Chapter 5 Reflection and Transmission

5.1 Introduction

→ moving on from unbounded media to 2 semi-infinite media divided by a planar boundary

→ Key parameters: * reflection & transmission coefficients
* Brewster angle (no reflection)
* critical angle (100% reflection)

5.2 Normal Incidence – Lossless Media

→ see Fig 5-1

We'll assume the electric field(s) are in the x-direction

$$\bar{E}^i = \hat{a}_x E_0 e^{-j\beta_1 z} \quad \text{incident} \quad z \le 0$$

$$\bar{E}^r = \hat{a}_x \Gamma^b E_0 e^{+j\beta_1 z} \quad \text{reflected} \quad z \le 0$$

$$\bar{E}^t = \hat{a}_x T^b E_0 e^{-j\beta_2 z} \quad \text{transmitted} \quad z \ge 0$$

where Γ^b & T^b are the reflection and transmission coefficients @ the boundary ($z=0$). Note that all the electric fields are assumed to be oriented the same.

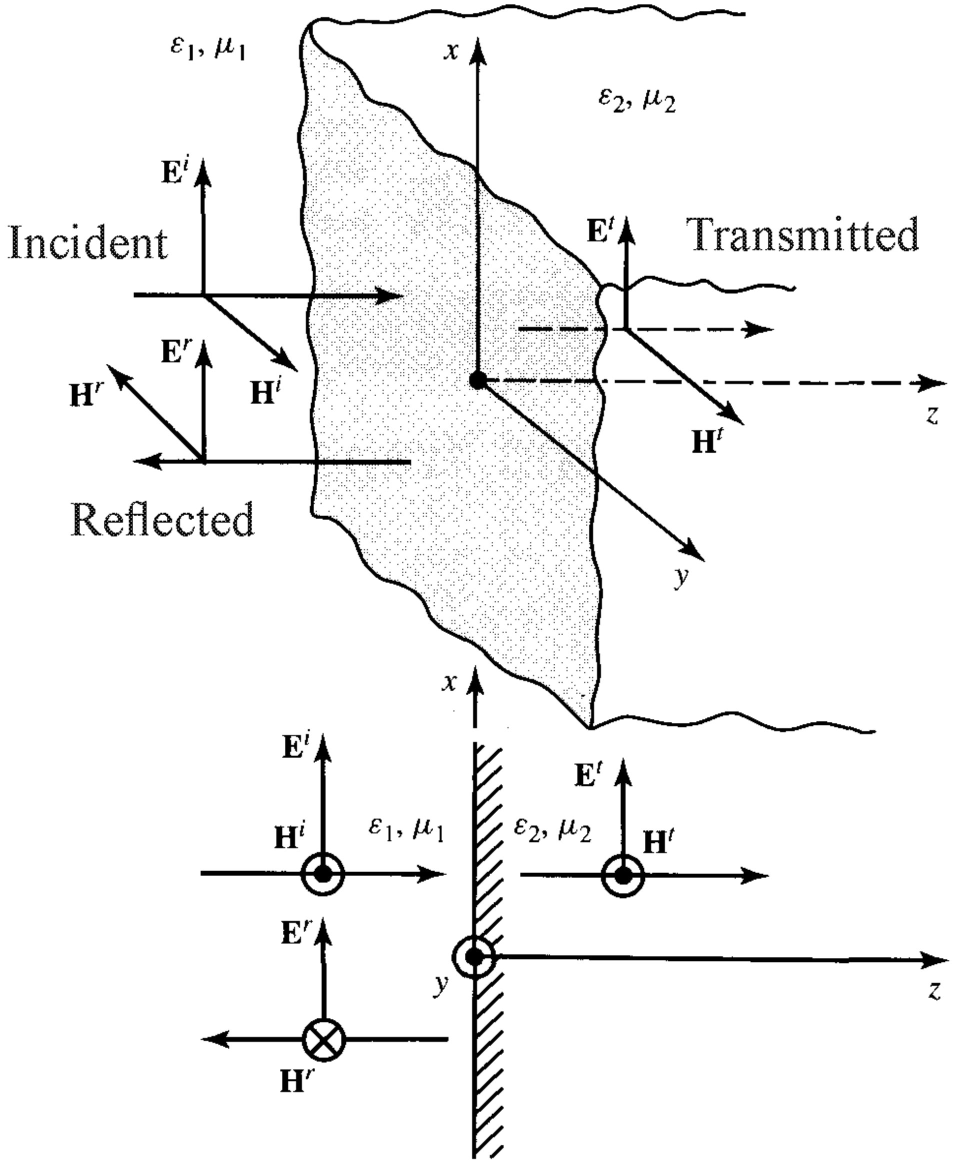


Figure 5-1 Wave reflection and transmission at normal incidence by a planar interface.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.2 cont.

Using $\bar{H} = \frac{\hat{a}_k \times \bar{E}}{\eta}$ or Faraday's Law, we can find the corresponding magnetic fields

$$\bar{H}^i = \frac{+\hat{a}_z \times \bar{E}^i}{\eta_1} = \hat{a}_y \frac{E_0}{\eta_1} e^{-j\beta_1 z} \qquad z \le 0$$

$$\bar{H}^r = \frac{-\hat{a}_z \times \bar{E}^r}{\eta_1} = -\hat{a}_y \frac{\Gamma^b E_0}{\eta_1} e^{j\beta_1 z} \qquad z \le 0$$

$$\bar{H}^t = \frac{+\hat{a}_z \times \bar{E}^t}{\eta_2} = \hat{a}_y \frac{T^b E_0}{\eta_2} e^{-j\beta_2 z} \qquad z \ge 0$$

Assuming no sources on the boundary, we apply the tangential boundary conditions @ $z = 0$

$$\bar{E}^i(z=0) + \bar{E}^r(z=0) = \bar{E}^t(z=0)$$

$$\bar{H}^i(z=0) + \bar{H}^r(z=0) = \bar{H}^t(z=0)$$

$$\Downarrow$$

$$E_0 + E_0 \Gamma^b = E_0 T^b \rightarrow 1 + \Gamma^b = T^b$$

$$\frac{E_0}{\eta_1} - \frac{\Gamma^b E_0}{\eta_1} = \frac{T^b E_0}{\eta_2} \rightarrow \frac{1}{\eta_1}(1 - \Gamma^b) = T^b/\eta_2$$

Solve 2 eq'ns - 2 unknowns $\Downarrow$

5.2 cont.

$$\boxed{\begin{aligned}\Gamma^b &= \frac{E^r}{E^i} = \frac{-H^r}{H^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\\ T^b &= \frac{E^t}{E^i} = \frac{\eta_2}{\eta_1}\frac{H^t}{H^i} = 1 + \Gamma^b = \frac{2\eta_2}{\eta_2 + \eta_1}\end{aligned}}$$

Note: $\Gamma^b = 0$ only if $\eta_1 = \eta_2 \Rightarrow \epsilon_2 = \epsilon_1$ for non-magnetic materials

We can define reflection + transmission coefficients away from the boundary @ locations $z = -\ell_1$ and $z = \ell_2$ as

$$\Gamma(z=-\ell_1) = \left.\frac{E^r(z)}{E^i(z)}\right|_{z=-\ell_1} = \frac{\Gamma_b E_0 e^{-j\beta_1 \ell_1}}{E_0 e^{+j\beta_1 \ell_1}}$$

$$\underline{\Gamma(z=-\ell_1) = \Gamma_b e^{-j2\beta_1 \ell_1}}$$

$$T(z_2=\ell_2, z_1=-\ell_1) = \frac{E^t(z_2=\ell_2)}{E^i(z_1=-\ell_1)} = \frac{T_b E_0 e^{-j\beta_2 \ell_2}}{E_0 e^{+j\beta_1 \ell_1}}$$

$$\underline{T(z_2=\ell_2, z_1=-\ell_1) = T^b e^{-j(\beta_2 \ell_2 + \beta_1 \ell_1)}}$$

Note: $\ell_1 + \ell_2 > 0$

5.2 cont.

What about power flow? The time-ave Poynting vectors are:

$$\bar{S}^i_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\bar{E}^i \times \bar{H}^{i*}\right) = \underline{\hat{a}_z \frac{|E_0|^2}{2\eta_1}} \qquad z<0$$

$$\bar{S}^r_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\bar{E}^r \times \bar{H}^{r*}\right) = -\hat{a}_z |\Gamma^b|^2 \frac{|E_0|^2}{2\eta_1}$$

$$= \underline{-\hat{a}_z |\Gamma^b|^2 |\bar{S}^i_{ave}|} \qquad z<0$$

$$\bar{S}^t_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\bar{E}^t \times \bar{H}^{t*}\right) = \hat{a}_z |T^b|^2 \frac{|E_0|^2}{2\eta_2}$$

massage to get $|\bar{S}^i_{ave}|$

$$= \hat{a}_z |T^b|^2 \frac{\eta_1}{\eta_2} \frac{|E_0|^2}{2\eta_1}$$

$$= \hat{a}_z |T^b|^2 \frac{\eta_1}{\eta_2} |\bar{S}^i_{ave}| \quad \text{Not obvious}$$

$$= \underline{\hat{a}_z \left(1 - |\Gamma^b|^2\right) |\bar{S}^i_{ave}|} \quad \text{Makes sense} \quad z>0$$

Comparing the underlined equations, we see that power <u>is</u> conserved.

$\Rightarrow$ In media 1, a standing wave will exist

$$\bar{E}^1 = \bar{E}^i + \bar{E}^r = \hat{a}_x E_0 e^{-j\beta_1 z}\left(1 + \Gamma^b e^{j2\beta_1 z}\right) \qquad z<0$$

$$= \hat{a}_x E_0 e^{-j\beta_1 z}\left(1 + \Gamma(z)\right)$$

7

5.2 cont.

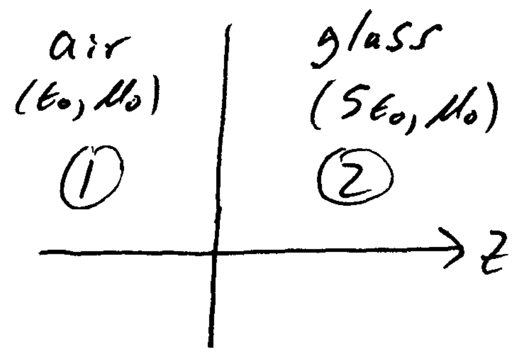
Comparing the max + min values of $\bar{E}'$, as done in Chap 4, leads to the standing wave ratio:

$$SWR = \frac{|\bar{E}'|_{max}}{|\bar{E}'|_{min}} = \frac{1 + |\Gamma^b|}{1 - |\Gamma^b|}$$

For non-magnetic media (e.g., $\mu_1 = \mu_2 = \mu_0$) or media w/ identical magnetic properties, the SWR simplifies to

$$SWR = \begin{cases} \sqrt{\epsilon_1/\epsilon_2} & \epsilon_1 > \epsilon_2 \\ \sqrt{\dfrac{\epsilon_2}{\epsilon_1}} & \epsilon_2 > \epsilon_1 \end{cases}$$

Example- A UPW in air (ε_0, μ_0, $z < 0$) is normally incident on a glass half-space ($5\varepsilon_0$, μ_0, $z > 0$). The 2.4 GHz incident electric field is oriented in the x-direction and has a field strength of 0.6 V/m at $z = 0$. Analyze and determine the various associated fields, power densities, and other related quantities.



$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1} = 2\pi(2.4\times10^9)\sqrt{4\pi\times10^{-7}(8.8541878\times10^{-12})}$$

$$= 50.3003 \text{ rad/m}$$

$$\eta_1 = \eta_0 = 376.7303\,\Omega$$

$$\beta_2 = \omega\sqrt{\mu_2\epsilon_2} = 2\pi(2.4\times10^9)\sqrt{4\pi\times10^{-7}(5)(8.8541878\times10^{-12})}$$

$$= 112.47485 \text{ rad/m}$$

$$\eta_2 = \sqrt{\mu_2/\epsilon_2} = \eta_0/\sqrt{\epsilon_{r2}} = \frac{376.7303}{\sqrt{5}} = 168.4789\,\Omega$$

$$\Gamma^b = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{168.48-376.73}{168.48+376.73} = -0.381966$$

$$T^b = 1+\Gamma^b = 1-0.381966 = 0.618034$$

ex. cont.

$$\underline{\bar{E}^i = \hat{a}_x\, 0.6\, e^{-j50.3z} \;(V/m) \quad z \le 0} \quad (air)$$

$$\bar{E}^r = \hat{a}_x(-0.382)0.6\, e^{+j50.3z}$$

$$\underline{\bar{E}^r = -\hat{a}_x\, 0.2292\, e^{+j50.3z} \;(V/m) \quad z \le 0 \quad (air)}$$

$$\underline{\bar{E}_{air} = \bar{E}^i + \bar{E}^r = \hat{a}_x\left[0.6\, e^{-j50.3z} - 0.2292\, e^{j50.3z}\right] (V/m) \quad z \le 0}$$

$$\bar{E}^t = \hat{a}_x(0.618)0.6\, e^{-j112.475z}$$

$$\underline{\bar{E}_{glass} = \bar{E}^t = \hat{a}_x\, 0.3708\, e^{-j112.475z} \;(V/m) \quad z \ge 0 \quad (glass)}$$

$$\underline{\bar{H}^i = \hat{a}_y \frac{0.6}{376.73} e^{-j50.3z} = \hat{a}_y\, 1.59265\, e^{-j50.3z} \left(\frac{mA}{m}\right) \quad z \le 0 \quad (air)}$$

$$\bar{H}^r = -\hat{a}_y \frac{(-0.382)0.6}{376.73} e^{+j50.3z}$$

$$\underline{\bar{H}^r = \hat{a}_y\, 0.60834\, e^{+j50.3z} \;(mA/m) \quad z \le 0 \quad (air)}$$

$$\underline{\bar{H}_{air} = \bar{H}^i + \bar{H}^r = \hat{a}_y\left[1.593\, e^{-j50.3z} + 0.608\, e^{j50.3z}\right]\left(\frac{mA}{m}\right)} \quad z \le 0 \; (air)$$

$$\bar{H}^t = \hat{a}_y \frac{0.618(0.6)}{168.479} e^{-j112.475z}$$

$$\underline{\bar{H}_{glass} = \bar{H}^t = \hat{a}_y\, 2.201\, e^{-j112.475z} \left(\frac{mA}{m}\right) \quad z \ge 0 \quad (glass)}$$

ex. cont.

$$SWR_{air} = \frac{1+|-0.382|}{1-|-0.382|} = \underline{\underline{2.236}}$$

$$\bar{S}^{i}_{ave} = \tfrac{1}{2}\,Re\left(\hat{a}_x\, 0.6\, e^{-j50.3z} \times \hat{a}_y\, 1.59265\times10^{-3}\, e^{+j50.3z}\right)$$

$$\underline{\bar{S}^{i}_{ave} = \hat{a}_z\, 0.4778\ \frac{mW}{m^2} \quad z \le 0\ (air)}$$

$$\bar{S}^{r}_{ave} = \tfrac{1}{2}\,Re\left(-\hat{a}_x\, 0.2292\, e^{j50.3z} \times \hat{a}_y\, 6.0834\times10^{-4}\, e^{-j50.3z}\right)$$

$$\underline{\bar{S}^{r}_{ave} = -\hat{a}_z\, 0.0697\ \frac{mW}{m^2} \quad z \le 0\ (air)}$$

$$\bar{S}^{t}_{ave} = \tfrac{1}{2}\,Re\left(\hat{a}_x\, 0.3708\, e^{-j112.475z} \times \hat{a}_y\, 2.201\times10^{-3}\, e^{+j112.475z}\right)$$

$$\underline{\bar{S}^{t}_{ave} = \hat{a}_z\, 0.4081\ mW/m^2 \quad z \ge 0\ (glass)}$$

Check $\bar{S}^{i}_{ave} + \bar{S}^{r}_{ave} \stackrel{?}{=} \bar{S}^{t}_{ave}$

$$\hat{a}_z(0.4778 - 0.0697) \stackrel{?}{=} \hat{a}_z\, 0.4081$$

$$\underbrace{\hat{a}_z\, 0.4081\ \frac{mW}{m^2}}_{air} = \underbrace{\hat{a}_z\, 0.4081\ \frac{mW}{m^2}}_{glass} \ \therefore$$

5.3 Oblique Incidence - Lossless Media

→ Again, we'll place the boundary @ $z=0$ (x-y plane)

→ Now, the UPW will be incident at an oblique angle

→ We'll define the 'plane of incidence' as the plane defined by the surface normal to the boundary (i.e., $+\hat{a}_z$) and the vector in the direction of incidence (x- & z-directions). In this case, the plane of incidence is the x-z plane.

Two cases

1) Electric field is orthogonal/perpendicular to plane of incidence (i.e. $\bar{y}$-direction)
   ⇒ Perpendicular/horizontal/E polarization $\bar{E}_{\perp}$

2) Magnetic field is orthogonal to plane of incidence ⇒ electric is parallel $\bar{E}_{\parallel}$
   ⇒ Parallel/vertical/H polarization

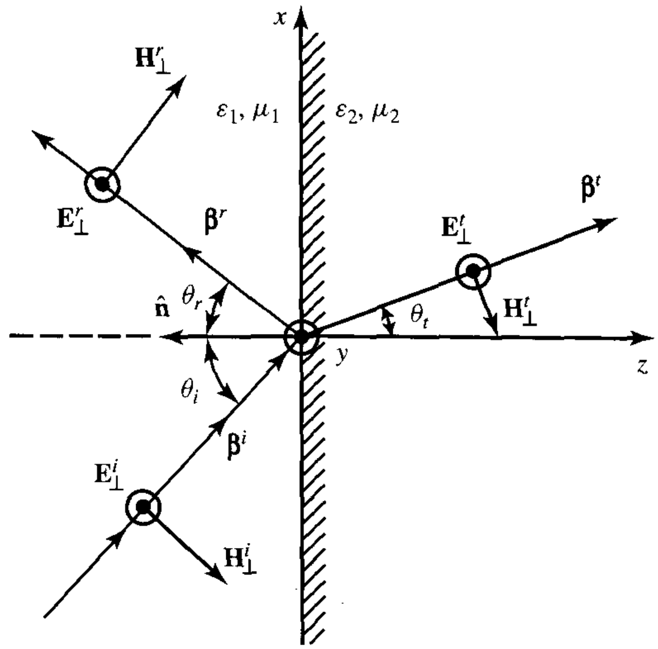


Figure 5-2 Perpendicular (horizontal) polarized uniform plane wave incident at an oblique angle on an interface.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.3.1 Perpendicular (Horizontal or E) Polarization

$$\bar{E}_\perp^i = \hat{a}_y E_\perp^i e^{-j\bar{\beta}_i\cdot\bar{r}} = \hat{a}_y E_o e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\bar{H}_\perp^i = (-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i) H_\perp^i e^{-j\bar{\beta}_i\cdot\bar{r}}$$

$$= (-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)\frac{E_o}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$\Rightarrow$ Similar to section 4.2.2 where we defined a $\bar{\beta}^+ = \beta(\hat{a}_x\sin\theta_i + \hat{a}_z\cos\theta_i)$

The reflected electric field will be in the y-direction ($\pm$ determined by a new reflection coefficient.

$$\bar{E}_\perp^r = \hat{a}_y E_\perp^r e^{-j\bar{\beta}_r\cdot\bar{r}} = \hat{a}_y \Gamma_\perp^b E_o e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\bar{H}_\perp^r = (\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r) H_\perp^r e^{-j\bar{\beta}_r\cdot\bar{r}}$$

$$= (\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r)\frac{\Gamma_\perp^b E_o}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Note: $E_\perp^r = \Gamma_\perp^b E^i = \Gamma_\perp^b E_o \Rightarrow \Gamma_\perp^b = \dfrac{E_\perp^r}{E_\perp^i}$

5.3.1 cont.

The transmitted fields are given as

$$\bar{E}_\perp^t = \hat{a}_y E_\perp^t e^{-j\bar{\beta}^t \cdot \bar{r}} = \hat{a}_y T_\perp^b E_0 e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\bar{H}_\perp^t = (-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_t) H_\perp^t e^{-j\bar{\beta}^t \cdot \bar{r}}$$

$$= (-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_t) \frac{T_\perp^b E_0}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Note: $E_\perp^t = T_\perp^b E_\perp^i = T_\perp^b E_0 \Rightarrow T_\perp^b = \dfrac{E_\perp^t}{E_\perp^i}$

Next, use our tangential boundary conditions (BCs) to find $\Gamma_\perp^b$ + $T_\perp^b$. First, apply tangential electric BC

$$\bar{E}_{1,t} = \left(\bar{E}_\perp^i + \bar{E}_\perp^r\right)\Big|_{z=0} = \left(\bar{E}_\perp^t\right)\Big|_{z=0} = \bar{E}_{2,t}$$

y-direction is tangential

$$\hat{a}_y E_0 e^{-j\beta_1 x\sin\theta_i} + \hat{a}_y \Gamma_\perp^b E_0 e^{-j\beta_1 x\sin\theta_r} = \hat{a}_y T_\perp^b E_0 e^{-j\beta_2 x\sin\theta_t}$$

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma_\perp^b e^{-j\beta_1 x\sin\theta_r} = T_\perp^b e^{-j\beta_2 x\sin\theta_t} \quad (5\text{-}14a)$$

Next, apply <u>tangential</u> magnetic BC (keep $\underline{\underline{\hat{a}_x}}$ terms)

$$\bar{H}_{1,t} = (\bar{H}_\perp^i + \bar{H}_\perp^r)\Big|_{z=0} = \bar{H}_\perp^t\Big|_{z=0} = \bar{H}_{2,t}$$

$\hat{a}_z$ terms are normal

$$-\hat{a}_x \frac{E_0}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \hat{a}_x \frac{\Gamma_\perp^b E_0}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}$$

$$= -\hat{a}_x \frac{T_\perp^b E_0}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

5.3.1 cont.

which simplifies to

$$\frac{1}{\eta_1}\left[-\cos\theta_i e^{-j\beta_1 x\sin\theta_i}+\Gamma_\perp^b\cos\theta_r e^{-j\beta_1 x\sin\theta_r}\right]=\frac{-T_\perp^b}{\eta_2}\cos\theta_t\, e^{-j\beta_2 x\sin\theta_t} \quad (5\text{-}14b)$$

Now (5-14a) can only hold true for all values of x if $\sin\theta_i = \sin\theta_r \Rightarrow \underline{\underline{\theta_i=\theta_r}}$ Snell's Law of reflection

$\Rightarrow$ incident + reflected wave must be 'in-phase' @ $z=0$ for all x

This, in turn, yields

$$\left(1+\Gamma_\perp^b\right)e^{-j\beta_1 x\sin\theta_i} = T_\perp^b e^{-j\beta_2 x\sin\theta_t}$$

which can only hold true for all values of x if

$$\underline{\underline{\beta_1\sin\theta_i = \beta_2\sin\theta_t}}$$ Snell's Law of refraction

With these two relationships, (5-14a) + (5-14b) have identical exponential terms that can be divided out to yield

$$1+\Gamma_\perp^b = T_\perp^b$$

$$\frac{\cos\theta_i}{\eta_1}\left(-1+\Gamma_\perp^b\right) = -\frac{\cos\theta_t}{\eta_2}T_\perp^b$$

2 eqns - 2 unknowns

5.3.1 cont.

Solving Fresnel reflection + transmission coefficients (perpend. pol.)

$$\boxed{\Gamma_\perp^b = \frac{E_\perp^r}{E_\perp^i} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}}$$

$$\boxed{T_\perp^b = \frac{E_\perp^t}{E_\perp^i} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}}$$

For dielectric media where $\mu_1 = \mu_2 \simeq \mu_0$, these become

$$\Gamma_\perp^b = \frac{\cos\theta_i - \sqrt{\epsilon_2/\epsilon_1}\sqrt{1-\left(\frac{\epsilon_1}{\epsilon_2}\right)\sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_2/\epsilon_1}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}}$$

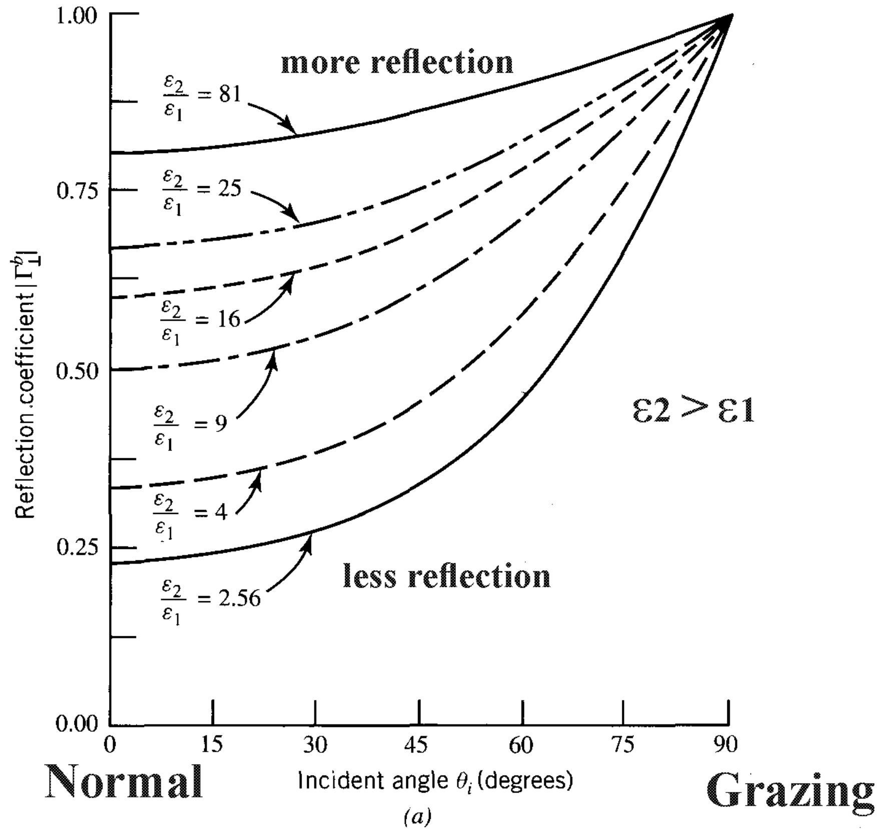
$$T_\perp^b = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\epsilon_2/\epsilon_1}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}}$$

When $\underline{\epsilon_2 > \epsilon_1}$ $\rightarrow$ $\Gamma_\perp^b$ is real + negative for all θ_i

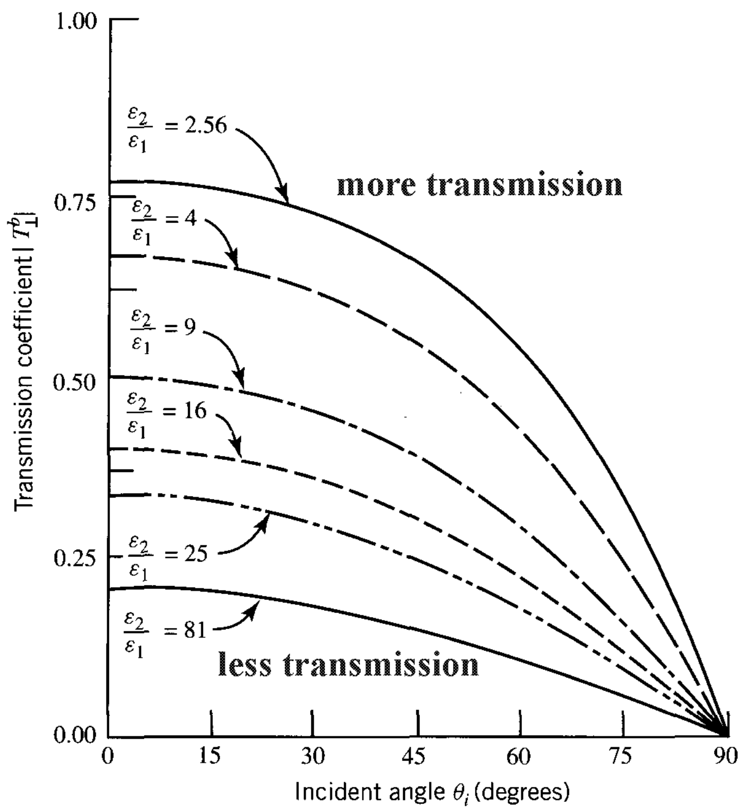
going from low ϵ_1 to higher ϵ_2 $\rightarrow$ $T_\perp^b$ is real + positive for all θ_i

See Figure 5-3 for trends

$\underline{\epsilon_2 = \epsilon_1}$ $\Rightarrow$ $\Gamma_\perp^b = 0$ + $T_\perp^b = 1$ (no boundary)



(a)



(b)

Figure 5-3 Magnitude of coefficients for perpendicular polarization as a function of incident angle. (*a*) Reflection. (*b*) Transmission.

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5.3.1 cont.

When $\epsilon_2 < \epsilon_1$ or $\frac{\epsilon_2}{\epsilon_1} < 1$, there is a range (high to low)

of angles $\theta_i \le \theta_c$ where $\Gamma_\perp^b$ & $T_\perp^b$ are real. However, at the critical angle θ_c,

$$\left|\Gamma_\perp^b\right|\Big|_{\frac{\epsilon_2}{\epsilon_1}<1 \;\&\; \theta_i=\theta_c} = 1$$

(Total internal, to ϵ_1 media, reflection!)

For $\theta_i > \theta_c$, $\Gamma_\perp^b$ & $T_\perp^b$ become complex.

We'll look at this more in section 5.3.4.

Lastly, $\bar{E}_\perp^1 = \bar{E}_\perp^i + \bar{E}_\perp^r$ $\quad z<0$ (media 1)

$$= \hat{a}_y E_0 \underbrace{e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}}_{\text{Traveling wave}} \underbrace{\left[1+\Gamma_\perp^b e^{j2\beta_1 z\cos\theta_i}\right]}_{\text{Standing wave}}$$

where we define

$$\Gamma_\perp(z) = \Gamma_\perp^b e^{j2\beta_1 z\cos\theta_i} \qquad z<0$$

5.3.2 Parallel (Vertical or H) Polarization

→ In this case, see Fig 5-4, $\bar{E}$ is in plane of incidence

$$\bar{H}_{\parallel}^{i} = \hat{a}_y H_{\parallel}^{i} e^{-j\bar{\beta}^{i}\cdot\bar{r}} = \hat{a}_y \frac{E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad z<0 \qquad (5\text{-}20b)$$

and

$$\bar{E}_{\parallel}^{i} = (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) E_0 e^{-j\bar{\beta}^{i}\cdot\bar{r}} \qquad z<0$$

$$= (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \qquad (5\text{-}20a)$$

where $E_{\parallel}^{i} = E_0$ & $H_{\parallel}^{i} = \frac{E_{\parallel}^{i}}{\eta_1} = \frac{E_0}{\eta_1}$

→ The reflected fields are defined

$$\bar{E}_{\parallel}^{r} = (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) E^{r} e^{-j\bar{\beta}^{r}\cdot\bar{r}}$$

$$= (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) \Gamma_{\parallel}^{b} E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \qquad (5\text{-}21a)$$

$$\bar{H}_{\parallel}^{r} = -\hat{a}_y H_{\parallel}^{r} e^{-j\bar{\beta}^{r}\cdot\bar{r}} = -\hat{a}_y \frac{\Gamma_{\parallel}^{b} E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad z<0 \qquad (5\text{-}21b)$$

where $E_{\parallel}^{r} = \Gamma_{\parallel}^{b} E^{i} = \Gamma_{\parallel}^{b} E_0$

& $H_{\parallel}^{r} = \frac{E_{\parallel}^{r}}{\eta_1} = \frac{\Gamma_{\parallel}^{b} E_0}{\eta_1}$

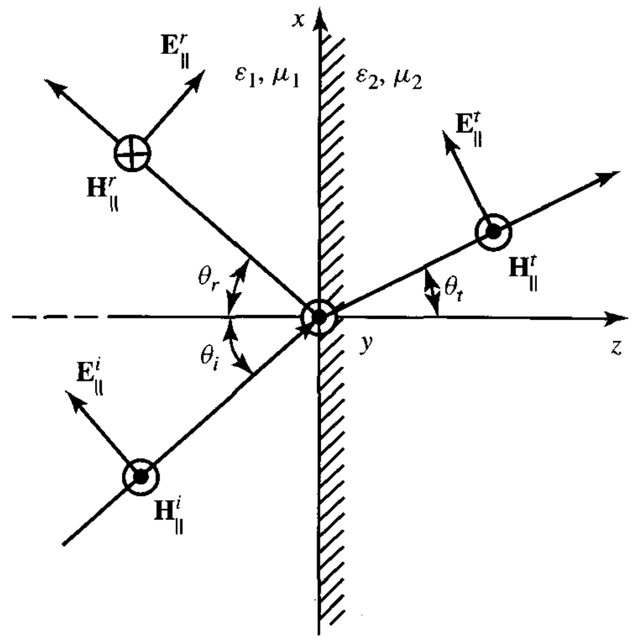

Figure 5-4 Parallel (vertical) polarized uniform plane wave incident at an oblique angle on an interface.

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5.3.2 cont.

The transmitted fields are defined

$$\bar{E}_{\parallel}^{t} = (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t) E_{\parallel}^{t} e^{-j\bar{\beta}^t \cdot \bar{r}} \quad (5\text{-}22a)$$

$$= (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t) T_{\parallel}^{b} E_o e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

$$\bar{H}_{\parallel}^{t} = \hat{a}_y H_{\parallel}^{t} e^{-j\bar{\beta}^t \cdot \bar{r}} = \hat{a}_y \frac{T_{\parallel}^{b} E_o}{\eta_2} e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)} \quad z > 0 \quad (5\text{-}22b)$$

where $E_{\parallel}^{t} = T_{\parallel}^{b} E^{i} = T_{\parallel}^{b} E_o$ & $H_{\parallel}^{t} = \frac{E_{\parallel}^{t}}{\eta_2} = \frac{T_{\parallel}^{b} E_o}{\eta_2}$

To determine $\Gamma_{\parallel}^{b}$, $T_{\parallel}^{b}$, θ_r & θ_i, we apply the tangential boundary conditions @ $z=0$

x & y components are tangential

$$\bar{E}_{1t}\Big|_{z=0} = \left(\bar{E}_{\parallel}^{i} + \bar{E}_{\parallel}^{r}\right)\Big|_{\substack{z=0\\ \tan}} = \bar{E}_{\parallel}^{t}\Big|_{\substack{z=0\\ \tan}} = \bar{E}_{2t}\Big|_{z=0}$$

$$\hat{a}_x \left(\cos\theta_i E_o e^{-j\beta_1 x \sin\theta_i} + \cos\theta_r \Gamma_{\parallel}^{b} E_o e^{-j\beta_1 x \sin\theta_r}\right)$$

$$= \hat{a}_x \cos\theta_t T_{\parallel}^{b} E_o e^{-j\beta_2 x \sin\theta_t}$$

$$\bar{H}_{1t}\Big|_{z=0} = \left(\bar{H}_{\parallel}^{i} + \bar{H}_{\parallel}^{r}\right)\Big|_{\substack{z=0\\ \tan}} = \bar{H}_{\parallel}^{t}\Big|_{\substack{z=0\\ \tan}} = \bar{H}_{2t}\Big|_{z=0}$$

$$\hat{a}_y \left(\frac{E_o}{\eta_1} e^{-j\beta_1 x \sin\theta_i} - \frac{\Gamma_{\parallel}^{b} E_o}{\eta_1} e^{-j\beta_1 x \sin\theta_r} \right) = \hat{a}_y \frac{T_{\parallel}^{b} E_o}{\eta_2} e^{-j\beta_2 x \sin\theta_t}$$

5.3.2 cont.

Removing common terms, we get

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma_{\parallel}^{b}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = T_{\parallel}^{b}\cos\theta_t e^{-j\beta_2 x\sin\theta_t} \qquad (5\text{-}23a)$$

$$\frac{1}{\eta_1}\left(e^{-j\beta_1 x\sin\theta_i} - \Gamma_{\parallel}^{b} e^{-j\beta_1 x\sin\theta_r}\right) = \frac{1}{\eta_2} T_{\parallel}^{b} e^{-j\beta_2 x\sin\theta_t} \qquad (5\text{-}23b)$$

Again, @ $z=0$, w/ $\eta_1, \eta_2, \Gamma_{\parallel}^{b}, + T_{\parallel}^{b}$ being constant, (5-23b) can only be true for all x if

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t$$

$$\Downarrow \qquad\qquad \Downarrow$$

Snell's laws of reflection + refraction $\quad \underline{\theta_i = \theta_r}$ (5-24a) $\quad \underline{\beta_1\sin\theta_i = \beta_2\sin\theta_t}$ (5-24b)

With the now identical exponentials removed, we get

$$\cos\theta_i + \Gamma_{\parallel}^{b}\cos\overset{\theta_i}{\cancel{\theta_r}} = T_{\parallel}^{b}\cos\theta_t$$

$$\frac{1}{\eta_1}\left(1 - \Gamma_{\parallel}^{b}\right) = \frac{1}{\eta_2} T_{\parallel}^{b}$$

which can be solved for $\Gamma_{\parallel}^{b} + T_{\parallel}^{b}$

5.3.2 cont.

$$\boxed{\begin{array}{c} \Gamma_{\parallel}^{b} = \dfrac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \\ \hline T_{\parallel}^{b} = \dfrac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \end{array}}$$

Fresnel reflection (5-24c) & transmission (5-24d) coefficients (parallel polarization)

For dielectric media where $\mu_1 \simeq \mu_2 \simeq \mu_0$

$$\Gamma_{\parallel}^{b}\Big|_{\mu_1=\mu_2} = \frac{-\cos\theta_i + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_1/\epsilon_2}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}} \quad (5\text{-}25c)$$

$$T_{\parallel}^{b}\Big|_{\mu_1=\mu_2} = \frac{2\sqrt{\epsilon_1/\epsilon_2}\cos\theta_i}{\cos\theta_i + \sqrt{\epsilon_1/\epsilon_2}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}} \quad (5\text{-}25d)$$

* For $\underline{\epsilon_2 > \epsilon_1}$, notice in Fig 5-5a that (low to high)
there is an angle where $|\Gamma_{\parallel}^{b}| = 0$,
i.e., the Brewster angle $\theta_B = \theta_i$.

* Note that θ_B gets larger as $\frac{\epsilon_2}{\epsilon_1}$ increases.

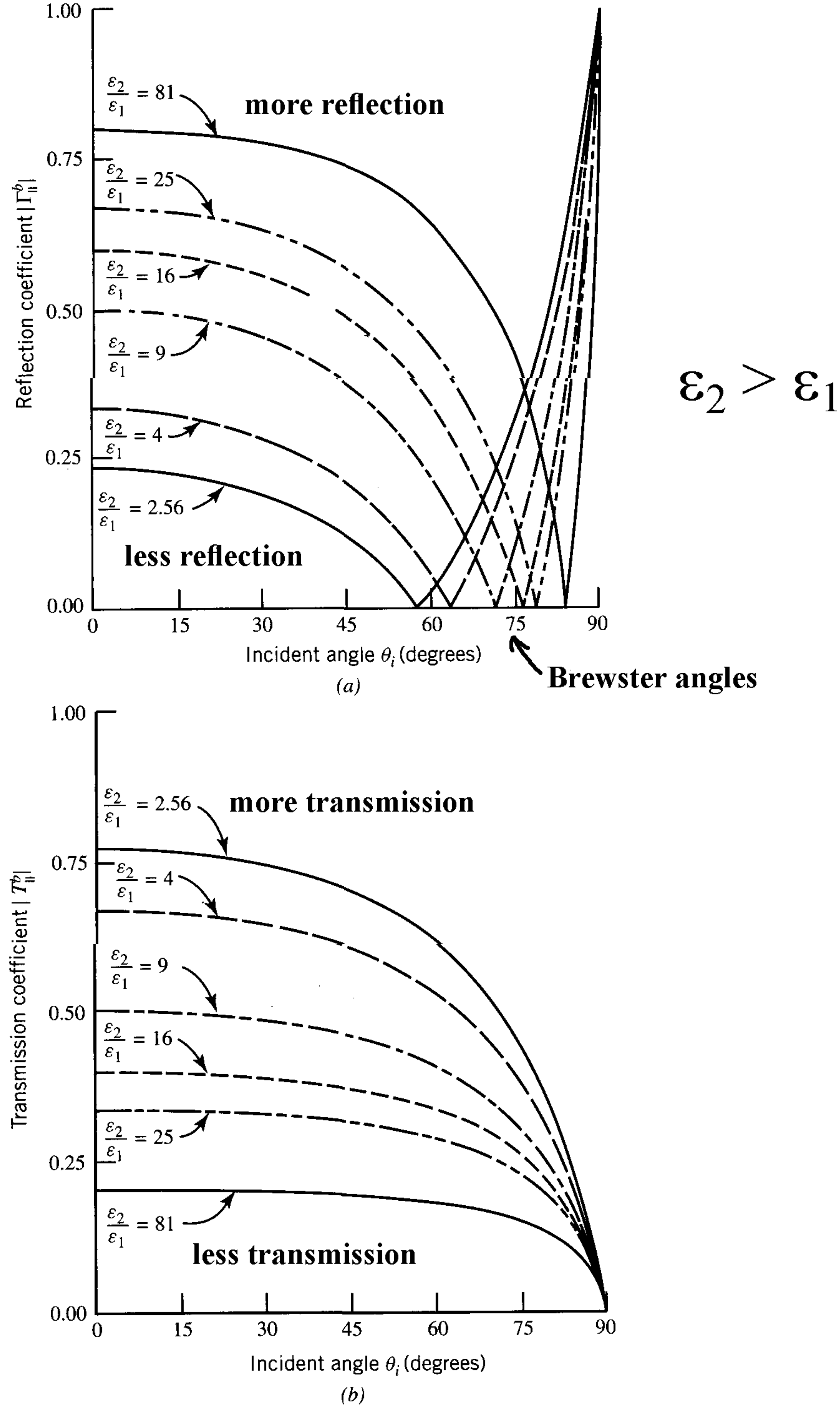


Figure 5-5 Magnitude of coefficients for parallel polarization as a function of incident angle. (*a*) Reflection. (*b*) Transmission.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.3.2 cont.

$\boxed{\epsilon_2 > \epsilon_1}$ → $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$ are both real. Further, for $\theta_i < \theta_B$, $\Gamma_{\parallel}^b < 0$, while for $\theta_i > \theta_B$, $\Gamma_{\parallel}^b > 0$. $T_{\parallel}^b$ is greater than zero for all θ_i

$\boxed{\epsilon_2 = \epsilon_1}$ ⇒ $\Gamma_{\parallel}^b = 0$ and $T_{\parallel}^b = 1$ (no boundary)

$\boxed{\epsilon_2 < \epsilon_1}$ → Again there will be a critical (high to low) angle $\underline{\theta_c}$ where $|\Gamma_{\parallel}^b|_{\epsilon_2/\epsilon_1 < 1} = 1$

→ $\theta_i < \theta_c$ $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$ are real

→ $\theta_i > \theta_c$ $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$ are complex

$$\bar{E}_{\parallel}^1 = \bar{E}_{\parallel}^i + \bar{E}_{\parallel}^r = \hat{a}_x \cos\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left(1 + \Gamma_{\parallel}^b e^{j2\beta_1 z\cos\theta_i}\right)$$

$$- \hat{a}_z \sin\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left(1 - \Gamma_{\parallel}^b e^{j2\beta_1 z\cos\theta_i}\right)$$

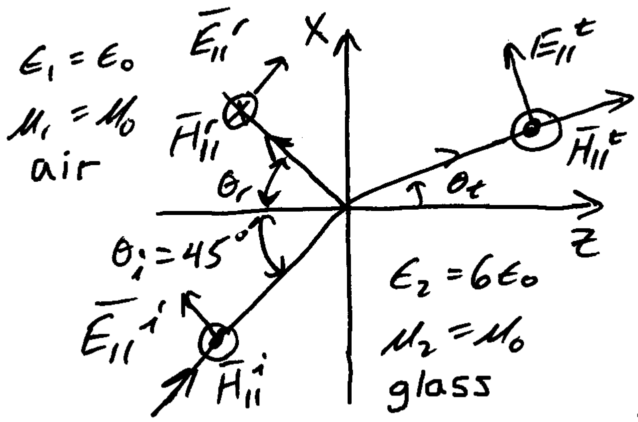
↑ Traveling ↑ standing

$$= \hat{a}_x \cos\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left[1 + \Gamma_{\parallel}(z)\right]$$

$$- \hat{a}_z \sin\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left[1 - \Gamma_{\parallel}(z)\right]$$

where $\Gamma_{\parallel}(z) = \Gamma_{\parallel}^b e^{j2\beta_1 z\cos\theta_i}$ $\underline{z < 0}$

Example- A UPW in air (ε_0, μ_0, $z < 0$) is obliquely incident on a glass half-space ($6\varepsilon_0$, μ_0, $z > 0$) at an angle of 45°. The 800 MHz incident magnetic field is oriented in the y-direction and has a field strength of 0.4 mA/m at $z = 0$. Analyze and determine the various associated fields, power densities, and other related quantities.



$$\eta_1 = \eta_0 = \underline{376.7303\ \Omega}$$

$$\beta_1 = \omega\sqrt{\mu_0\epsilon_0} = \frac{2\pi\, 800\times10^6}{2.9979\times10^8} = \underline{16.76676\ \frac{\text{rad}}{\text{m}}}$$

$$\eta_2 = \eta_0/\sqrt{\epsilon_{r2}} = \frac{376.7303}{\sqrt{6}} = \underline{153.7995\ \Omega}$$

$$\beta_2 = \omega\sqrt{\mu_0 6\epsilon_0} = 2\pi\,(800\times10^6)\sqrt{4\pi\times10^{-7}(6)\,8.8541878\times10^{-12}}$$

$$\underline{\beta_2 = 41.070007\ \frac{\text{rad}}{\text{m}}}$$

Per (5-24a), $\underline{\theta_r = \theta_i = 45°}$

Per (5-24b), $\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$

$$16.76676 \sin 45° = 41.07 \sin\theta_t$$

$$\hookrightarrow \theta_t = \sin^{-1}\left(\frac{16.76676 \sin 45°}{41.07001}\right) = \sin^{-1}(0.288675)$$

$$= 16.778655° \Rightarrow \underline{\theta_t = 16.7787°}$$

Per (5-24c), $$\Gamma_{\parallel}^b = \frac{-\eta_1\cos\theta_i + \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

$$\Gamma_{\parallel}^{b} = \frac{-376.7303\cos 45° + 153.7995\cos 16.7787°}{376.7303\cos 45° + 153.7995\cos 16.7787°} = \frac{-124.432}{413.64}$$

$$\underline{\underline{\Gamma_{\parallel}^{b} = -0.28802}}$$

$$T_{\parallel}^{b} = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{2(153.7995)\cos 45°}{413.64}$$

$$\underline{\underline{T_{\parallel}^{b} = 0.52583}}$$

Per (5-20d), $H_{\parallel}^{i} = \frac{E_0}{\eta_1} \Rightarrow E_0 = 376.7303(0.4\times 10^{-3})$

$$\underline{E_0 = 0.150692 \text{ V/m}}$$

(5-20b) $\underline{\bar{H}_{\parallel}^{i} = \hat{a}_y\, 0.4\, e^{-j16.767(0.707x + 0.707z)}\ (\text{mA/m})}$

(5-20a) $\underline{\bar{E}_{\parallel}^{i} = (0.707\hat{a}_x - 0.707\hat{a}_z)\, 0.1507\, e^{-j16.767(0.707x + 0.707z)}\ (\text{V/m})}$

(5-21a) $\bar{E}_{\parallel}^{r} = (0.707\hat{a}_x + 0.707\hat{a}_z)\, 0.1507(-0.288)\, e^{-j16.767(0.707x - 0.707z)}$

$$\underline{\bar{E}_{\parallel}^{r} = (0.707\hat{a}_x + 0.707\hat{a}_z)(-43.402)\, e^{-16.767(0.707x - 0.707z)}\ \left(\frac{\text{mV}}{\text{m}}\right)}$$

(5-21b) $\bar{H}_{\parallel}^{r} = -\hat{a}_y \frac{-43.402\times 10^{-3}}{376.7307} e^{-j16.767(0.707x - 0.707z)}$

$$\underline{\bar{H}_{\parallel}^{r} = +\hat{a}_y\, 0.1152\, e^{-j16.767(0.707x - 0.707z)}\ \left(\text{mA/m}\right)}$$

$$(5\text{-}22a)\quad \bar{E}_{11}^{\,t} = \left(\hat{a}_x \cos 16.78^\circ - \hat{a}_z \sin 16.78^\circ\right)(0.526)(0.1507)\, e^{-j41.07(x \sin 16.8^\circ + z \cos 16.8^\circ)}$$

$$\underline{\bar{E}_{11}^{\,t} = \left(0.9574\,\hat{a}_x - 0.2887\,\hat{a}_z\right)(79.268)\, e^{-j41.07(0.2887x + 0.9574z)} \left(\frac{mV}{m}\right)}$$

$$(5\text{-}22b)\quad \bar{H}_{11}^{\,t} = \hat{a}_y \frac{0.52583(0.150692)}{153.7995}\, e^{-j41.07(0.2887x + 0.9574z)}$$

$$\underline{\bar{H}_{11}^{\,t} = \hat{a}_y\, 0.5152\, e^{-j41.07(0.2887x + 0.9574z)} \left(\frac{mA}{m}\right)}$$

$$(5\text{-}26a)\quad \Gamma_{11}(z) = \Gamma_{11}^{\,b}\, e^{j2\beta_1 z \cos\theta_i}$$

$$= -0.28802\, e^{j2(16.76676)\cos 45^\circ z}$$

$$\underline{\Gamma_{11}(z) = -0.28802^{\,j23.7118z} \qquad z \le 0}$$

$$(5\text{-}26)\quad \bar{E}_{11}' = \hat{a}_x \cos 45^\circ (0.1507)\, e^{-j16.767(0.707x + 0.707z)}\left[1 + \Gamma_{11}(z)\right]$$

$$- \hat{a}_z \sin 45^\circ (0.1507)\, e^{-j16.767(0.707x + 0.707z)}\left[1 - \Gamma_{11}(z)\right]$$

$$\bar{E}_{11}' = \hat{a}_x\, 0.10656\, e^{-j16.767(0.707x + 0.707z)}\left[1 + \Gamma_{11}(z)\right]$$

$$- \hat{a}_z\, 0.10656\, e^{-j16.767(0.707x + 0.707z)}\left[1 - \Gamma_{11}(z)\right] \left(\frac{V}{m}\right)$$

$$z \le 0$$

5.3.3 Total Transmission - Brewster Angle

Is there angle of incidence or set of conditions where the reflection coefficient goes to zero?

A. Perpendicular (Horizontal) Polarization

Here set $$\Gamma_\perp^b = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = 0 \quad (5\text{-}27)$$

algebra ↳ $$\cos\theta_i = \sqrt{\frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}}\cos\theta_t \quad (5\text{-}27a)$$

use $\cos^2 A = 1 - \sin^2 A$ and square both sides

$$1 - \sin^2\theta_i = \frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}\left(1 - \sin^2\theta_t\right)$$

Next use $\beta_1 \sin\theta_i = \beta_2 \sin\theta_t \Rightarrow \sin^2\theta_t = \frac{\mu_1}{\mu_2}\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i$

$$1 - \sin^2\theta_i = \frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}\left(1 - \frac{\mu_1}{\mu_2}\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)$$

algebra ↳
$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\mu_1/\mu_2 - \mu_2/\mu_1}} \quad (5\text{-}28a)$$

For most dielectrics, $\mu_1 = \mu_2 = \mu_0$, which implies $\sin\theta_i \Rightarrow \infty \Rightarrow$ NOT POSSIBLE

$\Rightarrow \Gamma_\perp^b = 0$ can't be achieved.

5.3.3 cont.

B. Parallel (Vertical) Polarization

Here, set $\Gamma_{\parallel}^{b} = \dfrac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = 0$ (5-30)

algebra ↳ $\cos\theta_i = \sqrt{\dfrac{\mu_2}{\mu_1}\left(\dfrac{\epsilon_1}{\epsilon_2}\right)} \cos\theta_t$ (5-30a)

$$\cos^2\theta_i = 1 - \sin^2\theta_i = \frac{\mu_2}{\mu_1}\left(\frac{\epsilon_1}{\epsilon_2}\right)\cos^2\theta_t = \frac{\mu_2}{\mu_1}\left(\frac{\epsilon_1}{\epsilon_2}\right)\left(1 - \sin^2\theta_t\right)$$

and use $\sin^2\theta_t = \dfrac{\mu_1}{\mu_2}\dfrac{\epsilon_1}{\epsilon_2}\sin^2\theta_i$

$$1 - \sin^2\theta_i = \frac{\mu_2}{\mu_1}\frac{\epsilon_1}{\epsilon_2}\left(1 - \frac{\mu_1}{\mu_2}\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)$$

algebra ↳

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2}} \qquad (5\text{-}31a)$$

θ_i real if $\dfrac{\mu_2}{\mu_1} \geq \epsilon_1/\epsilon_2$

⇒ If $\mu_2 \simeq \mu_1 \simeq \mu_0$, this means $\epsilon_1/\epsilon_2 < 1$ or $\epsilon_1 < \epsilon_2$, i.e., the EM wave w/ parallel polarization is going from lower to higher permittivity media

Solve for Brewster angle θ_B

$$\boxed{\theta_i = \theta_B = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1 + \epsilon_2}}} \qquad (5\text{-}33)$$

$$\boxed{= \cos^{-1}\sqrt{\frac{\epsilon_1}{\epsilon_1 + \epsilon_2}}} \qquad (5\text{-}33a)$$

$$\boxed{= \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}} \qquad (5\text{-}33b)$$

Example- For a parallel polarized UPW in air (ε_0, μ_0, $z < 0$) obliquely incident on a glass half-space ($6\varepsilon_0$, μ_0, $z > 0$), find the Brewster angle. Then, determine the reflection and refraction angles as well as the reflection and transmission coefficients.

From earlier, $\eta_1 = 376.7303\,\Omega$ + $\eta_2 = 153.7995\,\Omega$

$\beta_1 = 16.76676$ rad/m $\quad \beta_2 = 41.070007$ rad/m

Per (5-33), $\theta_i = \theta_B = \sin^{-1}\sqrt{\dfrac{6\varepsilon_0}{\varepsilon_0 + 6\varepsilon_0}} = \sin^{-1}\sqrt{6/7}$

$$\underline{\underline{\theta_i = \theta_B = 67.792346^\circ}}$$

Per (5-24a), $\underline{\underline{\theta_r = \theta_i = \theta_B = 67.792346^\circ}}$

Per (5-24b), $\theta_t = \sin^{-1}\left(\dfrac{16.76676 \sin 67.79^\circ}{41.070007}\right) = \sin^{-1}(0.378)$

$$\underline{\underline{\theta_t = 22.20765^\circ}}$$

Per (5-24c), $\Gamma_{\parallel}^b = \dfrac{-376.73 \cos 67.79^\circ + 153.8 \cos 22.21^\circ}{376.73 \cos 67.79 + 153.8 \cos 22.21^\circ}$

$$\underline{\underline{\Gamma_{\parallel}^b = 0}}$$

Per (5-24d), $T_{\parallel}^b = \dfrac{2(153.8)\cos 67.79^\circ}{376.73 \cos 67.79^\circ + 153.8 \cos 22.21^\circ}$

$$T_{\parallel}^b = \frac{116.261}{284.7818}$$

$$\underline{\underline{T_{\parallel}^b = 0.40825}}$$