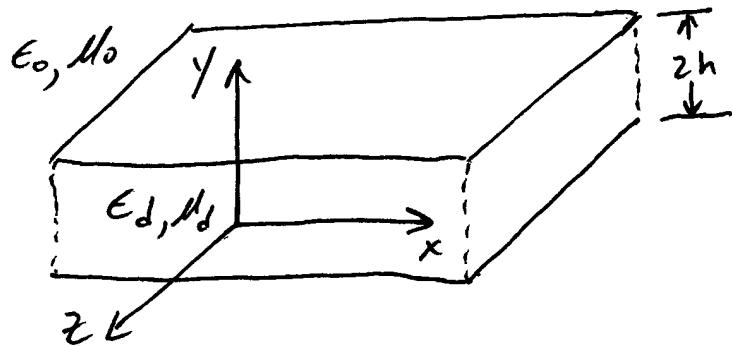


## 8.7 Dielectric Waveguide

- Dielectric slabs (Chap 8) or rods (Chap 9) can also guide EM waves
- May or may not have metal in structure
- Field modes supported are known as surface wave modes.
- Depending on the application, surface wave modes may be desirable (e.g., fiber optic cables) or to be avoided (e.g., PCBs and microstrip patch antennas).

### 8.7.1 Dielectric Slab Waveguide

We'll start w/ a dielectric slab of height  $2h$  that is infinite in the  $x$ - &  $z$ -directions.

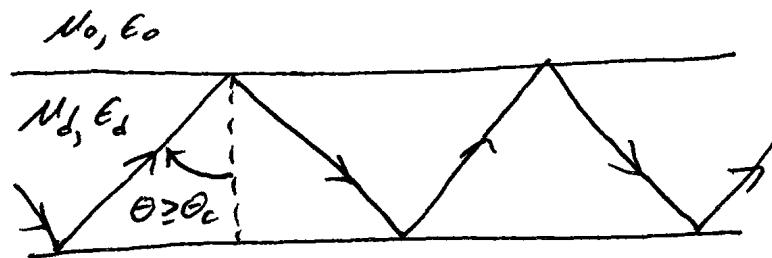


For simplicity, we'll assume the axes are rotated so that the EM waves guided are propagating in the  $\pm z$ -direction and nothing is changing in the  $\pm x$ -directions (i.e.,  $\partial/\partial x = 0$ ).

→ Reality : A rectangular slab waveguide would have some width  $a$  ( $-a/2 \leq x \leq a/2$ ).

### 8.7.1 cont.

- \* For a guided wave, we want it to stay inside the dielectric, i.e., bounce back-and-forth between the top and bottom at an angle greater than or equal to the critical angle  $\theta_c$ . Then, the refracted fields outside the dielectric are evanescent in form (remember Chapter 5) and all real power/power flow stays w/in the dielectric.



Analysis  $\rightarrow$  boundary-value problem for applicable wave eq'n w/ modal solutions

$\rightarrow$  ray-tracing / optics techniques

- \* Our dielectric slab can support  $TE^z$ ,  $TM^z$ ,  $TE^y$ , +  $TM^y$  modes. We'll consider the  $TE^z$  +  $TM^z$  for our  $\pm z$ -propagating waves.

## 8.7.2 Transverse Magnetic ( $TM_z$ ) Modes

Looking back to section 8.2.2 -

$$\begin{aligned} E_x &= -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_z}{\partial x \partial z} & H_x &= \frac{1}{\mu} \frac{\partial A_z}{\partial y} \\ E_y &= -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_z}{\partial y \partial z} & H_y &= -\frac{1}{\mu} \frac{\partial A_z}{\partial x} \end{aligned} \quad (8-24)$$

$$E_z = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z \right) \quad H_z = 0$$

where  $\nabla^2 A_z + \beta^2 A_z = 0$  (8-25) and

$$A_z = [C_1 \cos(\beta_x x) + C_2 \sin(\beta_x x)] [C_3 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \times [A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}] \quad (8-26)$$

$-h \leq y \leq h$

\* For our dielectric slab, we assumed nothing is changing in the  $x$ -direction  $\Rightarrow$  replace the first term of (8-26), i.e.,  $h(x)$ , w/ a constant.

\* Again, we will consider only waves propagating in the  $+z$ -direction, i.e., set  $B_3 = 0$  in (8-26).

This leaves

$$\begin{aligned} A_z^d &= [C_2^d \cos(\beta_y d) + D_2^d \sin(\beta_y d)] A_3^d e^{-j\beta_z z} \\ &= A_{ze}^d + A_{zo}^d \end{aligned} \quad (8-148)$$

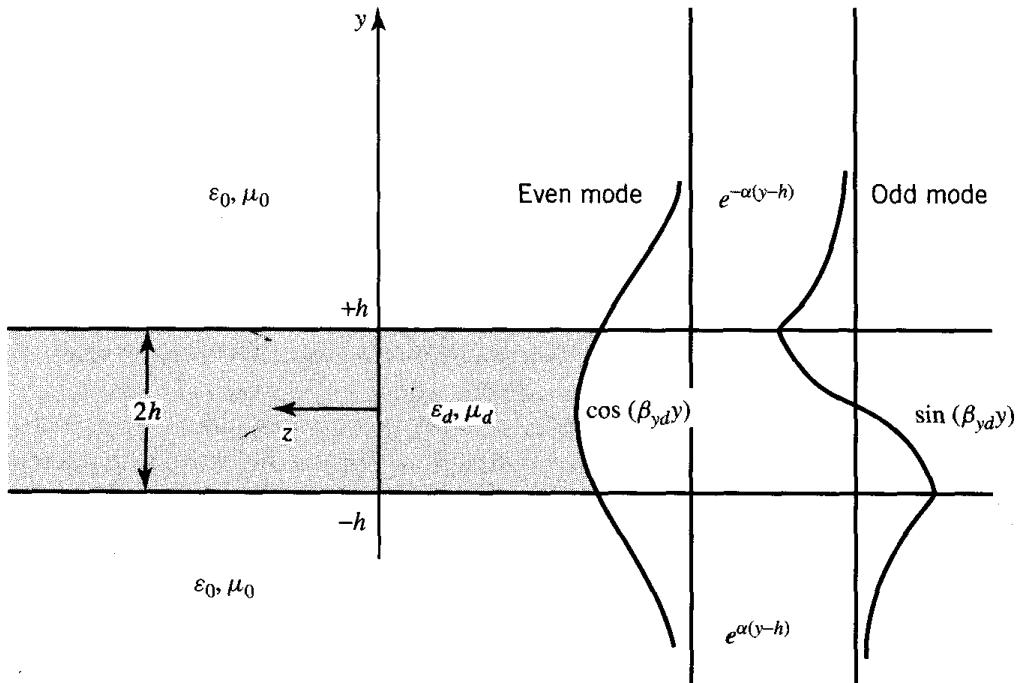
$\uparrow$   $\uparrow$   
 even modes odd modes

8.7.2 cont.

$$\text{where } A_{ze}^d = C_2^d A_3^d \cos(\beta_{yd} y) e^{-j\beta_z z} \quad (8-148a)$$

$$A_{zo}^d = D_2^d A_3^d \sin(\beta_{yd} y) e^{-j\beta_z z} \quad (8-148b)$$

$$\text{with } \beta_{yd}^2 + \beta_z^2 = \beta_d^2 = \omega^2 \mu_d \epsilon_d \quad (8-148c)$$



**Figure 8-20** Even and odd mode field distributions in a dielectric slab waveguide. (Source: M. Zahn, *Electromagnetic Field Theory*, 1979. Reprinted with permission of John Wiley & Sons, Inc.)

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

Again, for a waveguide, we assumed the fields above ( $y \geq h$ ) and below ( $y \leq -h$ ) to be evanescent in form. This will necessitate replacing  $h(y) = C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)$  in the expressions for  $A_z$  outside the dielectric slab (see Table 3-1).

8.7.2 cont.

$y \geq h$  (above dielectric)

$$\begin{aligned}
 A_z^{0+} &= \underbrace{(A_{ze}^{0+} e^{-j\beta_{y0}y} + B_{z0}^{0+} e^{-j\beta_{y0}y})}_{\text{even term}} \underbrace{A_{30}^{0+} e^{-j\beta_z z}}_{\text{odd term}} \\
 &= (A_{ze}^{0+} e^{-\alpha_{y0}y} + B_{z0}^{0+} e^{-\alpha_{y0}y}) A_{30}^{0+} e^{-j\beta_z z} \\
 &= A_{ze}^{0+} + A_{z0}^{0+} \tag{8-149}
 \end{aligned}$$

where

$$\begin{aligned}
 A_{ze}^{0+} &= A_{re}^{0+} A_{30}^{0+} e^{-\alpha_{y0}y} e^{-j\beta_z z} \\
 &= A_{me}^{0+} e^{-\alpha_{y0}y} e^{-j\beta_z z} \tag{8-149a}
 \end{aligned}$$

~mode even

$$\begin{aligned}
 A_{z0}^{0+} &= B_{z0}^{0+} A_{30}^{0+} e^{-\alpha_{y0}y} e^{-j\beta_z z} \\
 &= B_{mo}^{0+} e^{-\alpha_{y0}y} e^{-j\beta_z z} \tag{8-149b}
 \end{aligned}$$

~mode odd

$$\beta_{y0}^2 + \beta_z^2 = -\alpha_{y0}^2 + \beta_z^2 = \beta_0^2 = \omega^2 \mu_0 \epsilon_0 \tag{8-149c}$$

For the evanescent wave to exponentially decay for  $y \geq h$ , we expect  $\alpha_{y0}$  to be a real & positive number

## 8.7.2 cont.

$y \leq -h$  (below dielectric)

$$\begin{aligned} A_z^{0-} &= (A_{2e}^{0-} e^{+j\beta y_0 y} + B_{20}^{0-} e^{+j\beta y_0 y}) A_{30}^{0-} e^{-j\beta z^2} \\ &= (A_{2e}^{0-} e^{+\alpha y_0 y} + B_{20}^{0-} e^{+\alpha y_0 y}) A_{30}^{0-} e^{-j\beta z^2} \quad (8-150) \end{aligned}$$

$$A_{2e}^{0-} = A_{2e}^{0-} A_{30}^{0-} e^{+\alpha y_0 y} e^{-j\beta z^2} = A_{me}^{0-} e^{+\alpha y_0 y} e^{-j\beta z^2} \quad (8-150a)$$

$$A_{20}^{0-} = B_{20}^{0-} A_{30}^{0-} e^{+\alpha y_0 y} e^{-j\beta z^2} = B_{mo}^{0-} e^{+\alpha y_0 y} e^{-j\beta z^2} \quad (8-150b)$$

$$\beta_{y_0}^2 + \beta_z^2 = -\alpha_{y_0}^2 + \beta_z^2 = \beta_0^2 = \omega^2 \mu_0 \epsilon_0 \quad (8-150c)$$

EM boundary conditions @  $y = \pm h$

$$E_z^d(y=h, z) = E_z^{0+}(y=h, z) \quad (8-151a)$$

$$E_z^d(y=-h, z) = E_z^{0-}(y=-h, z) \quad (8-151b)$$

$$H_x^d(y=h, z) = H_x^{0+}(y=h, z) \quad (8-151c)$$

$$H_x^d(y=-h, z) = H_x^{0-}(y=-h, z) \quad (8-151d)$$

$\Rightarrow$  will apply to even and odd modes separately.

## 8.7.2 cont.

### A. $TM^2$ (Even)

Apply (6-59) or (8-24) to get the field components for the even mode

$$-\underline{h \leq y \leq h} \quad A_{ze}^d = A_{me}^d \cos(\beta_y y) e^{-j\beta_z z} \quad (8-148a)$$

$$E_{xe}^d = \frac{-j}{\omega \mu_d \epsilon_d} \frac{\partial^2 A_{ze}^d}{\partial x \partial z} = 0 \quad \left( \frac{\partial}{\partial x} = 0 \right) \quad (8-152a)$$

$$\begin{aligned} E_{ye}^d &= \frac{-j}{\omega \mu_d \epsilon_d} \frac{\partial^2 A_{ze}^d}{\partial y \partial z} = \frac{-j A_{me}^d}{\omega \mu_d \epsilon_d} \frac{\partial \cos(\beta_y y)}{\partial y} \frac{\partial e^{-j\beta_z z}}{\partial z} \\ &= \frac{\beta_y d \beta_z A_{me}^d}{\omega \mu_d \epsilon_d} \sin(\beta_y d y) e^{-j\beta_z z} \end{aligned} \quad (8-152b)$$

$$\begin{aligned} E_{ze}^d &= \frac{-j}{\omega \mu_d \epsilon_d} \left( \frac{\partial^2 A_{ze}^d}{\partial z^2} + \beta_d^2 A_{ze}^d \right) = \frac{-j}{\omega \mu_d \epsilon_d} (-\beta_z^2 + \beta_d^2) A_{ze}^d \\ &= -j \frac{(\beta_d^2 - \beta_z^2)}{\omega \mu_d \epsilon_d} A_{me}^d \cos(\beta_y d y) e^{-j\beta_z z} \end{aligned} \quad (8-152c)$$

$$H_{xe}^d = \frac{1}{\mu_d} \frac{\partial A_{ze}^d}{\partial y} = -\frac{\beta_y d}{\mu_d} A_{me}^d \sin(\beta_y d y) e^{-j\beta_z z} \quad (8-152d)$$

$$H_{ye}^d = \frac{-1}{\mu_d} \frac{\partial A_{ze}^d}{\partial x} = 0 \quad \left( \frac{\partial}{\partial x} = 0 \right) \quad (8-152e)$$

$$H_{ze}^d = 0 \quad (TM^2 \text{ mode}) \quad (8-152f)$$

### 8.7.2 A. cont.

$$\underline{y \geq +h} \quad A_{ze}^{0+} = A_{me}^{0+} e^{-\alpha_{y0} y} e^{-j\beta_z z} \quad (8-149a)$$

$$E_{xe}^{0t} = \frac{-j}{\omega_{M0}\epsilon_0} \frac{\partial^2 A_{ze}^{0+}}{\partial x \partial z} = 0 \quad (\frac{\partial}{\partial x} = 0) \quad (8-153a)$$

$$E_{ye}^{0t} = \frac{-j}{\omega_{M0}\epsilon_0} \frac{\partial^2 A_{ze}^{0+}}{\partial y \partial z} = \frac{\alpha_{y0}\beta_z}{\omega_{M0}\epsilon_0} A_{me}^{0+} e^{-\alpha_{y0} y} e^{-j\beta_z z} \quad (8-153b)$$

$$\begin{aligned} E_{ze}^{0t} &= \frac{-j}{\omega_{M0}\epsilon_0} \left( \frac{\partial^2 A_{ze}^{0+}}{\partial z^2} + \beta_0^2 A_{ze}^{0+} \right) \\ &= -j \frac{(\beta_0^2 - \beta_z^2)}{\omega_{M0}\epsilon_0} A_{me}^{0+} e^{-\alpha_{y0} y} e^{-j\beta_z z} \end{aligned} \quad (8-153c)$$

$$H_{xe}^{0t} = \frac{1}{\mu_0} \frac{\partial A_{ze}^{0+}}{\partial y} = -\frac{\alpha_{y0}}{\mu_0} A_{me}^{0+} e^{-\alpha_{y0} y} e^{-j\beta_z z} \quad (8-153d)$$

$$H_{ye}^{0t} = \frac{-1}{\mu_0} \frac{\partial A_{ze}^{0+}}{\partial x} = 0 \quad (\frac{\partial}{\partial x} = 0) \quad (8-153e)$$

$$H_{ze}^{0+} = 0 \quad (TM^z \text{ mode}) \quad (8-153f)$$

$$\underline{y \leq -h} \quad A_{ze}^{0-} = A_{me}^{0-} e^{+\alpha_{y0} y} e^{-j\beta_z z} \quad (8-150a)$$

$$E_{xe}^{0-} = \frac{-j}{\omega_{M0}\epsilon_0} \frac{\partial^2 A_{ze}^{0-}}{\partial x \partial z} = 0 \quad (\frac{\partial}{\partial x} = 0) \quad (8-154a)$$

$$E_{ye}^{0-} = \frac{-j}{\omega_{M0}\epsilon_0} \frac{\partial^2 A_{ze}^{0-}}{\partial y \partial z} = -\frac{\alpha_{y0}\beta_z}{\omega_{M0}\epsilon_0} A_{me}^{0-} e^{+\alpha_{y0} y} e^{-j\beta_z z} \quad (8-154b)$$

### 8.7.2 A. cont.

$$E_{ze}^{0-} = \frac{-j}{\omega \mu_0 \epsilon_0} \left( \frac{\partial^2 A_{ze}^{0-}}{\partial z^2} + \beta_0^2 A_{ze}^{0-} \right)$$

$$= -j \frac{(\beta_0^2 - \beta_z^2)}{\omega \mu_0 \epsilon_0} A_{me}^{0-} e^{+dy_0 y} e^{-j\beta_z z} \quad (8-154c)$$

$$H_{xe}^{0-} = \frac{1}{\mu_0} \frac{\partial A_{ze}^{0-}}{\partial y} = \frac{\alpha_{y0}}{\mu_0} A_{me}^{0-} e^{+dy_0 y} e^{-j\beta_z z} \quad (8-154d)$$

$$H_{ye}^{0-} = \frac{-1}{\mu_0} \frac{\partial A_{ze}^{0-}}{\partial x} = 0 \quad (\frac{\partial}{\partial x} = 0) \quad (8-154e)$$

$$H_{ze}^{0-} = 0 \quad (\text{TM}^2 \text{ mode}) \quad (8-154f)$$

Now, we can apply the tangential electric field boundary condition @ y=h (8-151a)

$$E_z^d(h, z) = E_z^{0+}(h, z)$$

$$-j \frac{(\beta_d^2 - \beta_z^2)}{\omega \mu_d \epsilon_d} A_{me}^d \cos(\beta_{yd} h) e^{-j\beta_z z} = -j \frac{(\beta_0^2 - \beta_z^2)}{\omega \mu_0 \epsilon_0} A_{me}^{0+} e^{-dy_0 h} e^{-j\beta_z z}$$

$$\frac{\beta_d^2 - \beta_z^2}{\mu_d \epsilon_d} A_{me}^d \cos(\beta_{yd} h) = \frac{\beta_0^2 - \beta_z^2}{\mu_0 \epsilon_0} A_{me}^{0+} e^{-dy_0 h}$$

Per (8-148c),  $\beta_{yd}^2 = \beta_d^2 - \beta_z^2$ . Per (8-149c),  $-dy_0^2 = \beta_0^2 - \beta_z^2$ .

$$\frac{\beta_{yd}^2}{\mu_d \epsilon_d} A_{me}^d \cos(\beta_{yd} h) = \frac{-dy_0^2}{\mu_0 \epsilon_0} A_{me}^{0+} e^{-dy_0 h} \quad (8-155a)$$

### 8.7.2 A. cont.

Next, apply the tangential electric field boundary condition @  $y=h$  (8-151b)

$$E_z^d(-h, z) = E_z^{0-}(-h, z)$$

$$-\frac{-j(\beta_d^2 - \beta_z^2)}{\omega M_d E_d} A_{me}^d \cos(\beta_{yd}(-h)) e^{-j\beta_z z} = -\frac{(\beta_0^2 - \beta_z^2)}{\omega M_0 E_0} A_{me}^{0-} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

Divide out common terms, use  $\cos(-A) = \cos(A)$ , use (8-148c), & use (8-149c) to get:

$$\frac{\beta_{yd}^2}{M_d E_d} A_{me}^d \cos(\beta_{yd} h) = \frac{-\alpha_{yo}^2}{M_0 E_0} A_{me}^{0-} e^{-\alpha_{yo} h} \quad (8-155b)$$

In order for (8-155a) & (8-155b) to hold true

$$A_{me}^{0+} = A_{me}^{0-} \Rightarrow A_{me}^0 \quad (8-155c).$$

This reduces (8-155a) & (8-155b) to a single

$$\underline{\frac{\beta_{yd}^2}{M_d E_d} A_{me}^d \cos(\beta_{yd} h) = \frac{-\alpha_{yo}^2}{M_0 E_0} A_{me}^0 e^{-\alpha_{yo} h}} \quad (8-156)$$

Continuing, we apply the tangential magnetic field boundary condition @  $y=h$  (8-151c)

$$H_x^d(h, z) = H_x^{0+}(h, z)$$

$$-\frac{\beta_{yd}}{M_d} A_{me}^d \sin(\beta_{yd} h) e^{-j\beta_z z} = -\frac{\alpha_{yo}}{M_0} A_{me}^{0+} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

### 8.7.2 A. cont.

$$\frac{\beta_{yd}}{M_d} A_{me}^d \sin(\beta_{yd} h) = \frac{\alpha_{yo}}{M_o} A_{me}^{o+} e^{-\alpha_{yo} h} \quad (1)$$

Applying the tangential magnetic field boundary condition @  $y = -h$  (8-151d)

$$H_x^d(-h, z) = H_x^{o-}(-h, z)$$

$$-\frac{\beta_{yd}}{M_d} A_{me}^d \sin(-\beta_{yd} h) e^{-j\beta_z z} = \frac{\alpha_{yo}}{M_o} A_{me}^{o-} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

use  $\sin(-A) = -\sin(A)$  and divide out common terms to get

$$+\frac{\beta_{yd}}{M_d} A_{me}^d \sin(\beta_{yd} h) = \frac{\alpha_{yo}}{M_o} A_{me}^{o-} e^{-\alpha_{yo} h} \quad (2)$$

(1) + (2) can only be true if  $A_{me}^{o+} = A_{me}^{o-} = \underline{A_{me}^o}$ ,

giving  $\underline{\frac{\beta_{yd}}{M_d} A_{me}^d \sin(\beta_{yd} h)} = \frac{\alpha_{yo}}{M_o} A_{me}^o e^{-\alpha_{yo} h} \quad (8-157)}$

Divide (8-157) by (8-156) to get:

$$\frac{\frac{\beta_{yd}}{M_d} A_{me}^d \sin(\beta_{yd} h)}{\frac{\beta_{yd}^2}{M_d E_d} A_{me}^d \cos(\beta_{yd} h)} = \frac{\frac{\alpha_{yo}}{M_o} A_{me}^o e^{-\alpha_{yo} h}}{-\frac{\alpha_{yo}^2}{M_o E_o} A_{me}^o e^{-\alpha_{yo} h}}$$

∴

$$\frac{E_d}{\beta_{yd}} \tan(\beta_{yd} h) = -\frac{E_o}{\alpha_{yo}}.$$

### 8.7.2 A. cont.

This can be rearranged into

$$-\frac{\epsilon_0}{\epsilon_d} \cot(\beta_{yd} h) = \frac{\alpha_{yo}}{\beta_{yd}}$$

↳ multiply by  $\beta_{yd} h$  to get

$$-\frac{\epsilon_0}{\epsilon_d} (\beta_{yd} h) \cot(\beta_{yd} h) = \alpha_{yo} h \quad (8-158)$$

$$\text{where } \beta_{yd}^2 = \beta_d^2 - \beta_z^2 = \omega^2 M_d \epsilon_d - \beta_z^2 \quad (8-158a)$$

$$\text{+ } \alpha_{yo}^2 = \beta_z^2 - \beta_0^2 = \beta_z^2 - \omega^2 M_0 \epsilon_0 \quad (8-158b)$$

Looking down at the dielectric slab,  
we can define a wave impedance

$$\begin{aligned} Z_w^{-yo} &= -\frac{E_{ze}^{0+}}{H_{xe}^{0+}} = \frac{E_{ze}^{0-}}{H_{xe}^{0-}} \\ &= -j \frac{(\beta_0^2 - \beta_z^2)}{\omega \epsilon_0 \alpha_{yo}} = j \frac{\alpha_{yo}}{\omega \epsilon_0} \quad (8-153c) \end{aligned}$$

From the last term, we see that this  
impedance has/is an inductive reactance

⇒ TM mode surface waves are  
supported by inductive surfaces  
(implications for metamaterials).

### 8.7.2 cont.

B.  $TM_2$  (odd)  $A_{z0}^d = A_{mo}^d \sin(\beta_{yd} y) e^{-j\beta_z z}$  (8-148b)

$$-h \leq y \leq h$$

$$E_{x0}^d = \frac{-j}{\omega \mu_d \epsilon_d} \frac{\partial^2 A_{z0}^d}{\partial x \partial z} = 0$$

$$E_{y0}^d = \frac{-j}{\omega \mu_d \epsilon_d} \frac{\partial^2 A_{z0}^d}{\partial y \partial z} = \frac{-j A_{mo}^d}{\omega \mu_d \epsilon_d} \beta_{yd} \cos(\beta_{yd} y) (-j\beta_z) e^{-j\beta_z z}$$

$$= -\frac{\beta_{yd} \beta_z A_{mo}^d}{\omega \mu_d \epsilon_d} \cos(\beta_{yd} y) e^{-j\beta_z z}$$

$$E_{z0}^d = \frac{-j}{\omega \mu_d \epsilon_d} \left( \frac{\partial^2 A_{z0}^d}{\partial z^2} + \beta_d^2 A_{z0}^d \right) = \frac{-j}{\omega \mu_d \epsilon_d} (-\beta_z^2 + \beta_d^2) A_{z0}^d$$

$$= -j \frac{(\beta_d^2 - \beta_z^2)}{\omega \mu_d \epsilon_d} A_{mo}^d \sin(\beta_{yd} y) e^{-j\beta_z z}$$

$$H_{x0}^d = \frac{1}{\mu_d} \frac{\partial A_{z0}^d}{\partial y} = \frac{\beta_{yd}}{\mu_d} A_{mo}^d \cos(\beta_{yd} y) e^{-j\beta_z z}$$

$$H_{y0}^d = -\frac{1}{\mu_d} \frac{\partial A_{z0}^d}{\partial x} = 0$$

$$H_{z0}^d = 0$$

Note: per (8-148c),  $\beta_{yd}^2 = \beta_d^2 - \beta_z^2$ .

### 8.7.2 B<sub>0</sub> cont.

$$y \geq h^+ \quad A_{z0}^{0+} = B_{m0}^{0+} e^{-\alpha_{y0} y} e^{-j\beta z^2} \quad (8-149b)$$

$$E_{x0}^{0+} = \frac{-j}{\omega \mu_0 \epsilon_0} \frac{\partial^2 A_{z0}^{0+}}{\partial x \partial z} = 0$$

$$E_{y0}^{0+} = \frac{-j}{\omega \mu_0 \epsilon_0} \frac{\partial^2 A_{z0}^{0+}}{\partial y \partial z} = \frac{\alpha_{y0} \beta z}{\omega \mu_0 \epsilon_0} B_{m0}^{0+} e^{-\alpha_{y0} y} e^{-j\beta z^2}$$

$$E_{z0}^{0+} = \frac{-j}{\omega \mu_0 \epsilon_0} \left( \frac{\partial^2 A_{z0}^{0+}}{\partial z^2} + f_0^2 A_{z0}^{0+} \right)$$

$$= -j \frac{(f_0^2 - \beta z^2)}{\omega \mu_0 \epsilon} B_{m0}^{0+} e^{-\alpha_{y0} y} e^{-j\beta z^2}$$

$$H_{x0}^{0+} = \frac{1}{N_0} \frac{\partial A_{z0}^{0+}}{\partial y} = -\frac{\alpha_{y0}}{N_0} B_{m0}^{0+} e^{-\alpha_{y0} y} e^{-j\beta z^2}$$

$$H_{y0}^{0+} = \frac{-1}{N_0} \frac{\partial A_{z0}^{0+}}{\partial x} = 0$$

$$H_{z0}^{0+} = 0 \quad \text{Note: Per (8-149c), } -\alpha_{y0}^2 = f_0^2 - \beta z^2$$

$$y \leq -h \quad A_{z0}^{0-} = B_{m0}^{0-} e^{\alpha_{y0} y} e^{-j\beta z^2} \quad (8-150b)$$

$$E_{x0}^{0-} = \frac{-j}{\omega \mu_0 \epsilon_0} \frac{\partial^2 A_{z0}^{0-}}{\partial x \partial z} = 0$$

$$E_{y0}^{0-} = \frac{-j}{\omega \mu_0 \epsilon_0} \frac{\partial^2 A_{z0}^{0-}}{\partial y \partial z} = -\frac{\alpha_{y0} \beta z}{\omega \mu_0 \epsilon_0} B_{m0}^{0-} e^{\alpha_{y0} y} e^{-j\beta z^2}$$

### 8.7.2 B. cont.

$$E_{z0}^{0-} = \frac{-j}{\omega M_0 \epsilon_0} \left( \frac{\partial^2 A_{z0}^{0-}}{\partial z^2} + \beta_0^2 A_{z0}^{0-} \right)$$

$$= -j \frac{(\beta_0^2 - \beta_z^2)}{\omega M_0 \epsilon_0} B_{mo}^{0-} e^{\alpha_{yo} y} e^{-j\beta_z z}$$

$$H_{x0}^{0-} = \frac{1}{M_0} \frac{\partial A_{z0}^{0-}}{\partial y} = \frac{\alpha_{yo}}{M_0} B_{mo}^{0-} e^{\alpha_{yo} y} e^{-j\beta_z z}$$

$$H_{y0}^{0-} = -\frac{1}{M_0} \frac{\partial A_{z0}^{0-}}{\partial x} = 0$$

$$H_{z0}^{0-} = 0$$

Apply the tangential boundary conditions

$$\textcircled{1} \quad E_z^d(h, z) = E_z^{0+}(h, z) \quad (8-151a)$$

$$-j \frac{(\beta_d^2 - \beta_z^2)}{\omega M_d \epsilon_d} A_{mo}^d \sin(\beta_{yd} h) e^{-j\beta_z z} = -j \frac{(\beta_0^2 - \beta_z^2)}{\omega M_0 \epsilon_0} B_{mo}^{0+} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

$$\frac{\beta_{yd}^2}{M_d \epsilon_d} A_{mo}^d \sin(\beta_{yd} h) = \frac{-\alpha_{yo}^2}{M_0 \epsilon_0} B_{mo}^{0+} e^{-\alpha_{yo} h}$$

$$\textcircled{2} \quad E_z^d(-h, z) = E_z^{0-}(-h, z) \quad (8-151b)$$

$$-j \frac{\beta_{yd}^2}{\omega M_d \epsilon_d} A_{mo}^d \sin(\beta_{yd}(-h)) e^{-j\beta_z z} = -j \frac{-\alpha_{yo}^2}{\omega M_0 \epsilon_0} B_{mo}^{0-} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

### 8.7.2 B. cont.

$$\text{use } \sin(-A) = -\sin(A)$$

$$\frac{+\beta_{yd}^2}{M_d E_d} A_{mo}^d \sin(\beta_{yd} h) = \frac{+\alpha_{yo}^2}{M_o E_o} B_{mo}^{0-} e^{-\alpha_{yo} h}$$

In order for the results of ① & ② to hold true on the RHS -

$$B_{mo}^{0-} = -B_{mo}^{0+} = B_{mo}^0$$

This reduces the two equations to

$$\underline{\frac{\beta_{yd}^2}{M_d E_d} A_{mo}^d \sin(\beta_{yd} h) = \frac{\alpha_{yo}^2}{M_o E_o} B_{mo}^0 e^{-\alpha_{yo} h}}$$

$$③ H_x^d(h, z) = H_x^{0+}(h, z) \quad (8-151c)$$

$$\frac{\beta_{yd}}{M_d} A_{mo}^d \cos(\beta_{yd} h) e^{-j\beta_z z} = \frac{-\alpha_{yo}}{M_o} B_{mo}^{0+} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

$$\frac{\beta_{yd}}{M_d} A_{mo}^d \cos(\beta_{yd} h) = -\frac{\alpha_{yo}}{M_o} B_{mo}^{0+} e^{-\alpha_{yo} h}$$

$$④ H_x^d(-h, z) = H_x^{0-}(-h, z) \quad (8-151d)$$

$$\frac{\beta_{yd}}{M_d} A_{mo}^d \cos(-\beta_{yd} h) e^{-j\beta_z z} = \frac{\alpha_{yo}}{M_o} B_{mo}^{0-} e^{-\alpha_{yo} h} e^{-j\beta_z z}$$

$\cos(-A) = \cos(A)$

$$\frac{\beta_{yd}}{M_d} A_{mo}^d \cos(\beta_{yd} h) = \frac{\alpha_{yo}}{M_o} B_{mo}^{0-} e^{-\alpha_{yo} h}$$

### 8.7.2 B. cont.

In order for the results of ③ + ④ to hold true on the RHS -

$$B_{mo}^{o-} = -B_{mo}^{o+} = B_{mo}^o$$

This reduces the two equations to

$$\frac{\beta_{yd}}{M_d} A_{mo}^d \cos(\beta_{yd} h) = \frac{\alpha_{yo}}{M_o} B_{mo}^o e^{-\alpha_{yo} h}$$

Divide the eq'n from ③ + ④ by the ① + ② eq'n

$$\frac{\frac{\beta_{yd}}{M_d} A_{mo}^d \cos(\beta_{yd} h)}{\frac{\beta_{yd}^2}{M_d E_d} A_{mo}^d \sin(\beta_{yd} h)} = \frac{\frac{\alpha_{yo}}{M_o} B_{mo}^o e^{-\alpha_{yo} h}}{\frac{\alpha_{yo}^2}{M_o E_o} B_{mo}^o e^{-\alpha_{yo} h}}$$

$$\leftarrow \frac{\epsilon_d}{\beta_{yd}} \cot(\beta_{yd} h) = \frac{\epsilon_o}{\alpha_{yo}}$$

$$\leftarrow \frac{\epsilon_o}{\epsilon_d} (\beta_{yd} h) + \tan(\beta_{yd} h) = \alpha_{yo} h \quad (8-159)$$

$$\text{where } \beta_{yd}^2 = \beta_d^2 - \beta_z^2 = \omega^2 M_d E_d - \beta_z^2 \quad (8-158a)$$

$$\alpha_{yo}^2 = \beta_z^2 - \beta_o^2 = \beta_z^2 - \omega^2 M_o E_o \quad (8-158b)$$

## 8.7.2 cont.

### C. Summary of $TM^2$ (Even) & $TM^2$ (Odd) Modes

$$-\frac{\epsilon_0}{\epsilon_d} (\beta_{yd} h) \cot(\beta_{yd} h) = \alpha_{yo} h \quad TM^2(\text{even}) \quad (8-160a)$$

$$\frac{\epsilon_0}{\epsilon_d} (\beta_{yd} h) \tan(\beta_{yd} h) = \alpha_{yo} h \quad TM^2(\text{odd}) \quad (8-160b)$$

$$\left. \begin{array}{l} \beta_{yd}^2 + \beta_z^2 = \beta_d^2 \Rightarrow \beta_{yd}^2 = \beta_d^2 - \beta_z^2 \\ -\alpha_{yo}^2 + \beta_z^2 = \beta_o^2 \Rightarrow \alpha_{yo}^2 = \beta_z^2 - \beta_o^2 \end{array} \right\} \quad (8-160c) \quad (8-160d)$$

$Z_w^{-yo} = -\frac{E_z^{0+}}{H_x^{0+}} = \frac{E_z^{0-}}{H_x^{0-}} = j \frac{\alpha_{yo}}{\omega \epsilon_0} \quad (8-160e)$

Objectives:

- \* find mode(s) that can be supported by dielectric slab as a waveguide.
- \* Solve for  $\beta_{yd}$ ,  $\alpha_{yo}$ ,  $\beta_z$ , and the cutoff frequencies for the mode(s).

### 8.7.2 C. cont.

For a dielectric slab waveguide, we required  $\beta_{yd}$ ,  $\alpha_{yo}$ , and  $\beta_z$  to be real and positive. Let's examine different combinations of  $\beta_0$ ,  $\beta_d$  &  $\beta_z$  to see what occurs:

1) Assume  $\beta_z < \beta_0 < \beta_d$ , then

$$\beta_{yd} = \pm \sqrt{\beta_d^2 - \beta_z^2} \quad \text{real } \cup$$

$$\alpha_{yo} = \sqrt{\beta_z^2 - \beta_0^2} = \pm j\sqrt{\beta_0^2 - \beta_z^2} \quad \text{imaginary } \cap$$

$\Rightarrow$  This combination does NOT work.

2) Assume  $\beta_z > \beta_d > \beta_0$ , then

$$\beta_{yd} = \sqrt{\beta_d^2 - \beta_z^2} = \pm j\sqrt{\beta_z^2 - \beta_d^2} \quad \text{imaginary } \cap$$

$$\alpha_{yo} = \pm \sqrt{\beta_z^2 - \beta_0^2} \quad \text{real } \cup$$

$\Rightarrow$  This combination does NOT work.

\* 3) Assume  $\beta_0 < \beta_z < \beta_d$ , then \*

$$\beta_{yd} = \pm \sqrt{\beta_d^2 - \beta_z^2} \quad \text{real } \cup \quad (8-163a)$$

$$\alpha_{yo} = \pm \sqrt{\beta_z^2 - \beta_0^2} \quad \text{real } \cup \quad (8-163b)$$

### 8.7.2 C. cont.

So, for our dielectric slab to function as a waveguide in a  $TM^2$  mode

$$\omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} = \beta_0 < \beta_z < \beta_d = \omega \sqrt{\mu_d \epsilon_d} = \frac{\omega \sqrt{\epsilon_r}}{c} \quad (8-164)$$

$(\mu_d = \mu_0)$

Cutoff occurs when  $\beta_z = \beta_0$ .

$$\begin{aligned} \beta_{yd,c} &= \sqrt{\beta_d^2 - \beta_z^2} \Big|_{\beta_z = \beta_0} \\ &= \sqrt{\omega_c^2 \mu_d \epsilon_d - \omega_c^2 \mu_0 \epsilon_0} \\ &= \pm \omega_c \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0} \quad \leftarrow \begin{array}{l} \text{choose} \\ \text{positive sol'n} \end{array} \end{aligned}$$

$$\beta_{yd,c} = \omega_c \sqrt{\mu_0 \epsilon_0} \sqrt{\mu_r \epsilon_r - 1} \quad (8-165a)$$

$$\alpha_{y0,c} = \sqrt{\beta_z^2 - \beta_0^2} \Big|_{\beta_z = \beta_0} = 0 \quad (8-165b)$$

$TM_m^2$  (even) Per (8-160a), at cutoff

$$-\frac{\epsilon_0}{\epsilon_d} (\beta_{yd} h) \cot(\beta_{yd} h) = \cancel{\alpha_{y0} h} = 0$$

$$\Rightarrow \cot(\beta_{yd} h) = \frac{\cos(\beta_{yd} h)}{\sin(\beta_{yd} h)} = 0$$

### 8.7.2 C. cont.

From trigonometry, we know that

$$\cos(\beta_{yd}h) = 0 \text{ when } \beta_{yd}h = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots = \frac{m\pi}{2} \quad (m=1, 3, 5, \dots)$$

$$\beta_{yd}h = \omega_c h \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0} = \frac{m\pi}{2} \quad \text{where } \omega_c = 2\pi f_c$$

$$\boxed{(f_c)_m = \frac{m}{4h\sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}} \quad m=1, 3, 5, \dots \\ TM_m^2 \text{ (even)}} \quad (8-166a)$$

$$\boxed{TM_m^2 \text{ (odd)}} \quad \text{Per (8-160b), at cutoff}$$

$$\frac{\epsilon_0}{\epsilon_d} (\beta_{yd}h) \tan(\beta_{yd}h) = \cancel{k_{yo}h} = 0$$

$$\Rightarrow \tan(\beta_{yd}h) = \frac{\sin(\beta_{yd}h)}{\cos(\beta_{yd}h)} = 0$$

From trigonometry, we know that

$$\sin(\beta_{yd}h) = 0 \text{ when } \beta_{yd}h = 0, \pi, 2\pi, \dots$$

$$\beta_{yd}h = \omega_c h \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0} = \frac{m\pi}{2} \quad (m=0, 2, 4, \dots)$$

$$\boxed{(f_c)_m = \frac{m}{4h\sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}} \quad m=0, 2, 4, \dots \\ TM_m^2 \text{ (odd)}} \quad (8-166b)$$

### 8.7.2 C. cont.

Observations for  $TM_m^z$  modes:

- 1)  $TM_0^z$  odd mode has  $f_{c,0} = 0$   
and is always dominant
- 2) Cutoff frequencies alternate between odd and even modes.

I.e.,  $TM_0^z$  (odd)       $f_{c,0} = 0$

$$\downarrow \quad TM_1^z \text{ (even)} \quad f_{c,1} = \frac{1}{4h\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}$$

$$\downarrow \quad TM_2^z \text{ (odd)} \quad f_{c,2} = \frac{2}{4h\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}$$

$$\downarrow \quad TM_3^z \text{ (even)} \quad f_{c,3} = \frac{3}{4h\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}$$

- 3) Note, all cutoff frequencies are integer multiples of  $f_{c,1}$ . E.g.

$$f_{c,m} = m f_{c,1} \quad (m=0, 1, 2, \dots)$$

### 8.7.2 C. cont.

ex. A slab of Teflon ( $\mu_0, 2.1 \epsilon_0$ ) is  $\frac{1}{16}$ " thick. Determine the first six modes and corresponding cutoff frequencies.

$$\text{Per (8-166a), } f_{c,1} = \frac{1}{4(\frac{1}{16})0.0254\sqrt{\mu_0\epsilon_0}\sqrt{1(2.1)-1}} \\ = 45.01431394 \text{ GHz}$$

m	Mode	$f_c$ (GHz)
0	$TM_0^2$ odd	0
1	$TM_1^2$ even	45.014
2	$TM_2^2$ odd	90.029
3	$TM_3^2$ even	135.043
4	$TM_4^2$ odd	180.057
5	$TM_5^2$ even	225.072

⇒ For a thinner substrate w/ a lower  $\epsilon_r$ , I'm unlikely to excite surface waves other than  $TM_0^2$  odd.

### 8.7.2 C. cont.

How do we find  $\beta_{yd}$ ,  $\alpha_{yo}$ , and  $\beta_z$  for a particular  $TM_m^2$  mode @ a frequency  $f$  above its cutoff frequency?

#### Method 1

- Assume  $\frac{\epsilon_0}{\epsilon_d}$ ,  $h$ , and  $f$  are known
- Apply/use (8-160a) or (8-160b) to iteratively find  $\beta_{yd}$  +  $\alpha_{yo}$  that work
- Find  $\beta_z$  using either (8-160c) or (8-160d)

#### Method 2

- Graph  $\alpha_{yo}h$  as a function of  $\beta_{yd}h$  for the applicable model(s)
- Find appropriate point along curve (covered in next section)
- Read off  $\alpha_{yo}h$  +  $\beta_{yd}h$
- Find  $\beta_z$  using (8-160c) or (8-160d)

### 8.7.2 cont.

#### D. Graphical Solution for $TM_m^2$ (Even) and $TM_m^2$ (Odd) Modes

- Define/specify  $\epsilon_0/\epsilon_d$
- Use (8-160a) for  $TM_m^2$  (even) where  $m=1, 3, 5, \dots$   
or  
(8-160b) for  $TM_m^2$  (odd) where  $m=0, 2, 4, \dots$   
to plot  $\alpha_{y0} h$  as a function of  $\beta_{yd} h$   
for mode(s) of interest.
- For each mode  $TM_m^2$ , the range of  $\beta_{yd} h_m$  will  
be  $\frac{m\pi}{2} \leq \beta_{yd} h_m < \frac{(m+1)\pi}{2}$  when we use  
(8-160a) and/or (8-160b) to calculate  $\alpha_{y0} h_m$ .

How do we find correct point(s) along the curve(s)?

Add (8-160c) and (8-160d) to get -

$$\begin{aligned}\beta_{yd}^2 &= \beta_d^2 - \beta_z^2 \\ + \underline{\alpha_{y0}^2} &= \underline{\beta_z^2 - \beta_0^2} \\ \alpha_{y0}^2 + \beta_{yd}^2 &= \beta_d^2 - \beta_0^2 = \omega^2(\mu_d \epsilon_d - \mu_0 \epsilon_0) \quad (8-167)\end{aligned}$$

### 8.7.2 D. cont.

Multiply (8-167) by  $h^2$  to get-

$$(\alpha_{yo} h)^2 + (\beta_{yd} h)^2 = (\omega h)^2 (\mu_0 \epsilon_0) (\mu_r \epsilon_r - 1) \\ = a^2 \quad (8-168)$$

This is the eq'n for a circle !!

The radius  $a$  of the circle is

$$a = \omega h \sqrt{\mu_0 \epsilon_0} \sqrt{\mu_r \epsilon_r - 1} = \beta_0 h \sqrt{\mu_r \epsilon_r - 1} \quad (8-168a)$$

→ Calculate  $a$  and draw arc on plot of  $\alpha_{yo} h$  vs  $\beta_{yd} h$  curves

→ Read off values of  $(\alpha_{yo} h)_m + (\beta_{yd} h)_m$  for model(s) where arc intersects curve(s).

→ Since  $h$  is known,  $\alpha_{yo,m} + \beta_{yd,m}$  can then be calculated for these mode(s).

→ Use (8-160c) or (8-160d) to find  $\beta_z$  for each of the mode(s).

## Recreate Example 8-11 of the text using MathCad

**Example- Polystyrene dielectric slab  $\epsilon_r = 2.56$  and  $h = 1/8"$  in air**

$$h := .125 \cdot 0.0254 \quad h = 0.003175 \quad m \quad \epsilon_r := 2.56 \quad f := 30 \cdot 10^9 \quad \text{Hz}$$

$$\mu_0 := 4 \cdot \pi \cdot 10^{-7} \quad \text{H/m} \quad \epsilon_0 := 8.8541878 \cdot 10^{-12} \quad \text{F/m} \quad \omega := 2 \cdot \pi \cdot f$$

$$\mu_d := \mu_0 \quad \epsilon_d := \epsilon_r \cdot \epsilon_0 \quad \beta_0 := \omega \cdot \sqrt{\mu_0 \cdot \epsilon_0} \quad \beta_d := \omega \cdot \sqrt{\mu_d \cdot \epsilon_d}$$

$$\beta_0 = 628.754 \quad \text{rad/m} \quad \beta_d = 1006.006 \quad \text{rad/m}$$

Use (8-166a) for even modes where  $m = 1, 3, 5 \dots$  or  $f_c(m) := \frac{m}{4 \cdot h \cdot \sqrt{\mu_d \cdot \epsilon_d - \mu_0 \cdot \epsilon_0}}$   
use (8-166b) for odd modes where  $m = 0, 2, 4 \dots$

Use (8-160b) for TM $\zeta$  odd modes-  $m = 0, 2, 4 \dots$  and  $\beta_{ydh}$  start at  $0, \pi, 2\pi \dots$

$$\alpha_{y0h\_odd}(\beta_{ydh}) := \frac{\epsilon_0}{\epsilon_d} \cdot (\beta_{ydh}) \cdot \tan(\beta_{ydh})$$

Use (8-160a) for TM $\zeta$  even modes-  $m = 1, 3, 5 \dots$  and  $\beta_{ydh}$  start at  $0.5\pi, 1.5\pi, 2.5\pi \dots$

$$\alpha_{y0h\_even}(\beta_{ydh}) := \frac{-\epsilon_0}{\epsilon_d} \cdot (\beta_{ydh}) \cdot \cot(\beta_{ydh})$$

$$n := 0..1460$$

**For TM0**       $\beta_{ydh0_n} := n \cdot 0.001$        $\alpha_{y0h0_n} := \alpha_{y0h\_odd}(\beta_{ydh0_n})$

$$f_c(0) = 0 \quad \text{Hz}$$

**For TM1**       $\beta_{ydh1_n} := n \cdot 0.001 + 0.5 \cdot \pi$        $\alpha_{y0h1_n} := \alpha_{y0h\_even}(\beta_{ydh1_n})$

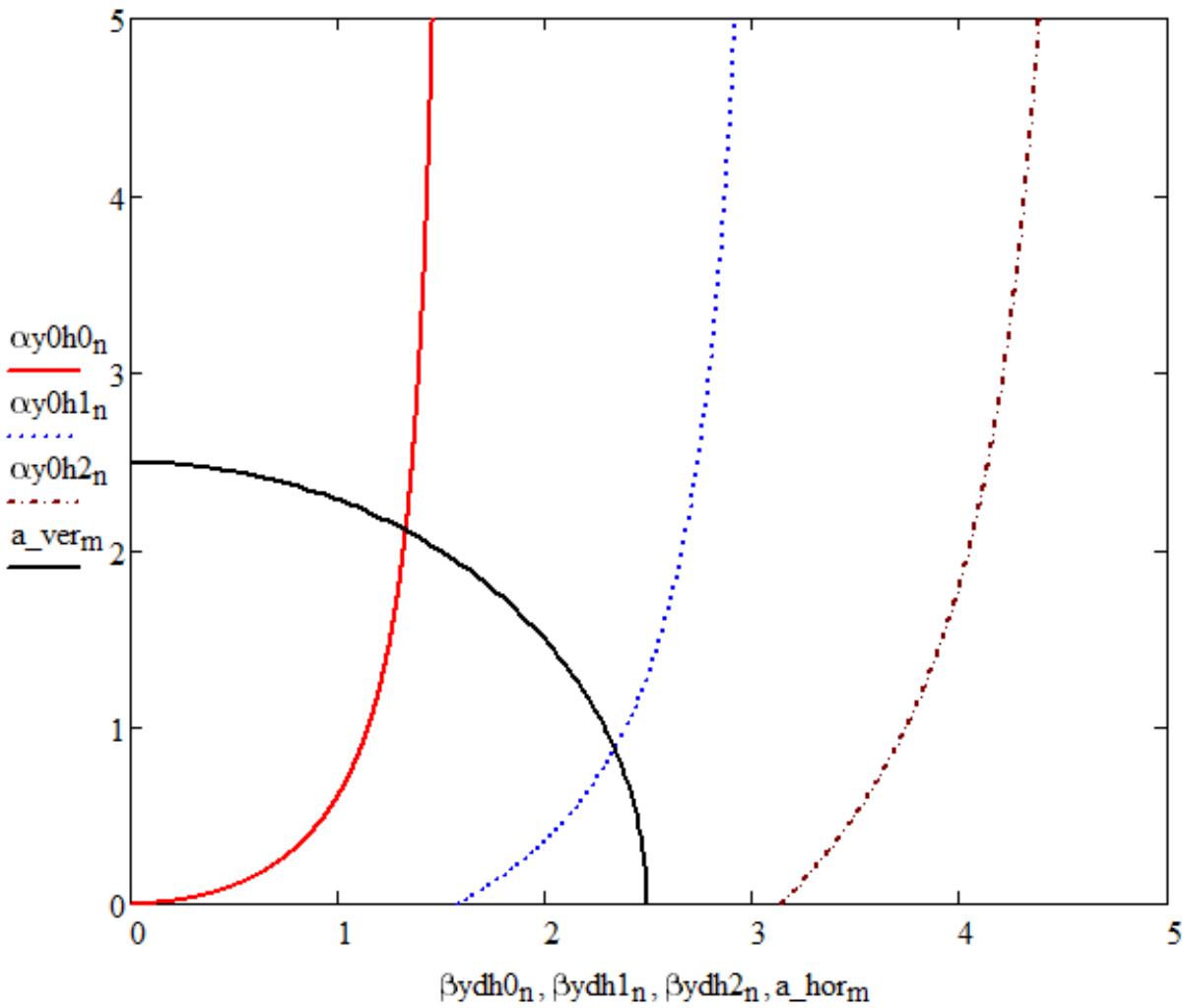
$$f_c(1) = 1.88997 \times 10^{10} \quad \text{Hz}$$

**For TM2**       $\beta_{ydh2_n} := n \cdot 0.001 + \pi$        $\alpha_{y0h2_n} := \alpha_{y0h\_odd}(\beta_{ydh2_n})$

$$f_c(2) = 3.77994 \times 10^{10} \quad \text{Hz}$$

**Per (8-168a)**       $a := \omega \cdot h \cdot \sqrt{\mu_d \cdot \epsilon_d - \mu_0 \cdot \epsilon_0}$        $a = 2.49337$

$$m := 0..90 \quad \theta_m := m \cdot \frac{\pi}{180} \quad a_{hor_m} := a \cdot \cos(\theta_m) \quad a_{ver_m} := a \cdot \sin(\theta_m)$$



Using trace function of MathCad, the intersection of arc  $a = 2.49337$  for  $f = 30\text{GHz}$  with the **TM<sub>0</sub> mode** (odd) trace is at  $\beta_{ydh} = 1.329$  and  $\alpha_{y0h} = 2.105$

$$\sqrt{1.329^2 + 2.105^2} = 2.4894$$

$$\alpha_{y0\_0} := \frac{2.105}{h} \quad \beta_{yd\_0} := \frac{1.329}{h} \quad \beta_{z\_0} := \sqrt{\beta d^2 - \beta_{yd\_0}^2}$$

$\alpha_{y0\_0} = 662.99$	Np/m	$\beta_{yd\_0} = 418.58$	rad/m	$\beta_{z\_0} = 914.79$	rad/m
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Using trace function of MathCad, the intersection of arc  $a = 2.49337$  for  $f = 30\text{GHz}$  with the **TM<sub>1</sub> mode** (even) trace is at  $\beta_{ydh} = 2.3388$  and  $\alpha_{y0h} = 0.88234$

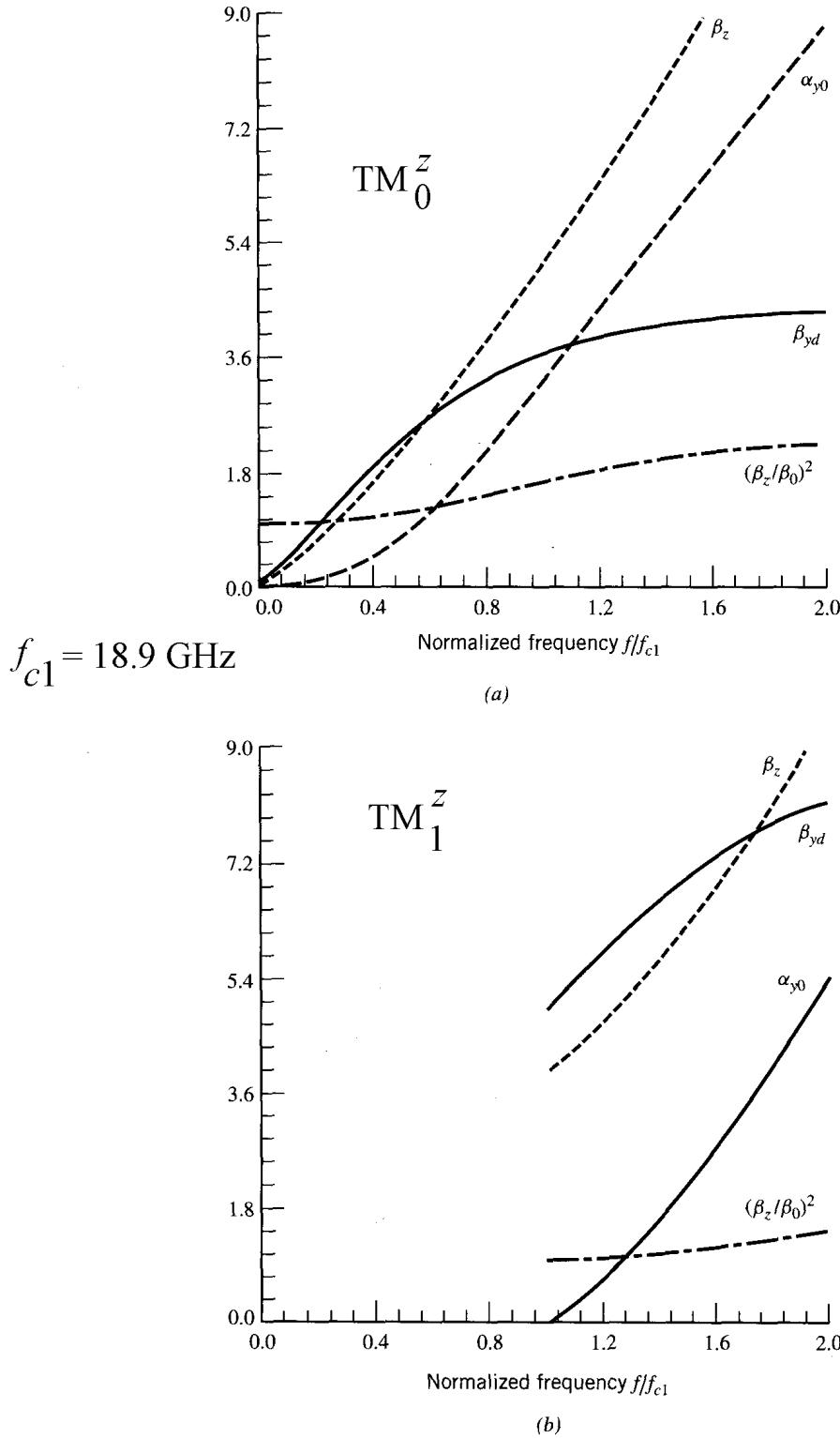
$$\sqrt{2.3388^2 + 0.88234^2} = 2.4997$$

$$\alpha_{y0\_1} := \frac{0.88234}{h} \quad \beta_{yd\_1} := \frac{2.3388}{h} \quad \beta_{z\_1} := \sqrt{\beta d^2 - \beta_{yd\_1}^2}$$

$\alpha_{y0\_1} = 277.9$	Np/m	$\beta_{yd\_1} = 736.63$	rad/m	$\beta_{z\_1} = 685.14$	rad/m
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### Example 8-11 Analytic results versus normalized frequency

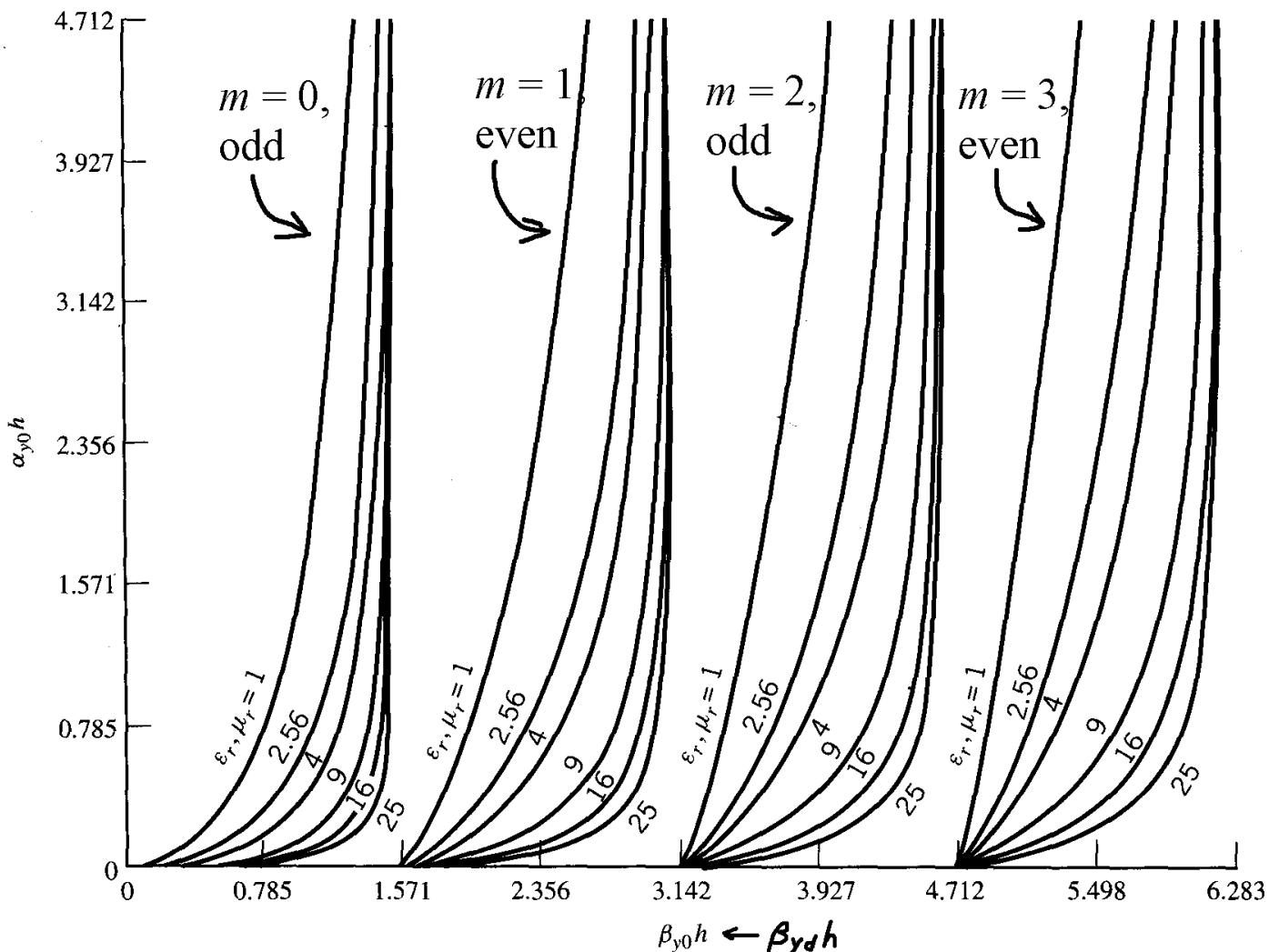
- Note that  $f = 30 \text{ GHz}$  in Example 8-11 is  $f/f_{c1} = 1.5873$  on Figure 8-23 where units are rad/cm ( $\beta_z$  &  $\beta_{yd}$ ) or Np/cm ( $\alpha_{y0}$ ).



**Figure 8-23** Attenuation and phase constants of  $\text{TM}_m^z$  modes in a dielectric slab waveguide ( $\epsilon_r = 2.56$ ,  $\mu_r = 1$ ,  $h = 0.3175 \text{ cm}$ ,  $f = 30 \text{ GHz}$ ). (a)  $\text{TM}_0^z$  mode. (b)  $\text{TM}_1^z$  mode.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

Figure 8-24 shows some trends as either  $\epsilon_r$  (TM $^z$  modes) or  $\mu_r$  (TE $^z$  modes) for the dielectric slab vary.



**Figure 8-24** Curves to be used for graphical solution of attenuation and phase constants for  $\text{TM}_m^z$  and  $\text{TE}_m^z$  modes in a dielectric slab waveguide.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

### 8.7.3 Transverse Electric (TE<sup>z</sup>) Modes

The procedure for analyzing the TE<sup>z</sup> modes for the dielectric slab waveguide follows a similar process to that for the TM<sup>z</sup> modes.

⇒ Look back to section 8.2.1 -

$$\begin{aligned} E_x &= -\frac{1}{\epsilon} \frac{\partial F_z}{\partial y} & H_x &= \frac{-j}{\omega \mu \epsilon} \frac{\partial^2 F_z}{\partial x \partial z} \\ E_y &= \frac{1}{\epsilon} \frac{\partial F_z}{\partial x} & H_y &= \frac{-j}{\omega \mu \epsilon} \frac{\partial^2 F_z}{\partial y \partial z} \\ E_z &= 0 & H_z &= \frac{-j}{\omega \mu \epsilon} \left( \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z \right) \quad (8-1) \end{aligned}$$

$$F_z = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)] [C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \times [A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}] \quad (8-6)$$

$-h \leq y \leq h$

⇒ In slab, assume no charge wrt x-dir.  
which simplifies first term of (8-6)  
to a constant

⇒ only consider waves in +z-direction

$$F_z^d = [C_2^d \cos(\beta_y d) + D_2^d \sin(\beta_y d)] A_3^d e^{-j\beta_z z}$$

$\uparrow$  even modes       $\uparrow$  odd modes

### 8.7.3 cont.

$$\text{Define } F_{2e}^d = C_2^d A_3^d \cos(\beta_{yd} y) e^{-j\beta_z z}$$

$$F_{2o}^d = D_2^d A_3^d \sin(\beta_{yd} y) e^{-j\beta_z z}$$

$$\text{w/ } \beta_{yd}^2 + \beta_z^2 = \beta_d^2 = \omega^2 \mu_d \epsilon_d$$

$\beta_{yd}$  &  $\beta_z$  are positive and real.

$y \geq h$  (above slab)

$$F_2^{0+} = (A_{2e}^{0+} e^{-j\beta_{y0} y} + B_{2o}^{0+} e^{-j\beta_{y0} y}) A_{3o}^{0+} e^{-j\beta_z z}$$

$\xrightarrow{e^{-\alpha_{y0} y}}$        $\xleftarrow{e^{-\alpha_{y0} y}} \text{(evanescent)}$

$$F_{2e}^{0+} = A_{me}^{0+} e^{-\alpha_{y0} y} e^{-j\beta_z z} \text{ (even)}$$

$$F_{2o}^{0+} = B_{mo}^{0+} e^{-\alpha_{y0} y} e^{-j\beta_z z} \text{ (odd)}$$

$$\text{w/ } -\alpha_{y0}^2 + \beta_z^2 = \beta_o^2 = \omega^2 \sqrt{\mu_0 \epsilon_0}$$

$\Rightarrow \alpha_{y0}$  is positive and real!

### 8.7.3 cont.

$y \leq -h$  (below slab)

$$F_z^{0-} = (A_{2e}^{0-} e^{+j\beta_{y0}y} + B_{2o}^{0-} e^{+j\beta_{y0}y}) A_{3o}^{0-} e^{-j\beta_z z}$$

$\downarrow e^{+\alpha_{y0}y}$        $\downarrow e^{+\alpha_{y0}y}$  (evanescent)

$$F_{ze}^{0-} = A_{me}^{0-} e^{+\alpha_{y0}y} e^{-j\beta_z z} \quad (\text{even})$$

$$F_{zo}^{0-} = B_{mo}^{0-} e^{+\alpha_{y0}y} e^{-j\beta_z z} \quad (\text{odd})$$

$$\text{w/ } -\alpha_{y0}^2 + \beta_z^2 = \beta_o^2 = \omega^2 \sqrt{\mu_0 \epsilon_0}$$

The same boundary conditions apply

$$E_z^d(y=h, z) = E_z^{0+}(y=h, z)$$

$$E_z^d(y=-h, z) = E_z^{0-}(y=-h, z)$$

$$H_x^d(y=h, z) = H_x^{0+}(y=h, z)$$

$$H_x^d(y=-h, z) = H_x^{0-}(y=-h, z)$$

Again, it is convenient to apply these to even & odd modes separately.

### 8.7.3 cont.

Skipping the intermediate steps, we get

$$-\frac{\mu_0}{\mu_d} (\beta_{yd} h) \cot(\beta_{yd} h) = \alpha_{y0} h \quad TE^2 \text{ (even)} \quad (8-169a)$$

$$\frac{\mu_0}{\mu_d} (\beta_{yd} h) \tan(\beta_{yd} h) = \alpha_{y0} h \quad TE^2 \text{ (odd)} \quad (8-169b)$$

$$\beta_{yd}^2 + \beta_z^2 = \beta_d^2 = \omega^2 \mu_d \epsilon_d \Rightarrow \beta_{yd}^2 = \beta_d^2 - \beta_z^2 \quad (8-169c)$$

for both  $TE^2$  (even + odd) in slab

$$-\alpha_{y0}^2 + \beta_z^2 = \beta_0^2 = \omega^2 \mu_0 \epsilon_0 \Rightarrow \alpha_{y0}^2 = \beta_z^2 - \beta_0^2 \quad (8-169d)$$

for both  $TE^2$  (even + odd) outside slab

$$f_{c,m} = \frac{m}{4h\sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}} \quad m=0,2,4,\dots \text{ (odd)} \quad (8-169e)$$

$$m=1,3,5,\dots \text{ (even)} \quad (8-169f)$$

Note:  $\beta_{yd} h = \frac{m\pi}{2}$  ( $m=0,2,4,\dots$ ) for odd modes

$\beta_{yd} h = \frac{m\pi}{2}$  ( $m=1,3,5,\dots$ ) for even modes

Since  $\mu_d = \mu_0$  in almost all practical cases, (8-169a) and (8-169b) will be slightly simpler than the  $TM^2$  eqns w/  $\epsilon_0/\epsilon_d$  terms (8-160a) & (8-160b).

### 8.7.3 cont.

For the  $TE^2$  modes, the wave impedance

$$Z_w^{-y0} = \frac{E_x^{0+}}{H_z^{0+}} = -\frac{E_x^{0-}}{H_z^{0-}} = -j \frac{\omega_{yo}}{\alpha_{yo}} \quad (8-169_i)$$

for both even & odd modes.

→ TE Surface waves are supported by capacitive surfaces w/  
implications for metamaterials.

The solution techniques are very similar.

#### Method 1 (Analytic)

→ Know  $\mu_d$  &  $\epsilon_d$ ,  $h$ , and  $f$ .

→ Apply/use (8-169a) &/or (8-169b)

to iteratively find  $\beta_{yo}$  &  $\alpha_{yo}$   
that work for modes of interest  
(need  $f$  above  $f_{c,m}$ )

→ Find  $\beta_z$  using (8-169c) or (8-169d)

### 8.7.3 cont.

#### Method 2 (Graphical)

- Know  $M_d$  &  $\epsilon_d$ ,  $h$ , and  $f$ .
- Use (8-169a) for  $TE^2$  even modes  
where  $m=1, 3, 5, \dots$  &  $\beta_{yd} h_{,m} = \frac{m\pi}{2}$

or

- use (8-169b) for  $TE^2$  odd modes  
where  $m=0, 2, 4, \dots$  &  $\beta_{yd} h_m = \frac{m\pi}{2}$   
to plot  $\alpha_{yo} h_m$  versus  $\beta_{yd} h_m$   
for each mode of interest

Note: For each mode the range of  $\beta_{yd} h_m$  will be

$$\frac{m\pi}{2} \leq \beta_{yd} h_m < \frac{(m+1)\pi}{2}$$

$\uparrow$   $\uparrow$   
 cut-off  $\alpha_{yo} h_m = 0$   $\alpha_{yo} h_m \rightarrow \infty$

Make sure  $\alpha_{yo} h$  and  $\beta_{yd} h$  scales  
are the same!! I.e., square plots.

### 8.7.3 cont.

#### Method 2 cont.

→ To find the correct point(s) along the curves of  $\alpha_{y0h}$  vs  $\beta_{yd}h$ , use

$$(\alpha_{y0h})^2 + (\beta_{yd}h)^2 = a^2 = \text{constant} \quad (8-169h)$$

$$\begin{aligned} \text{where } a &= wh\sqrt{\mu_0\epsilon_0\sqrt{\mu_r\epsilon_r - 1}} = wh\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0} \\ &= \beta_0 h \sqrt{\mu_r\epsilon_r - 1} \end{aligned} \quad (8-169i)$$

is the radius of an arc on the plots

→ Read off value(s) of  $\alpha_{y0h}$  &  $\beta_{yd}h$  where the arc intersects the  $\alpha_{y0h}$  vs  $\beta_{yd}h$  trace(s) for the various modes of interest (i.e., the modes where  $f \geq f_{c,m}$ ).

→ Calculate  $\alpha_{y0,m} = \frac{\alpha_{y0h_m}}{h}$  and  $\beta_{yd} = \frac{\beta_{yd}h_m}{h}$

→ USE (8-169c) or (8-169d) to find  $\beta_{z,m}$ .

## Recreate Example 8-12 of the text using MathCad

**Example- Polystyrene dielectric slab  $\epsilon_r = 2.56$ ,  $\mu_r = 1$ , and  $h = 1/8"$  in air**

$$h := .125 \cdot 0.0254 \quad h = 0.003175 \text{ m} \quad \epsilon_r := 2.56 \quad \mu_r := 1 \quad f := 30 \cdot 10^9 \text{ Hz}$$

$$\mu_0 := 4 \cdot \pi \cdot 10^{-7} \text{ H/m} \quad \epsilon_0 := 8.8541878 \cdot 10^{-12} \text{ F/m} \quad \omega := 2 \cdot \pi \cdot f$$

$$\mu_d := \mu_r \cdot \mu_0 \quad \epsilon_d := \epsilon_r \cdot \epsilon_0 \quad \beta_0 := \omega \cdot \sqrt{\mu_0 \cdot \epsilon_0} \quad \beta_d := \omega \cdot \sqrt{\mu_d \cdot \epsilon_d}$$

$$\beta_0 = 628.754 \text{ rad/m} \quad \beta_d = 1006.006 \text{ rad/m}$$

Use (8-169e) for even modes where  $m = 1, 3, 5 \dots$  or  
use (8-169f) for odd modes where  $m = 0, 2, 4 \dots$   $f_c(m) := \frac{m}{4 \cdot h \cdot \sqrt{\mu_d \cdot \epsilon_d - \mu_0 \cdot \epsilon_0}}$

**Use (8-169b) for TE<sup>z</sup> odd modes-  $m = 0, 2, 4 \dots$  and  $\beta_{ydh}$  start at 0,  $\pi$ ,  $2\pi \dots$**

$$\alpha_{y0h\_odd}(\beta_{ydh}) := \frac{\mu_0}{\mu_d} \cdot (\beta_{ydh}) \cdot \tan(\beta_{ydh})$$

**Use (8-169a) for TE<sup>z</sup> even modes-  $m = 1, 3, 5 \dots$  and  $\beta_{ydh}$  start at  $0.5\pi, 1.5\pi, 2.5\pi \dots$**

$$\alpha_{y0h\_even}(\beta_{ydh}) := \frac{-\mu_0}{\mu_d} \cdot (\beta_{ydh}) \cdot \cot(\beta_{ydh})$$

$$n := 0..1460$$

**For TE<sub>0</sub>**  $\beta_{ydh0_n} := n \cdot 0.001$   $\alpha_{y0h0_n} := \alpha_{y0h\_odd}(\beta_{ydh0_n})$

$$f_c(0) = 0 \text{ Hz}$$

**For TE<sub>1</sub>**  $\beta_{ydh1_n} := n \cdot 0.001 + 0.5 \cdot \pi$   $\alpha_{y0h1_n} := \alpha_{y0h\_even}(\beta_{ydh1_n})$

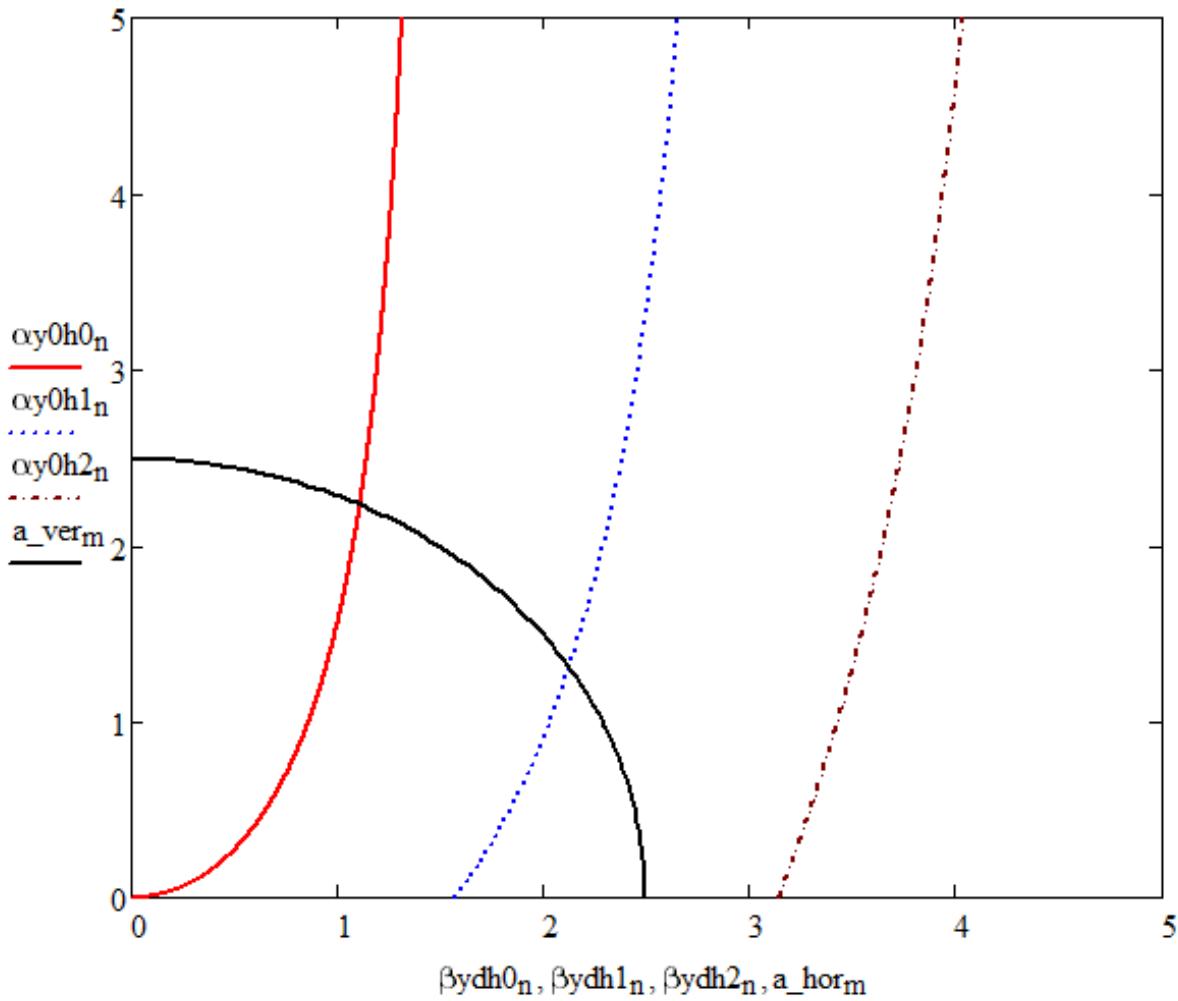
$$f_c(1) = 1.88997 \times 10^{10} \text{ Hz}$$

**For TE<sub>2</sub>**  $\beta_{ydh2_n} := n \cdot 0.001 + \pi$   $\alpha_{y0h2_n} := \alpha_{y0h\_odd}(\beta_{ydh2_n})$

$$f_c(2) = 3.77994 \times 10^{10} \text{ Hz}$$

**Per (8-169h)**  $a := \omega \cdot h \cdot \sqrt{\mu_d \cdot \epsilon_d - \mu_0 \cdot \epsilon_0}$  a = 2.49337

$$m := 0..90 \quad \theta_m := m \cdot \frac{\pi}{180} \quad a\_hor_m := a \cdot \cos(\theta_m) \quad a\_ver_m := a \cdot \sin(\theta_m)$$



Using trace function of MathCad, the intersection of arc  $a = 2.49337$  for  $f = 30\text{GHz}$  with the **TE<sub>0</sub> mode** (odd) trace is at  $\beta_{ydh} = 1.11$  and  $\alpha_{y0h} = 2.2359$ .

$$\sqrt{1.11^2 + 2.2359^2} = 2.4963$$

$$\alpha_{y0\_0} := \frac{2.2359}{h}$$

$$\beta_{yd\_0} := \frac{1.11}{h}$$

$$\beta_{z\_0} := \sqrt{\beta d^2 - \beta_{yd\_0}^2}$$

$$\boxed{\alpha_{y0\_0} = 704.22} \quad \text{Np/m}$$

$$\boxed{\beta_{yd\_0} = 349.61} \quad \text{rad/m}$$

$$\boxed{\beta_{z\_0} = 943.3} \quad \text{rad/m}$$

Using trace function of MathCad, the intersection of arc  $a = 2.49337$  for  $f = 30\text{GHz}$  with the **TE<sub>1</sub> mode** (even) trace is at  $\beta_{ydh} = 2.1228$  and  $\alpha_{y0h} = 1.3073$ .

$$\sqrt{2.1228^2 + 1.3073^2} = 2.4931$$

$$\alpha_{y0\_1} := \frac{1.3073}{h}$$

$$\beta_{yd\_1} := \frac{2.1228}{h}$$

$$\beta_{z\_1} := \sqrt{\beta d^2 - \beta_{yd\_1}^2}$$

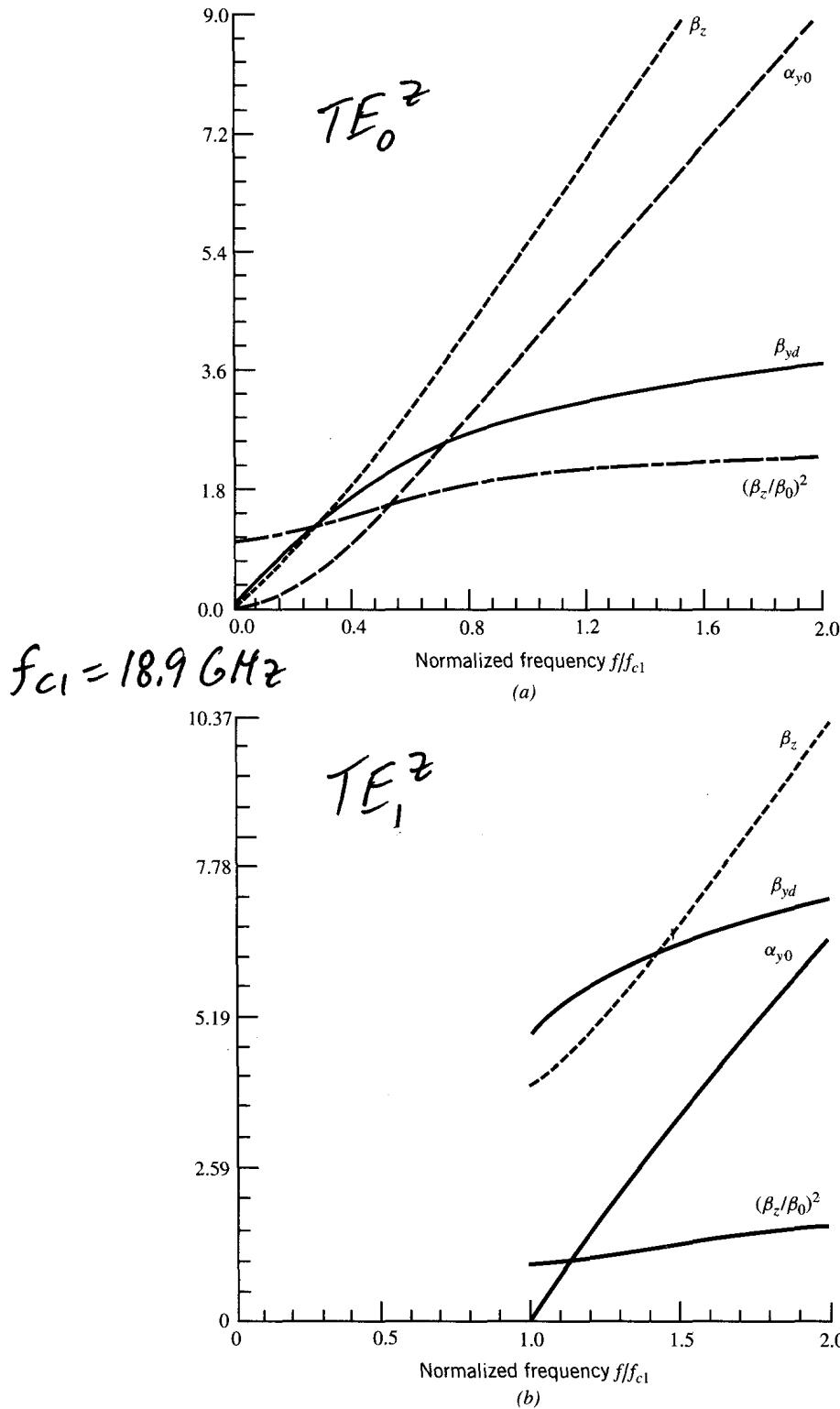
$$\boxed{\alpha_{y0\_1} = 411.75} \quad \text{Np/m}$$

$$\boxed{\beta_{yd\_1} = 668.6} \quad \text{rad/m}$$

$$\boxed{\beta_{z\_1} = 751.68} \quad \text{rad/m}$$

### Example 8-12 Analytic results versus normalized frequency

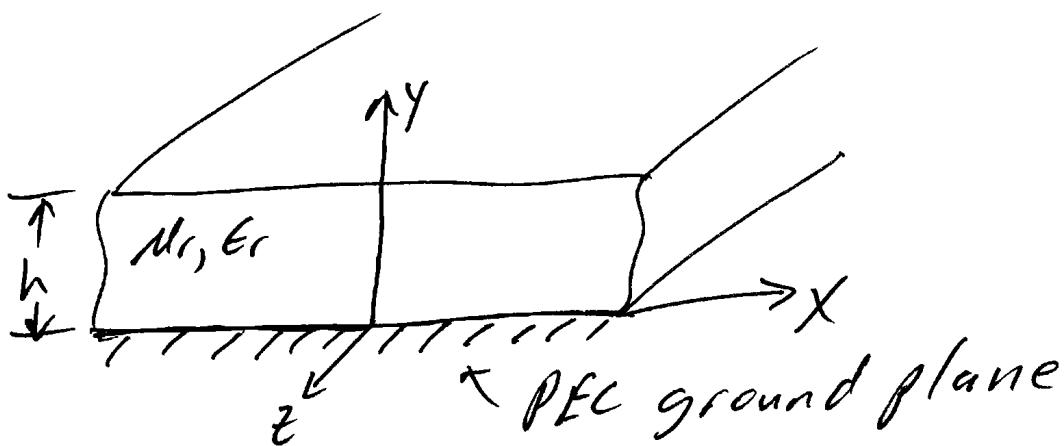
- Note that  $f = 30 \text{ GHz}$  in Example 8-12 is  $f/f_{c1} = 1.5873$  on Figure 8-23 where units are rad/cm ( $\beta_z$  &  $\beta_{yd}$ ) or Np/cm ( $\alpha_{y0}$ ).



**Figure 8-26** Attenuation and phase constants of  $\text{TE}_m^z$  modes in a dielectric slab waveguide ( $\epsilon_r = 2.56$ ,  $\mu_r = 1$ ,  $h = 0.3175 \text{ cm}$ ,  $f = 30 \text{ GHz}$ ). (a)  $\text{TE}_0^z$  mode. (b)  $\text{TE}_1^z$  mode.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

## 8.7.5 Dielectric-Covered Ground Plane



Note: We are now going from

$0 \leq y \leq h$  for our dielectric  
unlike the dielectric slab of  
Sections 8.7.1 – 8.7.4 where  
 $-h \leq y \leq h$  for the dielectric

$\Rightarrow$  Mode & field analysis follow that  
of dielectric slab in earlier sections.

$\Rightarrow$  Now  $\bar{E}_{tan}(y=0, z) = 0$

$$\bar{E}_{tan}^d(y=h, z) = E_{tan}^{ot}(y=h, z)$$

$$\bar{H}_{tan}^d(y=h, z) = H_{tan}^{ot}(y=h, z)$$

$\Rightarrow$  No fields for  $y \leq 0$ !

8.7.5 cont.

Following earlier work

$$\underline{TM^z \text{ (even)}} \quad |y| \leq h$$

$$E_{xc}^d = 0$$

$$E_{ye}^d = \frac{\beta_{yd}\beta_z}{\omega_{yd}E_d} A_{me}^d \sin(\beta_{yd}y) e^{-j\beta_z z}$$

$$E_{ze}^d = -j \frac{\beta_d^2 - \beta_z^2}{\omega_{yd}E_d} A_{me}^d \cos(\beta_{yd}y) e^{-j\beta_z z} \quad (8-177a)$$

$\Rightarrow$  NOT possible since  $E_{ze}^d(y=0) \neq 0$

due to  $\cos(\beta_{yd}y)|_{y=0}$  = 1 term.

$$\underline{TM^z \text{ (odd)}} \quad |y| \leq h$$

$$E_{xo}^d = 0$$

$$E_{yo}^d = \frac{-\beta_{yd}\beta_z}{\omega_{yd}E_d} A_{mo}^d \cos(\beta_{yd}y) e^{-j\beta_z z}$$

$$E_{zo}^d = -j \frac{\beta_d^2 - \beta_z^2}{\omega_{yd}E_d} A_{mo}^d \sin(\beta_{yd}y) e^{-j\beta_z z} \quad (8-177b)$$

$\Rightarrow$  This mode is possible since  $E_{zo}^d$  term

$\sin(\beta_{yd}y)|_{y=0} = 0$  satisfies  $E_{zo}^d(0,z) = 0$  boundary condition.

8.7.5 cont. $TE^2$  (even)     $|y| \leq h$ 

$$E_{xe}^d = \frac{\beta_{yd}}{\epsilon_d} B_{me}^d \sin(\beta_{yd}y) e^{-j\beta_z z}$$

$$E_{ye}^d = 0$$

$$E_{ze}^d = 0$$

(8-177c)

$\Rightarrow$  This mode is possible since  $E_{ze}^d = 0$

and  $E_{xe}^d = 0 @ y=0$  ( $\sin(\beta_{yd}y)|_{y=0} = 0$ ).

 $TE^2$  (odd)     $|y| \leq h$ 

$$E_{x0}^d = -\frac{\beta_{yd}}{\epsilon_d} B_{mo}^d \cos(\beta_{yd}y) e^{-j\beta_z z}$$

$$E_{yo}^d = 0$$

$$E_{zo}^d = 0$$

$\Rightarrow$  This mode NOT possible since

$E_{x0}^d(y=0) \neq 0$  due to  $\cos(\beta_{yd}y)|_{y=0} = 1$   
term.

ONLY TM<sup>2</sup> (odd) & TE<sup>2</sup> (even) modes

### 8.7.5 cont.

The controlling equations for our dielectric-covered ground plane surface waves are listed below.

$$\frac{\epsilon_0}{\epsilon_d} (\beta_{yd} h) \tan(\beta_{yd} h) = \alpha_{yo} h \quad TM^2 \text{ odd} \quad (8-178a)$$

$$-\frac{\mu_0}{\mu_d} (\beta_{yd} h) \cot(\beta_{yd} h) = \alpha_{yo} h \quad TE^2 \text{ even} \quad (8-178b)$$

$$\beta_{yd}^2 + \beta_z^2 = \beta_d^2 = \omega^2 \mu_d \epsilon_d \quad \text{Both} \quad (8-178c)$$

$$-\alpha_{yo}^2 + \beta_z^2 = \beta_o^2 = \omega^2 \mu_0 \epsilon_0 \quad \text{Both} \quad (8-178d)$$

$$f_{c,m} = \frac{m}{4h\sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}} \quad (8-178e)/(8-178f)$$

where  $m=0, 2, 4, \dots$  ( $\beta_{yd} h = 0, \pi, \dots$ )  $TM^2 \text{ odd}$

and  $m=1, 3, 5, \dots$  ( $\beta_{yd} h = 0.5\pi, 1.5\pi, \dots$ )  $TE^2 \text{ even}$

$\Rightarrow$  Note:  $TM_0^2$  is dominant w/  $f_{c,0} = 0$

$\Rightarrow$  Can use similar analytic & graphical techniques to find  $\alpha_{yo}$ ,  $\beta_{yd}$ , and  $\beta_z$ .

### 8.7.5 cont.

$$\text{Above cutoff } \alpha_{yo} = \sqrt{\beta_z^2 - \beta_0^2} = \sqrt{\beta_z^2 - \omega^2 M_0 E_0} \quad (8-17g)$$

If  $h \rightarrow \text{big}$ ,  $\beta_z \rightarrow \beta_d$  which means

$$\begin{aligned} \alpha_{yo}|_{h \rightarrow \text{big}} &= \sqrt{\omega^2 M_d E_d - \omega^2 M_0 E_0} \\ &= \omega \sqrt{M_0 E_0} \sqrt{\frac{M_d E_d}{M_0 E_0} - 1} \end{aligned} \quad (8-180)$$

which will be large (tightly bound).

If  $h \rightarrow \text{small}$ ,  $\beta_z \rightarrow \beta_0$  which means

$$\begin{aligned} \beta_{yd}^2|_{h \rightarrow \text{small}} &\approx \beta_d^2 - \beta_0^2 = \omega^2 (M_d E_d - M_0 E_0) \\ &= \omega^2 M_0 E_0 \left( \frac{M_d E_d}{M_0 E_0} - 1 \right) \end{aligned} \quad (8-181a)$$

and

$$\alpha_{yo}|_{h \rightarrow \text{small}} = \beta_z^2 - \beta_0^2 \rightarrow \text{small} \quad (8-181b)$$

(loosely bound).

$$\begin{aligned} \alpha_{yo}|_{h \rightarrow \text{small}} &= \beta_{yd} \frac{E_0}{E_d} \tan(\beta_{yd} h) \\ &\quad \xrightarrow{\beta_{yd} h} \\ &= h \frac{E_0}{E_d} (\beta_{yd})^2 \end{aligned} \quad (8-182)$$

### 8.7.5 cont.

Using (8-181a) expression for  $\beta \gamma d^2/h \rightarrow \text{small}$  in (8-182) gives

$$\begin{aligned} \omega_{y0/h \rightarrow \text{small}} &\approx h \frac{\epsilon_0}{\epsilon_d} \left[ \omega_{M0\epsilon_0}^2 \left( \frac{M_d \epsilon_d}{M_0 \epsilon_0} - 1 \right) \right] \\ &= h \beta_0^2 \left( \frac{M_d}{M_0} - \frac{\epsilon_0}{\epsilon_d} \right) \end{aligned}$$

$$= 2\pi \beta_0 \left( \frac{M_d}{M_0} - \frac{\epsilon_0}{\epsilon_d} \right) \frac{h}{\lambda_0} \quad (8-182a)$$

$\Rightarrow$  very small since  $h/\lambda_0 \ll 1$

Ex. Rogers Corp. RO4003C substrate

$\epsilon_r = 3.55$ ,  $h = 0.813 \text{ mm}$ . Find first 3 cutoff frequencies

$$TM_0^2 \Rightarrow f_{c,0} = \underline{\underline{0}}$$

$$TE_1^2 \Rightarrow f_{c,1} = \frac{1}{4(0.813 \times 10^{-3}) \sqrt{M_0 \epsilon_0} \sqrt{3.55(1) - 1}} \\ = \underline{\underline{57.73 \text{ GHz}}}$$

$$TM_2^2 \Rightarrow f_{c,2} = 2f_{c,1} = \underline{\underline{115.46 \text{ GHz}}}$$

Not too worried about higher modes.