed{fc_TE010 = 3.5456 \times 10^{9}} \text{ Hz}$$

$TE_{11\delta}$ mode $\quad amm := a \cdot 1000 \quad hmm := h \cdot 1000$

(Kajfez & Guillon, Dielectric Resonators, 1986, Artech House)

$$fc_TE11\delta := \frac{34}{amm \cdot \sqrt{\varepsilon r}} \cdot \left(\frac{a}{h} + 3.45\right) \cdot 10^{9}$$

$$\boxed{fc_TE11\delta = 4.8235 \times 10^{9}} \text{ Hz}$$

TE_{110} mode $\quad \chi 11 := 3.8318$

(9-101) $$fc_TE110 := \frac{1}{2 \cdot \pi \cdot \sqrt{\mu 0 \cdot \varepsilon r \cdot \varepsilon 0}} \cdot \sqrt{\left(\frac{\chi 11}{a}\right)^2 + \left(\frac{0 \cdot \pi}{h}\right)^2}$$

$$\boxed{fc_TE110 = 5.6493 \times 10^{9}} \text{ Hz}$$

TM_{111} mode $\quad \chi 11p := 1.8412$

(9-102a) $$fc_TM111 := \frac{1}{2 \cdot \pi \cdot \sqrt{\mu 0 \cdot \varepsilon r \cdot \varepsilon 0}} \cdot \sqrt{\left(\frac{\chi 11p}{a}\right)^2 + \left(\frac{1 \cdot \pi}{h}\right)^2}$$

$$\boxed{fc_TM111 = 5.9424 \times 10^{9}} \text{ Hz}$$

TE_{011} mode

(9-101) $$fc_TE011 := \frac{1}{2\cdot\pi\cdot\sqrt{\mu 0\cdot\varepsilon r\cdot\varepsilon 0}}\cdot\sqrt{\left(\frac{\chi 01}{a}\right)^2+\left(\frac{1\cdot\pi}{h}\right)^2}$$

$$\boxed{fc_TE011 = 6.3651\times 10^9}\ \text{Hz}$$

TE_{111} mode

(9-101) $$fc_TE111 := \frac{1}{2\cdot\pi\cdot\sqrt{\mu 0\cdot\varepsilon r\cdot\varepsilon 0}}\cdot\sqrt{\left(\frac{\chi 11}{a}\right)^2+\left(\frac{1\cdot\pi}{h}\right)^2}$$

$$\boxed{fc_TE111 = 7.7368\times 10^9}\ \text{Hz}$$

TE_{210} mode $\chi 21 := 5.1357$

(9-101) $$fc_TE210 := \frac{1}{2\cdot\pi\cdot\sqrt{\mu 0\cdot\varepsilon r\cdot\varepsilon 0}}\cdot\sqrt{\left(\frac{\chi 21}{a}\right)^2+\left(\frac{0\cdot\pi}{h}\right)^2}$$

$$\boxed{fc_TE210 = 7.5716\times 10^9}\ \text{Hz}$$