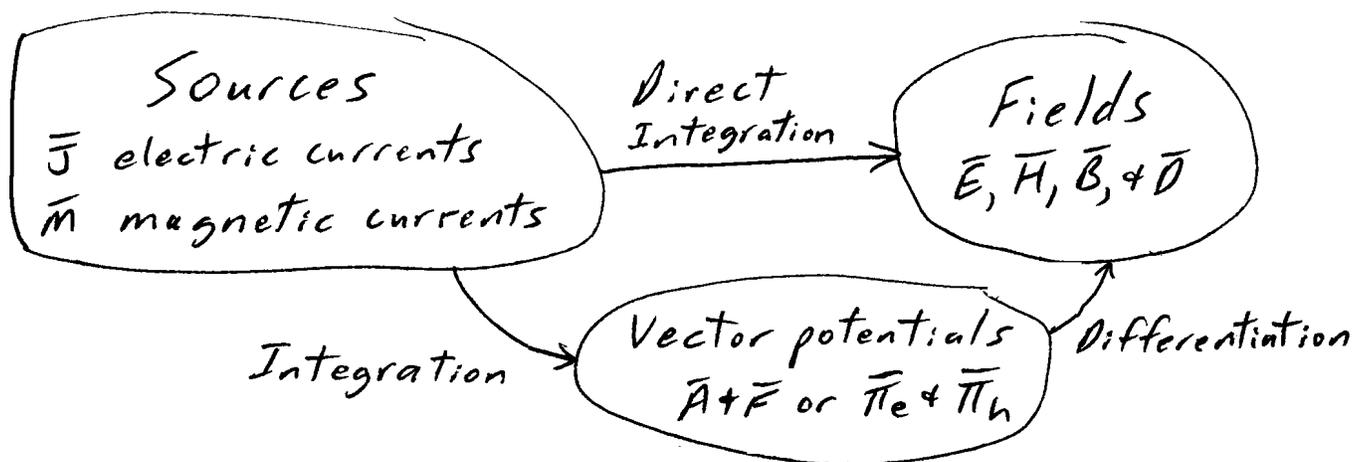


# Chapter 6 Auxiliary Vector Potentials, ...

## 6.1 Introduction

- \* Often antenna and boundary-value EM problems are made easier by using auxiliary vector potentials to find the electric ( $\vec{E}$ ) and magnetic ( $\vec{H}$ ) fields.
  - \* Most commonly used are the vector magnetic potential  $\vec{A}$  ( $\frac{Wb}{m}$ ) and the vector electric potential  $\vec{F}$  ( $\frac{V}{m}$ ).
  - \* Another pair of auxiliary vector potentials are the Hertz potentials  $\vec{\Pi}_e$  and  $\vec{\Pi}_h$ .
- Note,  $\vec{A}$ ,  $\vec{F}$ ,  $\vec{\Pi}_e$ , &  $\vec{\Pi}_h$  are mathematical tools and do not represent physically measurable quantities.



6.1 cont.

Maxwell's equations for homogenous medium

$$\text{Faraday} \quad \bar{\nabla} \times \bar{E} = -\bar{M} - j\omega\mu\bar{H} \quad (6-1a)$$

$$\text{Ampere} \quad \bar{\nabla} \times \bar{H} = \bar{J} + j\omega\epsilon\bar{E} \quad (6-1b)$$

$$\text{Gauss} \quad \bar{\nabla} \cdot \bar{E} = \frac{q_{ev}}{\epsilon} \quad (6-1c)$$

$$\bar{\nabla} \cdot \bar{H} = \frac{q_{mv}}{\mu} \quad (6-1d)$$

which lead to the vector wave equations

$$\bar{\nabla}^2 \bar{E} + \beta^2 \bar{E} = \bar{\nabla} \times \bar{M} + j\omega\mu\bar{J} + \frac{1}{\epsilon} \bar{\nabla} q_{ev} \quad (6-2a)$$

$$\bar{\nabla}^2 \bar{H} + \beta^2 \bar{H} = -\bar{\nabla} \times \bar{J} + j\omega\epsilon\bar{M} + \frac{1}{\mu} \bar{\nabla} q_{mv} \quad (6-2b)$$

$$\text{where} \quad \beta^2 = \omega^2 \mu \epsilon \quad (6-2c)$$

$\bar{J} \equiv$  electric current density ( $A/m^2$ )

\* can be real or equivalent sources

$\bar{M} \equiv$  magnetic current density ( $V/m^2$ )

\* only equivalent sources

## 6.2 The Vector Potential $\bar{A}$

In a source free region (e.g.,  $\rho_{mv} = 0$ ),

$$\bar{\nabla} \cdot \bar{B} = \bar{\nabla} \cdot \mu \bar{H} = 0.$$

By the vector identity  $\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0$ , we

can define  $\bar{B}_A = \mu \bar{H}_A = \bar{\nabla} \times \bar{A}$  (6-4)  $\star$

(subscript 'A' implies that this is the portion of  $\bar{B} + \bar{H}$  due to  $\bar{A}$ )

By Faraday's Law (w/  $\bar{M} = 0$ ), we get

$$\bar{\nabla} \times \bar{E}_A = -j\omega \mu \bar{H}_A = -j\omega \bar{\nabla} \times \bar{A} \quad (6-6)$$

$$\hookrightarrow \bar{\nabla} \times (\bar{E}_A + j\omega \bar{A}) = 0 \quad (6-7)$$

Using the vector identity  $\bar{\nabla} \times (-\bar{\nabla} \phi_e) = 0$ ,

we get

$$\bar{E}_A = -\bar{\nabla} \phi_e - j\omega \bar{A} \quad (6-9a)$$

Also, from Ampere's Law (w/  $\bar{J} = 0$ ), we

$$\text{get} \quad \bar{E}_A = \frac{1}{j\omega \epsilon} \bar{\nabla} \times \bar{H}_A \quad \star$$

6.2 cont.

In addition, we can use the vector identity

$$\bar{\nabla} \times \bar{\nabla} \times \bar{A} = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla}^2 \bar{A} \quad \text{on (6-4) to get}$$

$$\bar{\nabla} \times \mu \bar{H}_A = \bar{\nabla} \times \bar{B}_A = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla}^2 \bar{A} \quad (6-11)$$

For a homogenous medium

$$\mu(\bar{\nabla} \times \bar{H}_A) = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla}^2 \bar{A}$$

and plugging in Ampere's Law gives

$$\mu \bar{J} + j\omega \mu \epsilon \bar{E} = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla}^2 \bar{A}$$

using (6-9a) +  $\beta^2 = \omega^2 \mu \epsilon$  leads to

$$\begin{aligned} \bar{\nabla}^2 \bar{A} + \beta^2 \bar{A} &= -\mu \bar{J} + \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) + \bar{\nabla}(j\omega \mu \epsilon \phi_e) \\ &= -\mu \bar{J} + \bar{\nabla}[\bar{\nabla} \cdot \bar{A} + j\omega \mu \epsilon \phi_e] \quad (6-14) \end{aligned}$$

Now, we defined the curl of  $\bar{A}$ , but still have the freedom to define the divergence of  $\bar{A}$ .

To simplify (6-14), choose

Lorentz gauge condition  $\bar{\nabla} \cdot \bar{A} = -j\omega \mu \epsilon \phi_e \Rightarrow \phi_e = \frac{-1}{j\omega \mu \epsilon} \bar{\nabla} \cdot \bar{A} \quad (6-15)$

$$\text{Now, } \bar{\nabla}^2 \bar{A} + \beta^2 \bar{A} = -\mu \bar{J} \quad (6-16)$$

$$\text{and } \bar{E}_A = -\bar{\nabla} \phi_e - j\omega \bar{A} = -j\omega \bar{A} - j\frac{1}{\omega \mu \epsilon} \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) \quad (6-17)$$

## 6.3 The Vector Potential $\bar{F}$

In a source-free region (e.g.,  $\rho_{cv} = 0$ ),

$$\bar{\nabla} \cdot \bar{D} = \bar{\nabla} \cdot \epsilon \bar{E} = 0.$$

Using the vector identity  $\bar{\nabla} \cdot (-\bar{\nabla} \times \bar{F}) = 0$ ,

$$\text{define } \bar{D}_F = -\bar{\nabla} \times \bar{F} \quad (6-19)$$

$$\hookrightarrow \bar{E}_F = -\frac{1}{\epsilon} \bar{\nabla} \times \bar{F} \quad (6-19a) \quad \star$$

where the subscript 'F' implies that this is the part of  $\bar{D} + \bar{E}$  due to  $\bar{F}$ .

By Ampere's Law (w/  $\bar{J} = 0$ ), we get

$$\bar{\nabla} \times \bar{H}_F = j\omega \epsilon \bar{E}_F = j\omega \epsilon \left[ -\frac{1}{\epsilon} \bar{\nabla} \times \bar{F} \right]$$

$$\hookrightarrow \bar{\nabla} \times [\bar{H}_F + j\omega \bar{F}] = 0 \quad (6-21)$$

Using the vector identity  $\bar{\nabla} \times (-\bar{\nabla} \phi_m) = 0$ ,

$$\text{we get } \bar{H}_F = -\bar{\nabla} \phi_m - j\omega \bar{F} \quad (6-22)$$

Also, from Faraday's Law (w/  $\bar{m} = 0$ )

$$\bar{H}_F = \frac{-1}{j\omega \mu} \bar{\nabla} \times \bar{E}_F \quad \star$$

6.3 cont.

Also, we can use the vector identity

$$\nabla \times \nabla \times \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F} \text{ on (6-19a) to get}$$

$$\nabla \times \bar{E}_F = -\frac{1}{\epsilon} \nabla \times \nabla \times \bar{F} = -\frac{1}{\epsilon} [\nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}]$$

Plugging in Faraday's Law gives

$$-\bar{m} - j\omega\mu\bar{H}_F = -\frac{1}{\epsilon} [\nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}].$$

Then, using (6-22) &  $\beta^2 = \omega^2\mu\epsilon$ , we get

$$\nabla^2 \bar{F} + \beta^2 \bar{F} = -\epsilon\bar{m} + \nabla(\nabla \cdot \bar{F} + j\omega\mu\epsilon\phi_m) \quad (6-26)$$

The divergence of  $\bar{F}$  has not been defined.

To simplify (6-26), choose

$$\nabla \cdot \bar{F} = -j\omega\mu\epsilon\phi_m \Rightarrow \phi_m = \frac{-1}{j\omega\mu\epsilon} \nabla \cdot \bar{F} \quad (6-27)$$

to reduce (6-26) to

$$\nabla^2 \bar{F} + \beta^2 \bar{F} = -\epsilon\bar{m} \quad (6-28)$$

and (6-22) to

$$\bar{H}_F = -j\omega\bar{F} - \frac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \bar{F}) \quad (6-29) \star$$

## 6.4 The Vector Potentials $\bar{A}$ and $\bar{F}$

To find the overall field  $\bar{E}$  &  $\bar{H}$  due to contributions from  $\bar{A}$  &/or  $\bar{F}$ , follow the steps:

1) Define/specify the EM boundary-value problem.

2) a) Solve for  $\bar{A}$  using

$$\bar{\nabla}^2 \bar{A} + \beta^2 \bar{A} = -\mu \bar{J} \quad (6-30)$$

→ solutions for  $\bar{A}$  for rectangular, cylindrical, & spherical coordinates are discussed in section 3.4 of the text.

b) Solve for  $\bar{F}$  using

$$\bar{\nabla}^2 \bar{F} + \beta^2 \bar{F} = -\epsilon \bar{M} \quad (6-31)$$

→ solutions for  $\bar{F}$  for rectangular, cylindrical, & spherical coordinates are discussed in section 3.4 of the text.

6.4 cont.

3) a) Determine  $\bar{H}_A$  and  $\bar{E}_A$  using

$$\bar{H}_A = \frac{1}{\mu} \bar{\nabla} \times \bar{A} \quad (6-32a)$$

and

$$\bar{E}_A = -j\omega \bar{A} - j \frac{1}{\omega\mu\epsilon} \bar{\nabla} (\bar{\nabla} \cdot \bar{A}) \quad (6-32b)$$

or

$$\bar{E}_A = \frac{1}{j\omega\epsilon} \bar{\nabla} \times \bar{H}_A \quad (6-32c)$$

b) Determine  $\bar{E}_F$  and  $\bar{H}_F$  using

$$\bar{E}_F = -\frac{1}{\epsilon} \bar{\nabla} \times \bar{F} \quad (6-33a)$$

and

$$\bar{H}_F = -j\omega \bar{F} - j \frac{1}{\omega\mu\epsilon} \bar{\nabla} (\bar{\nabla} \cdot \bar{F}) \quad (6-33b)$$

or

$$\bar{H}_F = \frac{-1}{j\omega\mu} \bar{\nabla} \times \bar{E}_F \quad (6-33c)$$

4) The overall fields are then, by superposition,

$$\bar{E} = \bar{E}_A + \bar{E}_F \quad (6-34)$$

$$\bar{H} = \bar{H}_A + \bar{H}_F \quad (6-35)$$

## 6.5 Construction of Solutions

Depending on the problem geometry, there can be many solutions for the fields -

### Transverse Electromagnetic (TEM) modes

Both  $\vec{E}$  &  $\vec{H}$  are transverse to a direction (of wave propagation, most often)

### Transverse Electric (TE) modes

$\vec{E}$  is transverse to a direction

### Transverse Magnetic (TM) modes

$\vec{H}$  is transverse to a direction

\* We are going to be concerned with EM boundary-value problems, e.g., waveguides & resonators, as opposed to antenna-type problems.

\* For practical reasons, we will only examine the rectangular & cylindrical coordinate systems.

## 6.5.1 Transverse Electromagnetic Modes: Source-Free Region

### A. Rectangular Coordinate System

Assume vector potential  $\bar{A}$  has a solution of the form

$$\bar{A}(x, y, z) = \hat{a}_x A_x(x, y, z) + \hat{a}_y A_y(x, y, z) + \hat{a}_z A_z(x, y, z)$$

that will satisfy (6-30) w/  $\bar{J} = 0$

$$\bar{\nabla}^2 \bar{A} + \beta^2 \bar{A} = 0 \quad (6-38)$$

which can be broken down into 3 scalar equations

$$\nabla^2 A_x + \beta^2 A_x = 0 \quad (6-38a)$$

$$\nabla^2 A_y + \beta^2 A_y = 0 \quad (6-38b)$$

$$\nabla^2 A_z + \beta^2 A_z = 0 \quad (6-38c)$$

Similarly, assume  $\bar{F}$  has a solution

$$\bar{F}(x, y, z) = \hat{a}_x F_x(x, y, z) + \hat{a}_y F_y(x, y, z) + \hat{a}_z F_z(x, y, z)$$

that will satisfy (6-31) w/  $\bar{M} = 0$

$$\bar{\nabla}^2 \bar{F} + \beta^2 \bar{F} = 0 \quad (6-40)$$

### 6.5.1 A. cont.

(6-40) can be broken down into 3 scalar differential equations -

$$\nabla^2 F_x + \beta^2 F_x = 0 \quad (6-40a)$$

$$\nabla^2 F_y + \beta^2 F_y = 0 \quad (6-40b)$$

$$\nabla^2 F_z + \beta^2 F_z = 0 \quad (6-40c)$$

W/ these, the total electric field

$$\bar{E} = \bar{E}_A + \bar{E}_F = -j\omega\bar{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\bar{A}) - \frac{1}{\epsilon}\nabla\times\bar{F} \quad (6-36)$$

becomes (6-41)

$$\begin{aligned} \bar{E} = & \hat{a}_x \left[ -j\omega A_x - j\frac{1}{\omega\mu\epsilon} \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z} \right) - \frac{1}{\epsilon} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \right] \\ & + \hat{a}_y \left[ -j\omega A_y - j\frac{1}{\omega\mu\epsilon} \left( \frac{\partial^2 A_x}{\partial x\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\partial z} \right) - \frac{1}{\epsilon} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \right] \\ & + \hat{a}_z \left[ -j\omega A_z - j\frac{1}{\omega\mu\epsilon} \left( \frac{\partial^2 A_x}{\partial x\partial z} + \frac{\partial^2 A_y}{\partial y\partial z} + \frac{\partial^2 A_z}{\partial z^2} \right) - \frac{1}{\epsilon} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \right]. \end{aligned}$$

The total magnetic field

$$\bar{H} = \bar{H}_A + \bar{H}_F = \frac{1}{\mu}\nabla\times\bar{A} - j\omega\bar{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\bar{F}) \quad (6-42)$$

then becomes (6-43).

6.5.1 A, cont.

$$\begin{aligned} \bar{H} = & \hat{a}_x \left[ -j\omega F_x - j \frac{1}{\omega\mu\epsilon} \left( \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x\partial y} + \frac{\partial^2 F_z}{\partial x\partial z} \right) + \frac{1}{\mu} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \right] \\ & + \hat{a}_y \left[ -j\omega F_y - j \frac{1}{\omega\mu\epsilon} \left( \frac{\partial^2 F_x}{\partial x\partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y\partial z} \right) + \frac{1}{\mu} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] \\ & + \hat{a}_z \left[ -j\omega F_z - j \frac{1}{\omega\mu\epsilon} \left( \frac{\partial^2 F_x}{\partial x\partial z} + \frac{\partial^2 F_y}{\partial y\partial z} + \frac{\partial^2 F_z}{\partial z^2} \right) + \frac{1}{\mu} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \end{aligned}$$

Example 6-1 uses (6-41) & (6-43) to find  $\bar{A}$  &  $\bar{F}$  expressions for  $\bar{E}$  and  $\bar{H}$ , under 3 scenarios, that yield TEM<sup>z</sup> modes

$$A_x = A_y = F_x = F_y = 0 \quad \frac{\partial}{\partial x} \neq 0 \quad \frac{\partial}{\partial y} \neq 0 \quad (6-44)$$

$$A_z = A_z^+(x,y) e^{-j\beta z} + A_z^-(x,y) e^{+j\beta z} \quad (6-44a)$$

$$F_z = F_z^+(x,y) e^{-j\beta z} + F_z^-(x,y) e^{+j\beta z} \quad (6-44b)$$

No  $\bar{A}$   $A_x = A_y = A_z = F_x = F_y = 0 \quad \frac{\partial}{\partial x} \neq 0 \quad \frac{\partial}{\partial y} \neq 0 \quad (6-45)$

$$F_z = F_z^+(x,y) e^{-j\beta z} + F_z^-(x,y) e^{+j\beta z} \quad (6-45a)$$

No  $\bar{F}$   $A_x = A_y = F_x = F_y = F_z = 0 \quad \frac{\partial}{\partial x} \neq 0 \quad \frac{\partial}{\partial y} \neq 0 \quad (6-46)$

$$A_z = A_z^+(x,y) e^{-j\beta z} + A_z^-(x,y) e^{+j\beta z} \quad (6-46a)$$

6.5.1 cont.B. Cylindrical Coordinate System

Here we assume that  $\bar{A}$  +  $\bar{F}$  are

$$\bar{A}(\rho, \phi, z) = \hat{a}_\rho A_\rho(\rho, \phi, z) + \hat{a}_\phi A_\phi(\rho, \phi, z) + \hat{a}_z A_z(\rho, \phi, z) \quad (6-49a)$$

$$\bar{F}(\rho, \phi, z) = \hat{a}_\rho F_\rho(\rho, \phi, z) + \hat{a}_\phi F_\phi(\rho, \phi, z) + \hat{a}_z F_z(\rho, \phi, z) \quad (6-49b)$$

that satisfy  $\bar{\nabla}^2 \bar{A} + \beta^2 \bar{A} = 0$  and  $\bar{\nabla}^2 \bar{F} + \beta^2 \bar{F} = 0$

when  $\bar{J} = \bar{M} = 0$ . With these, the total

$\bar{E}$  (6-50) and  $\bar{H}$  (6-51) are:

$$\begin{aligned} \bar{E} = & \hat{a}_\rho \left\{ -j\omega A_\rho - j\frac{1}{\omega\mu\epsilon} \frac{\partial}{\partial \rho} \left[ \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \right] - \frac{1}{\epsilon} \left( \frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \right\} \\ & + \hat{a}_\phi \left\{ -j\omega A_\phi - j\frac{1}{\omega\mu\epsilon} \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[ \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \right] - \frac{1}{\epsilon} \left( \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) \right\} \\ & + \hat{a}_z \left\{ -j\omega A_z - j\frac{1}{\omega\mu\epsilon} \frac{\partial}{\partial z} \left[ \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \right] - \frac{1}{\epsilon} \frac{1}{\rho} \left[ \frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right] \right\} \end{aligned}$$

$$\begin{aligned} \bar{H} = & \hat{a}_\rho \left\{ -j\omega F_\rho - j\frac{1}{\omega\mu\epsilon} \frac{\partial}{\partial \rho} \left[ \frac{1}{\rho} \frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \right] + \frac{1}{\mu} \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \right\} \\ & + \hat{a}_\phi \left\{ -j\omega F_\phi - j\frac{1}{\omega\mu\epsilon} \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[ \frac{1}{\rho} \frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \right] + \frac{1}{\mu} \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \right\} \\ & + \hat{a}_z \left\{ -j\omega F_z - j\frac{1}{\omega\mu\epsilon} \frac{\partial}{\partial z} \left[ \frac{1}{\rho} \frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \right] + \frac{1}{\mu} \frac{1}{\rho} \left[ \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right] \right\} \end{aligned}$$

### 6.5.1 B. cont.

Example 6-2 uses (6-50) & (6-51) to find expressions for  $\bar{A}$  &  $\bar{F}$  that lead to  $\bar{E}$  &  $\bar{H}$  that yield TEMP modes under 3 scenarios

$$A_\phi = A_z = F_\phi = F_z = 0 \quad \frac{\partial}{\partial \phi} \neq 0 \quad \frac{\partial}{\partial z} \neq 0 \quad (6-52)$$

$$A_\rho = A_\rho^+(\phi, z) H_1^{(2)}(\beta\rho) + A_\rho^-(\phi, z) H_1^{(1)}(\beta\rho) \quad (6-52a)$$

$$F_\rho = F_\rho^+(\phi, z) H_1^{(2)}(\beta\rho) + F_\rho^-(\phi, z) H_1^{(1)}(\beta\rho) \quad (6-52b)$$

No  $\bar{A}$   $A_\rho = A_\phi = A_z = F_\phi = F_z = 0 \quad \frac{\partial}{\partial \phi} \neq 0 \quad \frac{\partial}{\partial z} \neq 0 \quad (6-53)$

$$F_\rho = F_\rho^+(\phi, z) H_1^{(2)}(\beta\rho) + F_\rho^-(\phi, z) H_1^{(1)}(\beta\rho) \quad (6-53a)$$

No  $\bar{F}$   $A_\phi = A_z = F_\rho = F_\phi = F_z = 0 \quad \frac{\partial}{\partial \phi} \neq 0 \quad \frac{\partial}{\partial z} \neq 0 \quad (6-54)$

$$A_\rho = A_\rho^+(\phi, z) H_1^{(2)}(\beta\rho) + A_\rho^-(\phi, z) H_1^{(1)}(\beta\rho) \quad (6-54a)$$

Note:  $\rightarrow H_1^{(1)}(\rho)$  is a Hankel function of first kind of order 1,  $H_1^{(1)}(x) = J_1(x) + jY_1(x)$

$\rightarrow H_1^{(2)}(\rho)$  is a Hankel function of second kind of order 1,  $H_1^{(2)}(x) = J_1(x) - jY_1(x)$

$\rightarrow$  See Appendix IV and (IV-14) & (IV-15) for more.

## 6.5.2 Transverse Magnetic Modes: Source-Free Region

From the expressions for  $\bar{H}$  given by (6-43) for rectangular and (6-51) for cylindrical coordinates, a transverse magnetic (TM) field for a given direction will have  $\bar{A}$  with a component in that direction only and  $\bar{F} = 0$ .

### A. Rectangular Coordinate System

$$\underline{TM^z} \Rightarrow \bar{A} = \hat{a}_z A_z(x, y, z) + \bar{F} = 0$$

This leads the wave equation for  $\bar{A}$  (6-38) to reduce down to

$$\nabla^2 A_z + \beta^2 A_z = 0 \quad (6-56)$$

In a manner similar to Chapter 3, use separation of variables to write  $A_z$  as

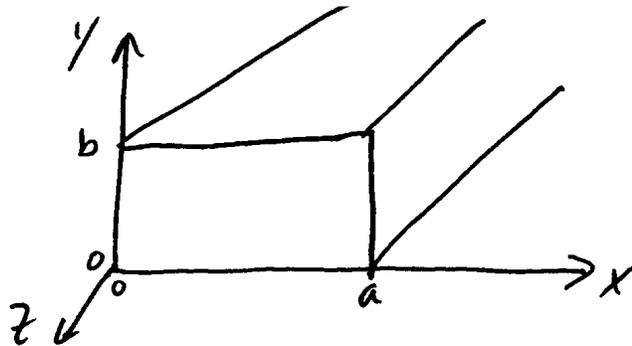
$$A_z(x, y, z) = f(x) g(y) h(z) \quad (6-57)$$

where functions  $f$ ,  $g$ , &  $h$  take the form(s) given by (3-28a) to (3-30b) in Chapter 3.

The choice of standing wave versus traveling wave functions will depend on the physical problem.

## 6.5.2 A. cont.

Example - What would be the appropriate  $\bar{A}$  for a  $TM^z$  mode for a rectangular waveguide oriented along the  $z$ -axis (see below)?



$\Rightarrow$  Due to the metal walls @  $x=0, x=a, y=0,$  and  $y=b,$  the  $TM^z$  wave can NOT propagate in the  $x$ - or  $y$ -directions  
So, use standing wave solutions for  $f(x)$  (3-28b) and  $g(y)$  (3-29b).

$\Rightarrow$  However, the  $TM^z$  wave could propagate in the  $\pm z$ -directions. Therefore, use a traveling wave solution for  $h(z)$  (3-30a)

$$A_z = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)] [C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \\ \times [A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}] \quad (6-58)$$

where  $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \epsilon$  (6-58a)

6.5.2 A. cont.

Once  $\bar{A} = \hat{a}_z A_z = \hat{a}_z f(x) g(y) h(z)$  is known or selected,  $\bar{E}$  can be found using (6-41) and  $\bar{H}$  using (6-43). These longish equations, with  $\bar{F} = 0$ , yield the following field components

TM <sup>z</sup> Rectangular Coord. (6-59)	
$E_x = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_z}{\partial x \partial z}$	$H_x = \frac{1}{\mu} \frac{\partial A_z}{\partial y}$
$E_y = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_z}{\partial y \partial z}$	$H_y = -\frac{1}{\mu} \frac{\partial A_z}{\partial x}$
$E_z = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z \right)$	$H_z = 0$ <small>by def'n</small>

where  $\nabla^2 A_z + \beta^2 A_z = 0$

TM<sup>x</sup>  $\bar{A} = \hat{a}_x A_x(x, y, z) = \hat{a}_x f(x) g(y) h(z)$  (6-60a)  
 $\bar{F} = 0$  (6-60b)

TM <sup>x</sup> Rectangular Coord. (6-61)	
$E_x = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 A_x}{\partial x^2} + \beta^2 A_x \right)$	$H_x = 0$ <small>by def'n</small>
$E_y = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_x}{\partial x \partial y}$	$H_y = \frac{1}{\mu} \frac{\partial A_x}{\partial z}$
$E_z = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_x}{\partial x \partial z}$	$H_z = -\frac{1}{\mu} \frac{\partial A_x}{\partial y}$

where  $\nabla^2 A_x + \beta^2 A_x = 0$

6.5.2 A. conti

$$\underline{TM^y} \quad \bar{A} = \hat{a}_y A_y(x, y, z) = \hat{a}_y f(x) g(y) h(z) \quad (6-63a)$$

$$\bar{F} = 0 \quad (6-63b)$$

TM<sup>y</sup> Rectangular Coord. (6-64)

$$E_x = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_y}{\partial x \partial y}$$

$$H_x = -\frac{1}{\mu} \frac{\partial A_y}{\partial z}$$

$$E_y = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 A_y}{\partial y^2} + \beta^2 A_y \right)$$

$$H_y = 0 \leftarrow \text{By def'n}$$

$$E_z = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 A_y}{\partial y \partial z}$$

$$H_z = \frac{1}{\mu} \frac{\partial A_y}{\partial x}$$

where  $\nabla^2 A_y + \beta^2 A_y = 0$

The actual values for  $\beta_x, \beta_y, \beta_z, \beta, \dots$  will depend on actual physical problem geometry, materials, & frequency.

6.5.2 cont.B. Cylindrical Coordinate System

⇒ Only going to consider  $TM^z$  as the  $TM^p$  and  $TM^\phi$  modes are not typically used (and more complex mathematically).

$$\underline{TM^z} \quad \bar{A} = \hat{a}_z A_z(\rho, \phi, z)$$

$$\bar{F} = 0$$

w/  $\bar{J} = 0$ , we get the scalar wave equation

$$\nabla^2 A_z + \beta^2 A_z = 0. \quad (6-67)$$

Again, use separation of variables, so

$$A_z = f(\rho) g(\phi) h(z) \quad (6-68)$$

where  $f(\rho)$ ,  $g(\phi)$ , &  $h(z)$  have solution forms given in Chapter 3 by (3-67a) to (3-69b).

The particular choices for  $f$ ,  $g$ , &  $h$  are guided by the physical problem(s) being considered.

### 6.5.2 B. cont.

With a form for  $A_z$  selected, the corresponding  $\bar{E}$  (6-50) and  $\bar{H}$  (6-51) can be found (remember  $\bar{F} = 0$  &  $A_x = A_y = 0$ ).

$TM^z$ Cylindrical Coord. (6-70)	
$E_\rho = -j\omega\mu\epsilon \frac{\partial^2 A_z}{\partial \rho \partial z}$	$H_\rho = \frac{1}{\mu} \frac{1}{\rho} \frac{\partial A_z}{\partial \phi}$
$E_\phi = -j\omega\mu\epsilon \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \phi \partial z}$	$H_\phi = -\frac{1}{\mu} \frac{\partial A_z}{\partial \rho}$
$E_z = -j\omega\mu\epsilon \left( \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z \right)$	$H_z = 0$ <small>By def'n</small>

⇒ Specific values for  $\beta_\rho$ ,  $\beta_z$ ,  $\beta$ , and the unknown constants will await physical problems.

Overall, these TM mode field equations can also be found by starting with the electric field component in the desired direction, e.g., use  $E_z$  for  $TM^z$ .

### 6.5.3 Transverse Electric Modes: Source-Free Region

In a fashion similar to the TM modes, an examination of the equations for  $\bar{E}$ , i.e., (6-41) for rectangular and (6-50) for cylindrical, leads to a transverse electric (TE) field in a given direction to have  $\bar{F}$  with a component only in that direction and  $\bar{A} = 0$ .

#### A. Rectangular Coordinate System

$$\underline{TE^x} \Rightarrow \bar{A} = 0, \bar{F} = \hat{a}_x F_x(x, y, z) = \hat{a}_x f(x) g(y) h(z)$$

$TE^x$ Rectangular Coord. (6-74)	
$E_x = 0$	$H_x = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 F_x}{\partial x^2} + \beta^2 F_x \right)$
$E_y = -\frac{1}{\epsilon} \frac{\partial F_x}{\partial z}$	$H_y = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_x}{\partial x \partial y}$
$E_z = \frac{1}{\epsilon} \frac{\partial F_x}{\partial y}$	$H_z = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_x}{\partial x \partial z}$

where  $\nabla^2 F_x + \beta^2 F_x = 0$  must be satisfied.

### 6.5.3 A. cont.

$$\underline{TEY} \Rightarrow \bar{A} = 0, \bar{F} = \hat{a}_y F_y(x, y, z) = \hat{a}_y f(x) g(y) h(z)$$

TEY Rectangular Coord. (6-77)

$$E_x = \frac{1}{\epsilon} \frac{\partial F_y}{\partial z} \quad H_x = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_y}{\partial x \partial y}$$

$$E_y = 0 \text{ or by def'n} \quad H_y = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 F_y}{\partial y^2} + \beta^2 F_y \right)$$

$$E_z = -\frac{1}{\epsilon} \frac{\partial F_y}{\partial x} \quad H_z = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_y}{\partial y \partial z}$$

where  $\nabla^2 F_y + \beta^2 F_y = 0$  must be satisfied.

$$\underline{TEz} \Rightarrow \bar{A} = 0, \bar{F} = \hat{a}_z F_z(x, y, z) = \hat{a}_z f(x) g(y) h(z)$$

TEz Rectangular Coord. (6-72)

$$E_x = -\frac{1}{\epsilon} \frac{\partial F_z}{\partial y} \quad H_x = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_z}{\partial x \partial z}$$

$$E_y = \frac{1}{\epsilon} \frac{\partial F_z}{\partial x} \quad H_y = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_z}{\partial y \partial z}$$

$$E_z = 0 \text{ or by def'n} \quad H_z = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z \right)$$

where  $\nabla^2 F_z + \beta^2 F_z = 0$  must be satisfied.

6.5.3 cont.B. Cylindrical Coordinate System

Again, for practical reasons only consider

the TE<sup>z</sup> mode  $\Rightarrow \bar{A} = 0$

$$\bar{F} = \hat{a}_z F_z(\rho, \phi, z)$$

TE<sup>z</sup> Cylindrical Coord. (6-80)

$$E_\rho = -\frac{1}{\epsilon} \frac{1}{\rho} \frac{\partial F_z}{\partial \phi}$$

$$H_\rho = -j \frac{1}{\omega \mu \epsilon} \frac{\partial^2 F_z}{\partial \rho \partial z}$$

$$E_\phi = \frac{1}{\epsilon} \frac{\partial F_z}{\partial \rho}$$

$$H_\phi = -j \frac{1}{\omega \mu \epsilon} \frac{1}{\rho} \frac{\partial^2 F_z}{\partial \phi \partial z}$$

$$E_z = 0 \text{ by def'n}$$

$$H_z = -j \frac{1}{\omega \mu \epsilon} \left( \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z \right)$$

where  $\nabla^2 F_z + \beta^2 F_z = 0$  must be satisfied.

Again, these TE mode field equations can also be found by starting with the magnetic field component in the same direction as desired for the mode, e.g., use  $H_x$  for TE<sup>x</sup>,