Chapter 5 Reflection and Transmission

5.1 Introduction

→ moving on from unbounded media to 2 semi-infinite media divided by a planar boundary

→ Key parameters: * reflection + transmission coefficients
* Brewster angle (no reflection)
* Critical angle (100% reflection)

5.2 Normal Incidence - Lossless Media

→ See Fig 5-1

We'll assume the electric field(s) are in the x-direction

$$\bar{E}^i = \hat{a}_x E_0 e^{-j\beta_1 z} \quad \text{incident} \quad z \le 0$$

$$\bar{E}^r = \hat{a}_x \Gamma^b E_0 e^{+j\beta_1 z} \quad \text{reflected} \quad z \le 0$$

$$\bar{E}^t = \hat{a}_x T^b E_0 e^{-j\beta_2 z} \quad \text{transmitted} \quad z \ge 0$$

where Γ^b + T^b are the reflection and transmission coefficients @ the boundary ($z=0$). Note that all the electric fields are assumed to be oriented the same.

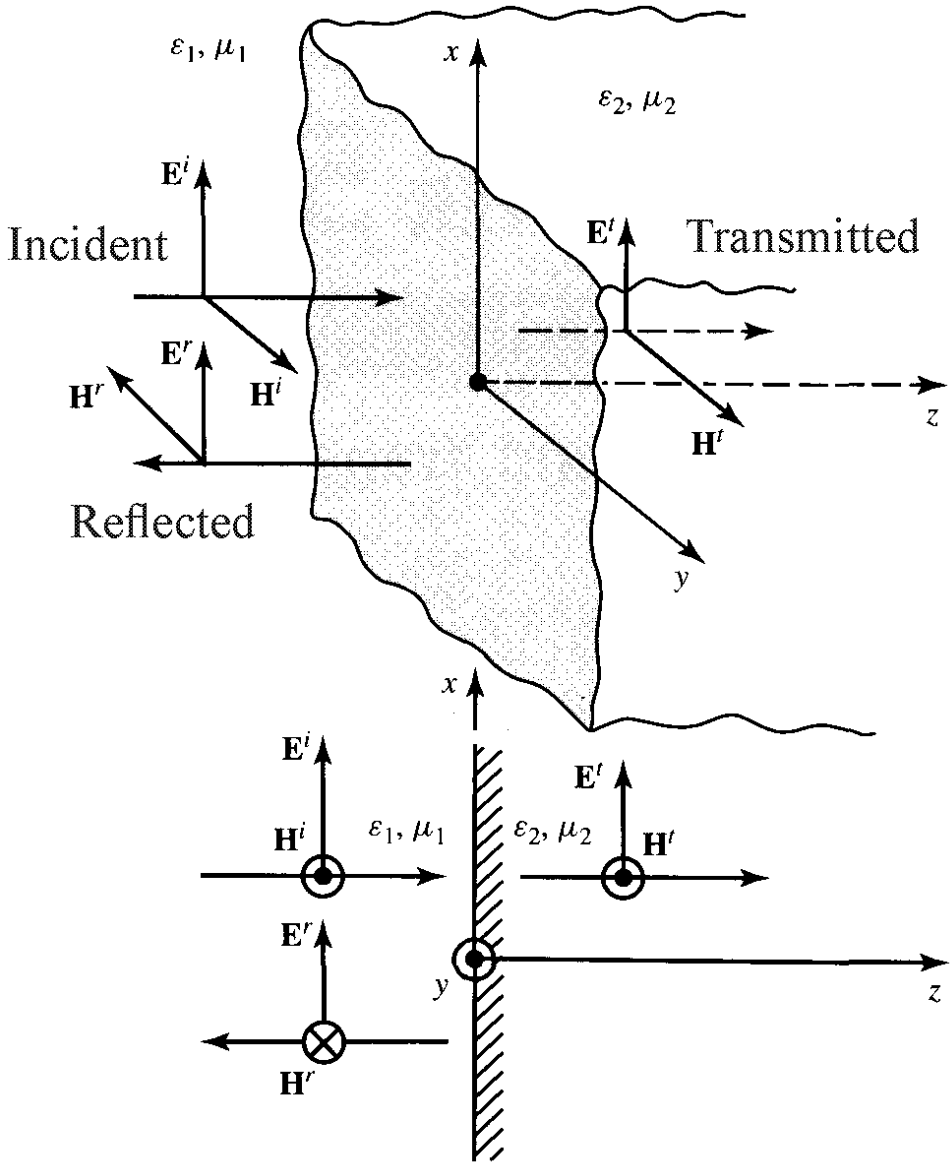

Figure 5-1 Wave reflection and transmission at normal incidence by a planar interface.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.2 cont.

Using $\bar{H} = \dfrac{\hat{a}_k \times \bar{E}}{\eta}$ or Faraday's Law, we can find the corresponding magnetic fields

$$\bar{H}^i = \frac{+\hat{a}_z \times \bar{E}^i}{\eta_1} = \hat{a}_y \frac{E_0}{\eta_1} e^{-j\beta_1 z} \qquad z \le 0$$

$$\bar{H}^r = \frac{-\hat{a}_z \times \bar{E}^r}{\eta_1} = -\hat{a}_y \frac{\Gamma^b E_0}{\eta_1} e^{j\beta_1 z} \qquad z \le 0$$

$$\bar{H}^t = \frac{+\hat{a}_z \times \bar{E}^t}{\eta_2} = \hat{a}_y \frac{T^b E_0}{\eta_2} e^{-j\beta_2 z} \qquad z \ge 0$$

Assuming no sources on the boundary, we apply the tangential boundary conditions @ $z=0$

$$\bar{E}^i(z=0) + \bar{E}^r(z=0) = \bar{E}^t(z=0)$$

$$\bar{H}^i(z=0) + \bar{H}^r(z=0) = \bar{H}^t(z=0)$$

$$\Downarrow$$

$$E_0 + E_0 \Gamma^b = E_0 T^b \rightarrow 1 + \Gamma^b = T^b$$

$$\frac{E_0}{\eta_1} - \frac{\Gamma^b E_0}{\eta_1} = \frac{T^b E_0}{\eta_2} \rightarrow \frac{1}{\eta_1}(1-\Gamma^b) = T^b/\eta_2$$

Solve 2 eq'ns - 2 unknowns $\Downarrow$

5.2 cont.

$$\boxed{\begin{aligned} \Gamma^b &= \frac{E^r}{E^i} = \frac{-H^r}{H^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\ T^b &= \frac{E^t}{E^i} = \frac{\eta_2}{\eta_1}\frac{H^t}{H^i} = 1 + \Gamma^b = \frac{2\eta_2}{\eta_2 + \eta_1} \end{aligned}}$$

Note: $\Gamma^b = 0$ only if $\eta_1 = \eta_2 \Rightarrow \epsilon_2 = \epsilon_1$ for non-magnetic materials

We can define reflection + transmission coefficients away from the boundary @ locations $z = -\ell_1$ and $z = \ell_2$ as

$$\Gamma(z = -\ell_1) = \left.\frac{E^r(z)}{E^i(z)}\right|_{z=-\ell_1} = \frac{\Gamma_b E_0 e^{-j\beta_1 \ell_1}}{E_0 e^{+j\beta_1 \ell_1}}$$

$$\underline{\Gamma(z = -\ell_1) = \Gamma_b e^{-j2\beta_1 \ell_1}}$$

$$T(z_2 = \ell_2, z_1 = -\ell_1) = \frac{E^t(z_2 = \ell_2)}{E^i(z_1 = -\ell_1)} = \frac{T_b E_0 e^{-j\beta_2 \ell_2}}{E_0 e^{+j\beta_1 \ell_1}}$$

$$\underline{T(z_2 = \ell_2, z_1 = -\ell_1) = T^b e^{-j(\beta_2 \ell_2 + \beta_1 \ell_1)}}$$

Note: $\ell_1 + \ell_2 > 0$

5.2 cont.

What about power flow? The time-ave Poynting vectors are:

$$\bar{S}^i_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\bar{E}^i \times \bar{H}^{i*}\right) = \underline{\hat{a}_z \frac{|E_0|^2}{2\eta_1}} \qquad z<0$$

$$\bar{S}^r_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\bar{E}^r \times \bar{H}^{r*}\right) = -\hat{a}_z |\Gamma^b|^2 \frac{|E_0|^2}{2\eta_1}$$

$$= \underline{-\hat{a}_z |\Gamma^b|^2 |\bar{S}^i_{ave}|} \qquad z<0$$

$$\bar{S}^t_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\bar{E}^t \times \bar{H}^{t*}\right) = \hat{a}_z |T^b|^2 \frac{|E_0|^2}{2\eta_2}$$

massage to get $|\bar{S}^i_{ave}|$

$$= \hat{a}_z |T^b|^2 \frac{\eta_1}{\eta_2} \frac{|E_0|^2}{2\eta_1}$$

$$= \hat{a}_z |T^b|^2 \frac{\eta_1}{\eta_2} |\bar{S}^i_{ave}| \qquad \text{Not obvious}$$

$$= \underline{\hat{a}_z \left(1 - |\Gamma^b|^2\right) |\bar{S}^i_{ave}|} \qquad \text{makes sense} \quad z>0$$

Comparing the underlined equations, we see that power <u>is</u> conserved.

$\Rightarrow$ In media 1, a standing wave will exist

$$\bar{E}^1 = \bar{E}^i + \bar{E}^r = \hat{a}_x E_0 e^{-j\beta_1 z}\left(1 + \Gamma^b e^{j2\beta_1 z}\right) \qquad z<0$$

$$= \hat{a}_x E_0 e^{-j\beta_1 z}\left(1 + \Gamma(z)\right)$$

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5.2 cont.

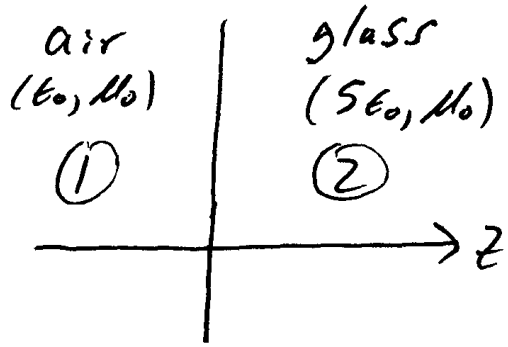
Comparing the max & min values of $\bar{E}'$, as done in Chap 4, leads to the standing wave ratio:

$$SWR = \frac{|\bar{E}'|_{max}}{|\bar{E}'|_{min}} = \frac{1+|\Gamma^b|}{1-|\Gamma^b|}$$

For non-magnetic media (e.g., $\mu_1 = \mu_2 = \mu_0$) or media w/ identical magnetic properties, the SWR simplifies to

$$SWR = \begin{cases} \sqrt{\epsilon_1/\epsilon_2} & \epsilon_1 > \epsilon_2 \\ \sqrt{\frac{\epsilon_2}{\epsilon_1}} & \epsilon_2 > \epsilon_1 \end{cases}$$

Example- A UPW in air (ε_0, μ_0, $z < 0$) is normally incident on a glass half-space ($5\varepsilon_0$, μ_0, $z > 0$). The 2.4 GHz incident electric field is oriented in the x-direction and has a field strength of 0.6 V/m at $z = 0$. Analyze and determine the various associated fields, power densities, and other related quantities.



$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1} = 2\pi(2.4\times10^9)\sqrt{4\pi\times10^{-7}(8.8541878\times10^{-12})}$$

$$= \underline{50.3003 \text{ rad/m}}$$

$$\eta_1 = \eta_0 = \underline{376.7303\,\Omega}$$

$$\beta_2 = \omega\sqrt{\mu_2\epsilon_2} = 2\pi(2.4\times10^9)\sqrt{4\pi\times10^{-7}(5)(8.8541878\times10^{-12})}$$

$$= \underline{112.47485 \text{ rad/m}}$$

$$\eta_2 = \sqrt{\mu_2/\epsilon_2} = \eta_0/\sqrt{\epsilon_{r2}} = \frac{376.7303}{\sqrt{5}} = \underline{168.4789\,\Omega}$$

$$\Gamma^b = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{168.48-376.73}{168.48+376.73} = \underline{-0.381966}$$

$$T^b = 1+\Gamma^b = 1-0.381966 = \underline{0.618034}$$

ex. cont.

$$\bar{E}^i = \hat{a}_x\, 0.6\, e^{-j50.3z} \ (V/m) \quad z \le 0 \ \text{(air)}$$

$$\bar{E}^r = \hat{a}_x (-0.382) 0.6\, e^{+j50.3z}$$

$$\bar{E}^r = -\hat{a}_x\, 0.2292\, e^{+j50.3z} \ (V/m) \quad z \le 0 \ \text{(air)}$$

$$\bar{E}_{air} = \bar{E}^i + \bar{E}^r = \hat{a}_x \left[0.6\, e^{-j50.3z} - 0.2292\, e^{j50.3z}\right] (V/m) \quad z \le 0$$

$$\bar{E}^t = \hat{a}_x (0.618) 0.6\, e^{-j112.475z}$$

$$\bar{E}_{glass} = \bar{E}^t = \hat{a}_x 0.3708\, e^{-j112.475z} \ (V/m) \quad z \ge 0 \ \text{(glass)}$$

$$\bar{H}^i = \hat{a}_y \frac{0.6}{376.73} e^{-j50.3z} = \hat{a}_y\, 1.59265\, e^{-j50.3z} \left(\frac{mA}{m}\right) \quad z \le 0 \ \text{(air)}$$

$$\bar{H}^r = -\hat{a}_y \frac{(-0.382)0.6}{376.73} e^{+j50.3z}$$

$$\bar{H}^r = \hat{a}_y\, 0.60834\, e^{+j50.3z} \ (mA/m) \quad z \le 0 \ \text{(air)}$$

$$\bar{H}_{air} = \bar{H}^i + \bar{H}^r = \hat{a}_y \left[1.593\, e^{-j50.3z} + 0.608\, e^{j50.3z}\right] \left(\frac{mA}{m}\right) \quad z \le 0 \ \text{(air)}$$

$$\bar{H}^t = \hat{a}_y \frac{0.618(0.6)}{168.479} e^{-j112.475z}$$

$$\bar{H}_{glass} = \bar{H}^t = \hat{a}_y\, 2.201\, e^{-j112.475z} \left(\frac{mA}{m}\right) \quad z \ge 0 \ \text{(glass)}$$

ex. cont.

$$SWR_{air} = \frac{1+|-0.382|}{1-|-0.382|} = \underline{\underline{2.236}}$$

$$\bar{S}^{i}_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\hat{a}_x\, 0.6\, e^{-j50.3z} \times \hat{a}_y\, 1.59265\times10^{-3}\, e^{+j50.3z}\right)$$

$$\underline{\bar{S}^{i}_{ave} = \hat{a}_z\, 0.4778\ \frac{mW}{m^2} \quad z \le 0 \ (air)}$$

$$\bar{S}^{r}_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(-\hat{a}_x\, 0.2292\, e^{j50.3z} \times \hat{a}_y\, 6.0834\times10^{-4}\, e^{-j50.3z}\right)$$

$$\underline{\bar{S}^{r}_{ave} = -\hat{a}_z\, 0.0697\ \frac{mW}{m^2} \quad z \le 0 \ (air)}$$

$$\bar{S}^{t}_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left(\hat{a}_x\, 0.3708\, e^{-j112.475z} \times \hat{a}_y\, 2.201\times10^{-3}\, e^{+j112.475z}\right)$$

$$\underline{\bar{S}^{t}_{ave} = \hat{a}_z\, 0.4081\ mW/m^2 \quad z \ge 0 \ (glass)}$$

Check $\bar{S}^{i}_{ave} + \bar{S}^{r}_{ave} \stackrel{?}{=} \bar{S}^{t}_{ave}$

$$\hat{a}_z(0.4778 - 0.0697) \stackrel{?}{=} \hat{a}_z\, 0.4081$$

$$\underbrace{\hat{a}_z\, 0.4081\ \frac{mW}{m^2}}_{air} = \underbrace{\hat{a}_z\, 0.4081\ \frac{mW}{m^2}}_{glass} \quad \therefore$$

5.3 Oblique Incidence - Lossless Media

→ Again, we'll place the boundary @ $z=0$ (x-y plane)

→ Now, the UPW will be incident at an oblique angle

→ We'll define the 'plane of incidence' as the plane defined by the surface normal to the boundary (i.e., $+\hat{a}_z$) and the vector in the direction of incidence (x- & z-directions). In this case, the plane of incidence is the x-z plane.

Two cases

1) Electric field is orthogonal/perpendicular to plane of incidence (i.e. $\bar{y}$-direction)
   ⇒ Perpendicular/horizontal/E polarization $\bar{E}_\perp$

2) Magnetic field is orthogonal to plane of incidence ⇒ electric is parallel $\bar{E}_\parallel$
   ⇒ Parallel/vertical/H polarization

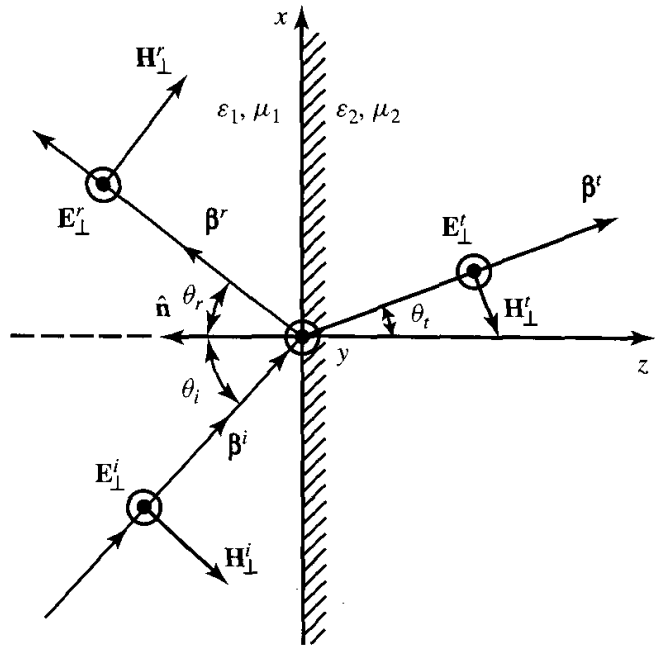


Figure 5-2 Perpendicular (horizontal) polarized uniform plane wave incident at an oblique angle on an interface.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.3.1 Perpendicular (Horizontal or E) Polarization

$$\bar{E}_\perp^i = \hat{a}_y E_\perp^i e^{-j\bar{\beta}_i \cdot \bar{r}} = \hat{a}_y E_0 e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$$\bar{H}_\perp^i = (-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i) H_\perp^i e^{-j\bar{\beta}_i \cdot \bar{r}}$$

$$= (-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i) \frac{E_0}{\eta_1} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$\Rightarrow$ Similar to section 4.2.2 where we defined a $\bar{\beta}^+ = \beta(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i)$

The reflected electric field will be in the y-direction ($\pm$ determined by a new reflection coefficient.

$$\bar{E}_\perp^r = \hat{a}_y E_\perp^r e^{-j\bar{\beta}_r \cdot \bar{r}} = \hat{a}_y \Gamma_\perp^b E_0 e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$\bar{H}_\perp^r = (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) H_\perp^r e^{-j\bar{\beta}_r \cdot \bar{r}}$$

$$= (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) \frac{\Gamma_\perp^b E_0}{\eta_1} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

Note: $E_\perp^r = \Gamma_\perp^b E^i = \Gamma_\perp^b E_0 \Rightarrow \Gamma_\perp^b = \dfrac{E_\perp^r}{E_\perp^i}$

5.3.1 cont.

The transmitted fields are given as

$$\bar{E}_\perp^t = \hat{a}_y E_\perp^t e^{-j\bar{\beta}^t \cdot \bar{r}} = \hat{a}_y T_\perp^b E_0 e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\bar{H}_\perp^t = (-\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t) H_\perp^t e^{-j\bar{\beta}^t \cdot \bar{r}}$$

$$= (-\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t) \frac{T_\perp^b E_0}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Note: $E_\perp^t = T_\perp^b E_\perp^i = T_\perp^b E_0 \Rightarrow T_\perp^b = \frac{E_\perp^t}{E_\perp^i}$

Next, use our tangential boundary conditions (BCs) to find $\Gamma_\perp^b$ + $T_\perp^b$. First, apply tangential electric BC

$$\bar{E}_{1,t} = \left(\bar{E}_\perp^i + \bar{E}_\perp^r\right)\Big|_{z=0} = \left(\bar{E}_\perp^t\right)\Big|_{z=0} = \bar{E}_{2,t}$$ y-direction is tangential

$$\hat{a}_y E_0 e^{-j\beta_1 x\sin\theta_i} + \hat{a}_y \Gamma_\perp^b E_0 e^{-j\beta_1 x\sin\theta_r} = \hat{a}_y T_\perp^b E_0 e^{-j\beta_2 x\sin\theta_t}$$

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma_\perp^b e^{-j\beta_1 x\sin\theta_r} = T_\perp^b e^{-j\beta_2 x\sin\theta_t} \quad (5\text{-}14a)$$

Next, apply <u>tangential</u> magnetic BC (Keep $\hat{a}_x$ terms)

$$\bar{H}_{1,t} = \left(\bar{H}_\perp^i + \bar{H}_\perp^r\right)\Big|_{z=0} = \bar{H}_\perp^t\Big|_{z=0} = \bar{H}_{2,t}$$ $\hat{a}_z$ terms are normal

$$-\hat{a}_x \frac{E_0}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \hat{a}_x \frac{\Gamma_\perp^b E_0}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}$$

$$= -\hat{a}_x \frac{T_\perp^b E_0}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

5.3.1 cont.

which simplifies to

$$\frac{1}{\eta_1}\left[-\cos\theta_i e^{-j\beta_1 x \sin\theta_i} + \Gamma_\perp^b \cos\theta_r e^{-j\beta_1 x \sin\theta_r}\right] = \frac{-T_\perp^b}{\eta_2}\cos\theta_t e^{-j\beta_2 x \sin\theta_t} \quad (5\text{-}14b)$$

Now (5-14a) can only hold true for <u>all</u> values of x if $\sin\theta_i = \sin\theta_r \Rightarrow \underline{\underline{\theta_i = \theta_r}}$ Snell's Law of reflection

$\Rightarrow$ incident + reflected wave <u>must</u> be 'in-phase' @ $z=0$ for all x

This, in turn, yields

$$\left(1+\Gamma_\perp^b\right) e^{-j\beta_1 x \sin\theta_i} = T_\perp^b e^{-j\beta_2 x \sin\theta_t}$$

which can only hold true for all values of x if

$$\underline{\underline{\beta_1 \sin\theta_i = \beta_2 \sin\theta_t}}$$ Snell's Law of refraction

With these two relationships, (5-14a) + (5-14b) have identical exponential terms that can be divided out to yield

$$1+\Gamma_\perp^b = T_\perp^b$$

$$\frac{\cos\theta_i}{\eta_1}\left(-1+\Gamma_\perp^b\right) = -\frac{\cos\theta_t}{\eta_2} T_\perp^b$$

2 eqns - 2 unknowns

5.3.1 cont.

Solving Fresnel reflection + transmission coefficients (perpend. pol.)

$$\boxed{\Gamma_\perp^b = \frac{E_\perp^r}{E_\perp^i} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}}$$

$$\boxed{T_\perp^b = \frac{E_\perp^t}{E_\perp^i} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}}$$

For dielectric media where $\mu_1 \simeq \mu_2 \simeq \mu_0$, these become

$$\Gamma_\perp^b = \frac{\cos\theta_i - \sqrt{\epsilon_2/\epsilon_1}\sqrt{1-\left(\frac{\epsilon_1}{\epsilon_2}\right)\sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_2/\epsilon_1}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}}$$

$$T_\perp^b = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\epsilon_2/\epsilon_1}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}}$$

When $\underline{\epsilon_2 > \epsilon_1}$ $\rightarrow$ $\Gamma_\perp^b$ is real & negative for all θ_i

$\rightarrow$ $T_\perp^b$ is real & positive for all θ_i

going from low ϵ_1 to higher ϵ_2

See Figure 5-3 for trends

$\underline{\epsilon_2 = \epsilon_1}$ $\Rightarrow$ $\Gamma_\perp^b = 0$ & $T_\perp^b = 1$ (no boundary)

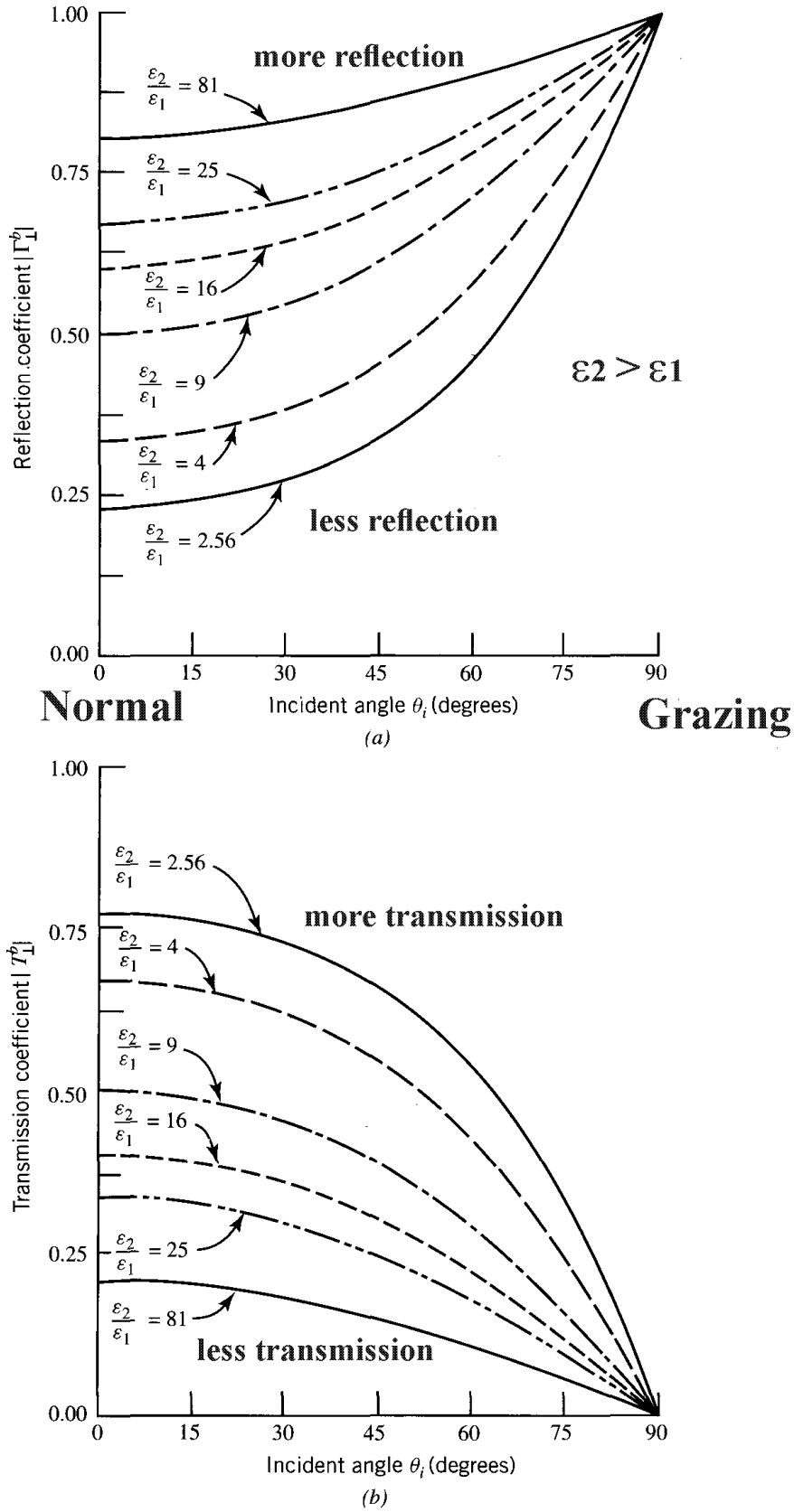


Figure 5-3 Magnitude of coefficients for perpendicular polarization as a function of incident angle. (*a*) Reflection. (*b*) Transmission.

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5.3.1 cont.

When $\epsilon_2 < \epsilon_1$ or $\frac{\epsilon_2}{\epsilon_1} < 1$, there is a range (high to low) of angles $\theta_i \le \theta_c$ where $\Gamma_\perp^b$ & $T_\perp^b$ are real. However, at the critical angle θ_c,

$$\left|\Gamma_\perp^b\right|\Big|_{\substack{\frac{\epsilon_2}{\epsilon_1}<1 \\ \& \\ \theta_i=\theta_c}} = 1$$

(Total internal, to ϵ_1 media, reflection!)

For $\theta_i > \theta_c$, $\Gamma_\perp^b$ & $T_\perp^b$ become complex.

We'll look at this more in section 5.3.4.

Lastly, $\bar{E}_\perp^1 = \bar{E}_\perp^i + \bar{E}_\perp^r$ $\qquad z<0$ (media 1)

$$= \hat{a}_y E_o^i \underbrace{e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}}_{\text{Traveling wave}} \underbrace{\left\{1 + \Gamma_\perp^b e^{j2\beta_1 z\cos\theta_i}\right\}}_{\text{Standing wave}}$$

where we define

$$\Gamma_\perp(z) = \Gamma_\perp^b e^{j2\beta_1 z\cos\theta_i} \qquad z<0$$

5.3.2 Parallel (Vertical or H) Polarization

→ In this case, see Fig 5-4, $\bar{E}$ is in plane of incidence

$$\bar{H}_{\parallel}^{i} = \hat{a}_y H_{\parallel}^{i} e^{-j\bar{\beta}^{i}\cdot\bar{r}} = \hat{a}_y \frac{E_0}{\eta_1} e^{-j\beta_1(x_1 \sin\theta_i + z\cos\theta_i)} \quad z<0 \qquad (5\text{-}20b)$$

and

$$\bar{E}_{\parallel}^{i} = (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) E_0 e^{-j\bar{\beta}^{i}\cdot\bar{r}} \qquad z<0$$

$$= (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \qquad (5\text{-}20a)$$

where $E_{\parallel}^{i} = E_0$ + $H_{\parallel}^{i} = \frac{E_{\parallel}^{i}}{\eta_1} = \frac{E_0}{\eta_1}$

→ The reflected fields are defined

$$\bar{E}_{\parallel}^{r} = (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) E^{r} e^{-j\bar{\beta}^{r}\cdot\bar{r}}$$

$$= (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) \Gamma_{\parallel}^{b} E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \qquad (5\text{-}21a)$$

$$\bar{H}_{\parallel}^{r} = -\hat{a}_y H_{\parallel}^{r} e^{-j\bar{\beta}^{r}\cdot\bar{r}} = -\hat{a}_y \frac{\Gamma_{\parallel}^{b} E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \qquad z<0 \qquad (5\text{-}21b)$$

where $E_{\parallel}^{r} = \Gamma_{\parallel}^{b} E^{i} = \Gamma_{\parallel}^{b} E_0$

+ $H_{\parallel}^{r} = \frac{E_{\parallel}^{r}}{\eta_1} = \frac{\Gamma_{\parallel}^{b} E_0}{\eta_1}$

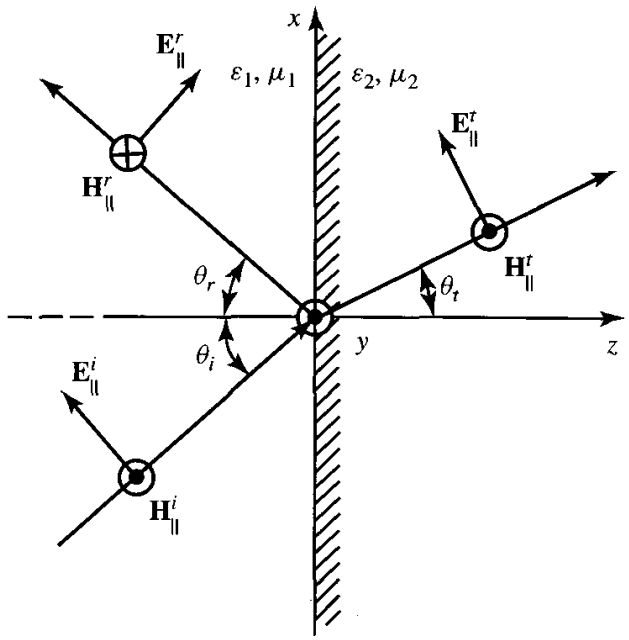

Figure 5-4 Parallel (vertical) polarized uniform plane wave incident at an oblique angle on an interface.

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5.3.2 cont.

The transmitted fields are defined

$$\bar{E}_{\parallel}^{t} = (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t) E_{\parallel}^{t} e^{-j\bar{\beta}^t \cdot \bar{r}} \quad (5\text{-}22a)$$

$$= (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t) T_{\parallel}^{b} E_0 e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)}$$

$$\bar{H}_{\parallel}^{t} = \hat{a}_y H_{\parallel}^{t} e^{-j\bar{\beta}^t \cdot \bar{r}} = \hat{a}_y \frac{T_{\parallel}^{b} E_0}{\eta_2} e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)} \quad z > 0 \quad (5\text{-}22b)$$

where $E_{\parallel}^{t} = T_{\parallel}^{b} E^{i} = T_{\parallel}^{b} E_0$ & $H_{\parallel}^{t} = \frac{E_{\parallel}^{t}}{\eta_2} = \frac{T_{\parallel}^{b} E_0}{\eta_2}$

To determine $\Gamma_{\parallel}^{b}$, $T_{\parallel}^{b}$, θ_r & θ_t, we apply the tangential boundary conditions @ $z = 0$

x & y components are tangential

$$\bar{E}_{1,t}\Big|_{z=0} = \left(\bar{E}_{\parallel}^{i} + \bar{E}_{\parallel}^{r}\right)\Big|_{\substack{z=0 \\ \tan}} = \bar{E}_{\parallel}^{t}\Big|_{\substack{z=0 \\ \tan}} = \bar{E}_{2,t}\Big|_{z=0}$$

$$\hat{a}_x \left(\cos\theta_i E_0 e^{-j\beta_1 x \sin\theta_i} + \cos\theta_r \Gamma_{\parallel}^{b} E_0 e^{-j\beta_1 x \sin\theta_r} \right)$$

$$= \hat{a}_x \cos\theta_t T_{\parallel}^{b} E_0 e^{-j\beta_2 x \sin\theta_t}$$

$$\bar{H}_{1,t}\Big|_{z=0} = \left(\bar{H}_{\parallel}^{i} + \bar{H}_{\parallel}^{r}\right)\Big|_{\substack{z=0 \\ \tan}} = \bar{H}_{\parallel}^{t}\Big|_{\substack{z=0 \\ \tan}} = \bar{H}_{2,t}\Big|_{z=0}$$

$$\hat{a}_y \left(\frac{E_0}{\eta_1} e^{-j\beta_1 x \sin\theta_i} - \frac{\Gamma_{\parallel}^{b} E_0}{\eta_1} e^{-j\beta_1 x \sin\theta_r} \right) = \hat{a}_y \frac{T_{\parallel}^{b} E_0}{\eta_2} e^{-j\beta_2 x \sin\theta_t}$$

5.3.2 cont.

Removing common terms, we get

$$\cos\theta_i e^{-j\beta_1 x \sin\theta_i} + \Gamma_{\parallel}^b \cos\theta_r e^{-j\beta_1 x \sin\theta_r} = T_{\parallel}^b \cos\theta_t e^{-j\beta_2 x \sin\theta_t} \quad (5\text{-}23a)$$

$$\frac{1}{\eta_1}\left(e^{-j\beta_1 x \sin\theta_i} - \Gamma_{\parallel}^b e^{-j\beta_1 x \sin\theta_r}\right) = \frac{1}{\eta_2} T_{\parallel}^b e^{-j\beta_2 x \sin\theta_t} \quad (5\text{-}23b)$$

Again, @ $z=0$, w/ $\eta_1, \eta_2, \Gamma_{\parallel}^b$, & $T_{\parallel}^b$ being constant, (5-23b) can only be true for all x if

$$\beta_1 x \sin\theta_i = \beta_1 x \sin\theta_r = \beta_2 x \sin\theta_t$$

$$\Downarrow \qquad\qquad \Downarrow$$

Snell's laws of reflection & refraction

$$\underline{\theta_i = \theta_r} \quad (5\text{-}24a) \qquad \underline{\beta_1 \sin\theta_i = \beta_2 \sin\theta_t} \quad (5\text{-}24b)$$

With the now identical exponentials removed, we get

$$\cos\theta_i + \Gamma_{\parallel}^b \cos\overset{\theta_i}{\cancel{\theta_r}} = T_{\parallel}^b \cos\theta_t$$

$$\frac{1}{\eta_1}\left(1 - \Gamma_{\parallel}^b\right) = \frac{1}{\eta_2} T_{\parallel}^b$$

which can be solved for $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$

5.3.2 cont.

$$\boxed{\Gamma_{\parallel}^{b} = \frac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}}$$

$$\boxed{T_{\parallel}^{b} = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}}$$

Fresnel reflection (5-24c) & transmission (5-24d) coefficients (parallel polarization)

For dielectric media where $\mu_1 \simeq \mu_2 \simeq \mu_0$

$$\Gamma_{\parallel}^{b}\Big|_{\mu_1=\mu_2} = \frac{-\cos\theta_i + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_1/\epsilon_2}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}} \quad (5\text{-}25c)$$

$$T_{\parallel}^{b}\Big|_{\mu_1=\mu_2} = \frac{2\sqrt{\epsilon_1/\epsilon_2}\cos\theta_i}{\cos\theta_i + \sqrt{\epsilon_1/\epsilon_2}\sqrt{1-(\epsilon_1/\epsilon_2)\sin^2\theta_i}} \quad (5\text{-}25d)$$

* For $\underline{\epsilon_2 > \epsilon_1}$, notice in Fig 5-5a that (low to high)
  there is an angle where $|\Gamma_{\parallel}^{b}| = 0$,
  i.e., the Brewster angle $\theta_B = \theta_i$.
* Note that θ_B gets larger as $\frac{\epsilon_2}{\epsilon_1}$ increases.

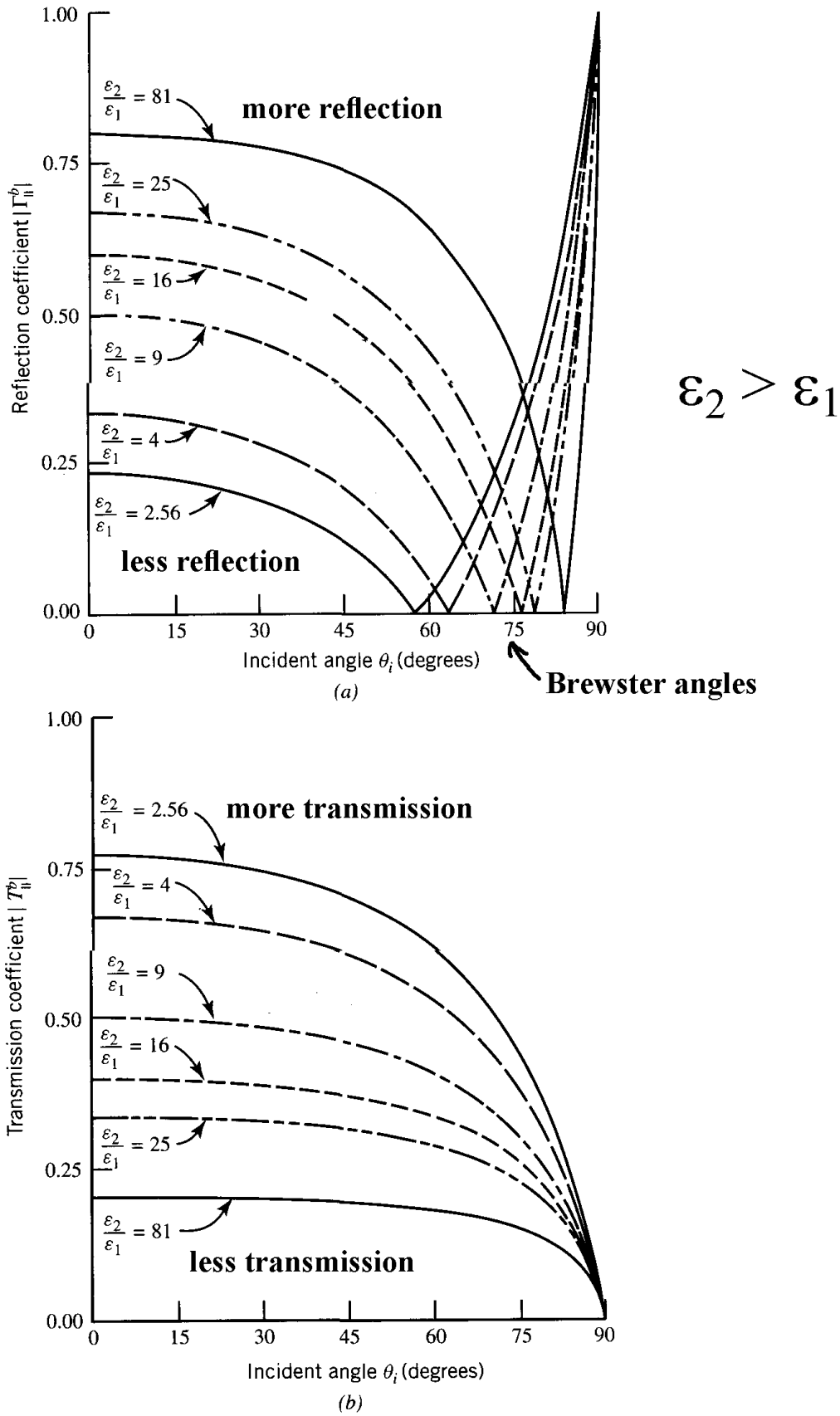


Figure 5-5 Magnitude of coefficients for parallel polarization as a function of incident angle. (*a*) Reflection. (*b*) Transmission.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.3.2 cont.

$\boxed{\epsilon_2 > \epsilon_1}$ → $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$ are both real. Further, for $\theta_i < \theta_B$, $\Gamma_{\parallel}^b < 0$, while for $\theta_i > \theta_B$, $\Gamma_{\parallel}^b > 0$. $T_{\parallel}^b$ is greater than zero for all θ_i

$\boxed{\epsilon_2 = \epsilon_1}$ ⇒ $\Gamma_{\parallel}^b = 0$ and $T_{\parallel}^b = 1$ (no boundary)

$\boxed{\epsilon_2 < \epsilon_1}$ → Again there will be a critical (high to low) angle $\underline{\theta_c}$ where $|\Gamma_{\parallel}^b|_{\epsilon_2/\epsilon_1 < 1} = 1$

→ $\theta_i < \theta_c$ $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$ are real

→ $\theta_i > \theta_c$ $\Gamma_{\parallel}^b$ + $T_{\parallel}^b$ are complex

$$\bar{E}_{\parallel}^1 = \bar{E}_{\parallel}^i + \bar{E}_{\parallel}^r = \hat{a}_x \cos\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left(1 + \Gamma_{\parallel}^b e^{j2\beta_1 z\cos\theta_i}\right)$$

$$- \hat{a}_z \sin\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left(1 - \Gamma_{\parallel}^b e^{j2\beta_1 z\cos\theta_i}\right)$$

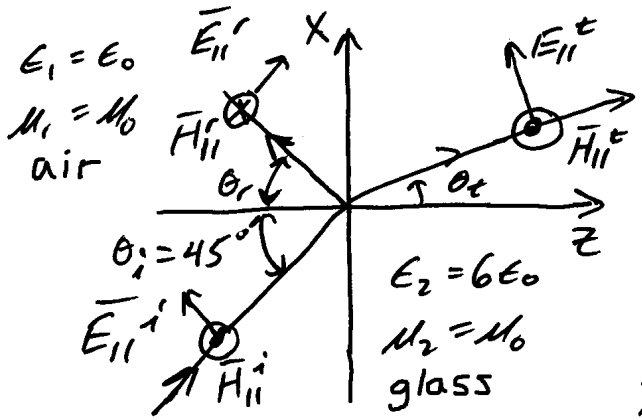
↑ Traveling ↑ standing

$$= \hat{a}_x \cos\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left[1 + \Gamma_{\parallel}(z)\right]$$

$$- \hat{a}_z \sin\theta_i E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\left[1 - \Gamma_{\parallel}(z)\right]$$

where $\Gamma_{\parallel}(z) = \Gamma_{\parallel}^b e^{j2\beta_1 z\cos\theta_i}$ $\quad \underline{z < 0}$

Example- A UPW in air (ε_0, μ_0, $z < 0$) is obliquely incident on a glass half-space ($6\varepsilon_0$, μ_0, $z > 0$) at an angle of 45°. The 800 MHz incident magnetic field is oriented in the y-direction and has a field strength of 0.4 mA/m at z = 0. Analyze and determine the various associated fields, power densities, and other related quantities.



$$\eta_1 = \eta_0 = \underline{376.7303\,\Omega}$$

$$\beta_1 = \omega\sqrt{\mu_0\epsilon_0} = \frac{2\pi\, 800\times10^6}{2.9979\times10^8} = \underline{16.76676\ \frac{\text{rad}}{\text{m}}}$$

$$\eta_2 = \eta_0/\sqrt{\epsilon_{r2}} = \frac{376.7303}{\sqrt{6}} = \underline{153.7995\,\Omega}$$

$$\beta_2 = \omega\sqrt{\mu_0 6\epsilon_0} = 2\pi(800\times10^6)\sqrt{4\pi\times10^{-7}(6)\,8.8541878\times10^{-12}}$$

$$\underline{\beta_2 = 41.070007\ \frac{\text{rad}}{\text{m}}}$$

Per (5-24a), $\underline{\theta_r = \theta_i = 45°}$

Per (5-24b), $\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$

$$16.76676 \sin 45° = 41.07 \sin\theta_t$$

$$\hookrightarrow \theta_t = \sin^{-1}\left(\frac{16.76676 \sin 45°}{41.07001}\right) = \sin^{-1}(0.288675)$$

$$= 16.778655° \Rightarrow \underline{\theta_t = 16.7787°}$$

Per (5-24c), $\Gamma_{\parallel}^b = \dfrac{-\eta_1\cos\theta_i + \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$

$$\Gamma_{\parallel}^{b} = \frac{-376.7303\cos 45^\circ + 153.7995\cos 16.7787^\circ}{376.7303\cos 45^\circ + 153.7995\cos 16.7787^\circ} = \frac{-124.432}{413.64}$$

$$\underline{\underline{\Gamma_{\parallel}^{b} = -0.28802}}$$

$$T_{\parallel}^{b} = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{2(153.7995)\cos 45^\circ}{413.64}$$

$$\underline{\underline{T_{\parallel}^{b} = 0.52583}}$$

Per (5-20d), $H_{\parallel}^{i} = \frac{E_0}{\eta_1} \Rightarrow E_0 = 376.7303(0.4\times10^{-3})$

$$\underline{E_0 = 0.150692 \text{ V/m}}$$

(5-20b) $\underline{\bar{H}_{\parallel}^{i} = \hat{a}_y\, 0.4\, e^{-j16.767(0.707x+0.707z)} \left(\text{mA/m}\right)}$

(5-20a) $\underline{\bar{E}_{\parallel}^{i} = (0.707\hat{a}_x - 0.707\hat{a}_z)\,0.1507\, e^{-j16.767(0.707x+0.707z)} \left(\text{V/m}\right)}$

(5-21a) $\bar{E}_{\parallel}^{r} = (0.707\hat{a}_x + 0.707\hat{a}_z)\,0.1507(-0.288)\,e^{-j16.767(0.707x-0.707z)}$

$$\underline{\bar{E}_{\parallel}^{r} = (0.707\hat{a}_x + 0.707\hat{a}_z)(-43.402)\,e^{-16.767(0.707x-0.707z)} \left(\frac{\text{mV}}{\text{m}}\right)}$$

(5-21b) $\bar{H}_{\parallel}^{r} = -\hat{a}_y \frac{-43.402\times10^{-3}}{376.7307}\, e^{-j16.767(0.707x-0.707z)}$

$$\underline{\bar{H}_{\parallel}^{r} = +\hat{a}_y\, 0.1152\, e^{-j16.767(0.707x-0.707z)} \left(\text{mA/m}\right)}$$

(5-22a) $\bar{E}_{\parallel}^{t} = (\hat{a}_x \cos 16.78^\circ - \hat{a}_z \sin 16.78^\circ)(0.526)(0.1507)\, e^{-j41.07(x \sin 16.8^\circ + z \cos 16.8^\circ)}$

$$\bar{E}_{\parallel}^{t} = (0.9574\,\hat{a}_x - 0.2887\,\hat{a}_z)(79.268)\, e^{-j41.07(0.2887x + 0.9574z)} \left(\frac{mV}{m}\right)$$

(5-22b) $\bar{H}_{\parallel}^{t} = \hat{a}_y \dfrac{0.52583(0.150692)}{153.7995}\, e^{-j41.07(0.2887x + 0.9574z)}$

$$\bar{H}_{\parallel}^{t} = \hat{a}_y\, 0.5152\, e^{-41.07(0.2887x + 0.9574z)} \left(\frac{mA}{m}\right)$$

(5-26a) $\Gamma_{\parallel}(z) = \Gamma_{\parallel}^{b}\, e^{j2\beta_1 z \cos\theta_i}$

$= -0.28802\, e^{j2(16.76676)\cos 45^\circ z}$

$$\Gamma_{\parallel}(z) = -0.28802\, e^{j23.7118z} \qquad z \le 0$$

(5-26) $\bar{E}_{\parallel}^{i} = \hat{a}_x \cos 45^\circ (0.1507)\, e^{-j16.767(0.707x + 0.707z)} [1 + \Gamma_{\parallel}(z)]$

$- \hat{a}_z \sin 45^\circ (0.1507)\, e^{-j16.767(0.707x + 0.707z)} [1 - \Gamma_{\parallel}(z)]$

$\bar{E}_{\parallel}^{i} = \hat{a}_x\, 0.10656\, e^{-j16.767(0.707x + 0.707z)} [1 + \Gamma_{\parallel}(z)]$

$- \hat{a}_z\, 0.10656\, e^{-j16.767(0.707x + 0.707z)} [1 - \Gamma_{\parallel}(z)] \left(\frac{V}{m}\right)$

$z \le 0$

5.3.3 Total Transmission - Brewster Angle

Is there angle of incidence or set of conditions where the reflection coefficient goes to zero?

A. Perpendicular (Horizontal) Polarization

Here set $\Gamma_\perp^b = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = 0$ (5-27)

algebra $\cos\theta_i = \sqrt{\dfrac{\mu_1}{\mu_2}\dfrac{\epsilon_2}{\epsilon_1}}\cos\theta_t$ (5-27a)

use $\cos^2 A = 1 - \sin^2 A$ and square both sides

$$1 - \sin^2\theta_i = \frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}\left(1 - \sin^2\theta_t\right)$$

Next use $\beta_1 \sin\theta_i = \beta_2 \sin\theta_t \Rightarrow \sin^2\theta_t = \dfrac{\mu_1}{\mu_2}\dfrac{\epsilon_1}{\epsilon_2}\sin^2\theta_i$

$$1 - \sin^2\theta_i = \frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}\left(1 - \frac{\mu_1}{\mu_2}\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)$$

algebra

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\mu_1/\mu_2 - \mu_2/\mu_1}} \qquad (5\text{-}28a)$$

For most dielectrics, $\mu_1 = \mu_2 = \mu_0$, which implies $\sin\theta_i \to \infty \Rightarrow$ NOT POSSIBLE

$\Rightarrow \Gamma_\perp^b = 0$ can't be achieved.

5.3.3 cont.

B. Parallel (Vertical) Polarization

Here, set $\Gamma_{\parallel}^{b} = \dfrac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = 0$ (5-30)

algebra ↳ $\cos\theta_i = \sqrt{\dfrac{\mu_2}{\mu_1}\left(\dfrac{\epsilon_1}{\epsilon_2}\right)}\cos\theta_t$ (5-30a)

$$\cos^2\theta_i = 1 - \sin^2\theta_i = \frac{\mu_2}{\mu_1}\left(\frac{\epsilon_1}{\epsilon_2}\right)\cos^2\theta_t = \frac{\mu_2}{\mu_1}\left(\frac{\epsilon_1}{\epsilon_2}\right)\left(1-\sin^2\theta_t\right)$$

and use $\sin^2\theta_t = \dfrac{\mu_1}{\mu_2}\dfrac{\epsilon_1}{\epsilon_2}\sin^2\theta_i$

$$1-\sin^2\theta_i = \frac{\mu_2}{\mu_1}\frac{\epsilon_1}{\epsilon_2}\left(1 - \frac{\mu_1}{\mu_2}\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)$$

algebra ↳

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2}} \qquad (5\text{-}31a)$$

θ_i real if $\left|\epsilon_2/\epsilon_1 - \mu_2/\mu_1\right| \le \left|\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2\right|$

Text typo (?) on equation (5-32) and (5-32a)

Solve for Brewster angle θ_B

$$\theta_i = \theta_B = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}} \qquad (5\text{-}33)$$

$$= \cos^{-1}\sqrt{\frac{\epsilon_1}{\epsilon_1+\epsilon_2}} \qquad (5\text{-}33a)$$

$$= \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad (5\text{-}33b)$$

Example- For a parallel polarized UPW in air (ε_0, μ_0, $z < 0$) obliquely incident on a glass half-space ($6\varepsilon_0$, μ_0, $z > 0$), find the Brewster angle. Then, determine the reflection and refraction angles as well as the reflection and transmission coefficients.

From earlier, $\eta_1 = 376.7303\,\Omega$ + $\eta_2 = 153.7995\,\Omega$

$\beta_1 = 16.76676$ rad/m $\quad \beta_2 = 41.070007$ rad/m

Per (5-33), $\theta_i = \theta_B = \sin^{-1}\sqrt{\dfrac{6\varepsilon_0}{\varepsilon_0 + 6\varepsilon_0}} = \sin^{-1}\sqrt{6/7}$

$$\underline{\underline{\theta_i = \theta_B = 67.792346^\circ}}$$

Per (5-24a), $\underline{\underline{\theta_r = \theta_i = \theta_B = 67.792346^\circ}}$

Per (5-24b), $\theta_t = \sin^{-1}\left(\dfrac{16.76676 \sin 67.79^\circ}{41.070007}\right) = \sin^{-1}(0.378)$

$$\underline{\underline{\theta_t = 22.20765^\circ}}$$

Per (5-24c), $\Gamma_{\parallel}^{b} = \dfrac{-376.73 \cos 67.79^\circ + 153.8 \cos 22.21^\circ}{376.73 \cos 67.79 + 153.8 \cos 22.21^\circ}$

$$\underline{\underline{\Gamma_{\parallel}^{b} = 0}}$$

Per (5-24d), $T_{\parallel}^{b} = \dfrac{2(153.8) \cos 67.79^\circ}{376.73 \cos 67.79^\circ + 153.8 \cos 22.21^\circ}$

$$T_{\parallel}^{b} = \frac{116.261}{284.7818}$$

$$\underline{\underline{T_{\parallel}^{b} = 0.40825}}$$

5.3.4 Total Reflection - Critical Angle

Now, consider if there is an angle or set of conditions where the reflection coefficient has a magnitude of 1.

A. Perpendicular (Horizontal) Polarization

Set $|\Gamma_\perp^b| = \left| \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \right| = 1$

This is true when $\eta_1 \cos\theta_t = jB$ ← some imaginary #

Since $\left| \dfrac{A \mp jB}{A \pm jB} \right| = \dfrac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2}} = 1$

Further, given that $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ is a real pos #, this requires $\cos\theta_t$ to be imaginary.

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} \quad \text{use } \beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$

$$= \sqrt{1 - \frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2} \sin^2\theta_i}$$

To be imaginary, the argument must be negative $\Rightarrow \dfrac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2} \sin^2\theta_i \geq 1$

5.3.4 cont.

A. cont. Solving for θ_i in this case yields

$$\theta_i \geq \theta_c = \sin^{-1}\sqrt{\frac{\mu_2 \epsilon_2}{\mu_1 \epsilon_1}} \qquad (5\text{-}35b)$$

where θ_c is the critical angle.

Further, this implies $\mu_1 \epsilon_1 \geq \mu_2 \epsilon_2$ for normal materials.

If $\mu_1 \simeq \mu_2 \simeq \mu_0$, $\theta_i \geq \theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}$

$+ \ \epsilon_1 \geq \epsilon_2$

$\Rightarrow$ Can have a critical angle when going from higher to lower permittivity!

At $\theta_i = \theta_c$, $\theta_t = 90°$ (parallel to boundary)

$$\Gamma_\perp^b \Big|_{\theta_i = \theta_c} = 1$$

$$T_\perp^b \Big|_{\theta_i = \theta_c} = 2$$

$$\left.\begin{aligned} \bar{E}_\perp^t &= \hat{a}_y \, 2 E_0 \, e^{-j\beta_2 x} \\ \bar{H}_\perp^t &= \hat{a}_z \frac{2E_0}{\eta_2} e^{-j\beta_2 x} \end{aligned}\right\} \begin{array}{l} \underline{\text{Surface wave}} \\ \text{traveling in } +x\text{-dir} \\ \text{parallel to boundary} \end{array}$$

5.3.4 cont.

A. cont. $\bar{S}^t_{ave}\Big|_{\theta_i=\theta_c} = \hat{a}_x \dfrac{2|E_0|^2}{\eta_2}$ ← Power density is parallel to boundary!

⇒ No power across $z=0$ plane in $+z$-direction

$$\bar{S}^i_{ave}\Big|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}\left(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right)$$

$$\bar{S}^r_{ave}\Big|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}\left(\hat{a}_x \sin\theta_i - \hat{a}_z \cos\theta_i\right)$$

Since $\cos^2 A + \sin^2 A = 1$,

$$\left|\bar{S}^i_{ave}\right|_{\theta_i=\theta_c} = \left|\bar{S}^r_{ave}\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}$$

What happens to the transmitted wave when $\theta_i > \theta_c$ under these conditions?

$$\sin\theta_t\Big|_{\theta_i>\theta_c} = \frac{\beta_1}{\beta_2}\sin\theta_i\Big|_{\theta_i>\theta_c} = \frac{\sqrt{\mu_1\epsilon_1}}{\sqrt{\mu_2\epsilon_2}}\sin\theta_i\Big|_{\theta_i>\theta_c} > 1$$

This can only hold if $\theta_t = \theta_R + j\theta_X$ is complex.

$$\cos\theta_t\Big|_{\theta_i>\theta_c} = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i}$$

$$= \pm j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}\Bigg|_{\theta_i>\theta_c}$$

5.3.4 cont.

A. cont. $$E_\perp^t\Big|_{\theta_i>\theta_c} = \hat{a}_y T_\perp^b E_0 e^{-j\beta_2 x \sin\theta_t} e^{-j\beta_2 z\cos\theta_t}$$

$$= \hat{a}_y T_\perp^b E_0 e^{-j\beta_2 x\left(\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i\right)} e^{-j\beta_2 z\left(-j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i-1}\right)}$$

sub sub

$$= \hat{a}_y T_\perp^b E_0 \underbrace{e^{-\beta_2 z\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i-1}}}_{\text{decays in z-dir.}} \underbrace{e^{-j\beta_2 x\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i}}_{\text{prop. in x-dir}}$$

let $$\alpha_e = \beta_2\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i-1} = \omega\sqrt{\mu_1\epsilon_1\sin^2\theta_i-\mu_2\epsilon_2}\Big|_{\theta_i>\theta_c}$$

$$\beta_e = \beta_2\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i = \omega\sqrt{\mu_1\epsilon_1}\sin\theta_i = \beta_1\sin\theta_i\Big|_{\theta_i>\theta_c}$$

So $$\underline{E_\perp^t\Big|_{\theta_i>\theta_c} = \hat{a}_y T_\perp^b E_0 e^{-\alpha_e z} e^{-j\beta_e x}}$$

where $$v_{pe} = \frac{\omega}{\beta_e} = \frac{\omega}{\beta_2\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i} = \frac{v_{p2}}{\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i}\Big\}_{\theta_i>\theta_c} > 1$$

$$= \frac{1}{\sqrt{\mu_1\epsilon_1}\sin\theta_i} < v_{p2}$$

We have what is called a <u>tightly bound slow surface wave</u> when $\theta_i > \theta_c$

$\Rightarrow$ slower than v_{p2} in $+x$-direction

$\Rightarrow$ decays rapidly in $+z$-direction

5.3.4 cont.

A. cont. $$\Gamma_\perp^b\Big|_{\theta_i \geq \theta_c} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}\Bigg|_{\theta_i \geq \theta_c}$$

use $$\cos\theta_t = -j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}\Bigg|_{\theta_i \geq \theta_c}$$

$$\Gamma_\perp^b\Big|_{\theta_i \geq \theta_c} = \frac{\sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_i + j\sqrt{\mu_1/\epsilon_1}\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}}{\sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_i - j\sqrt{\frac{\mu_1}{\epsilon_1}}\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}}$$

$$= |\Gamma_\perp^b| e^{j2\psi_\perp}$$

where $|\Gamma_\perp^b| = 1$ and $\psi_\perp = \tan^{-1}\left(\frac{X_\perp}{R_\perp}\right)$

w/ $$X_\perp = \sqrt{\frac{\mu_1}{\epsilon_1}}\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}$$

$$R_\perp = \sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_i$$

Similarly, $$T_\perp^b\Big|_{\theta_i \geq \theta_c} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}\Bigg|_{\theta_i \geq \theta_c}$$

$$= \frac{2\sqrt{\mu_2/\epsilon_2}\cos\theta_i}{\sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_i - j\sqrt{\mu_1/\epsilon_1}\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}}\Bigg|_{\theta_i \geq \theta_c}$$

$$= \underbrace{\frac{2R_\perp}{\sqrt{R_\perp^2 + X_\perp^2}}}_{|T_\perp^b|} e^{j\psi_\perp}$$

5.3.4 cont.

A. cont.

What happens to the transmitted power density when $\theta_i \geq \theta_c$?

$$\bar{S}_{ave}^{t}\Big|_{\theta_i \geq \theta_c} = \tfrac{1}{2} \mathrm{Re}\left(\bar{E}^t \times \bar{H}^{t*}\right)$$

$$= \tfrac{1}{2} \mathrm{Re}\Big[\left(\hat{a}_y T_\perp^b E_0 e^{-\alpha_e z} e^{-j\beta_e z}\right) \times \left(-\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t\right)^* \frac{T_\perp^{b*} E_0^*}{\eta_2} e^{-\alpha_e z} e^{+j\beta_e z}\Big]$$

$$= \frac{1}{2} \frac{|T_\perp^b|^2 |E_0|^2}{\eta_2} e^{-2\alpha_e z} \mathrm{Re}\Big[\hat{a}_z \underbrace{(\cos\theta_t)^*}_{+j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}} + \hat{a}_x \underbrace{(\sin\theta_t)^*}_{\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i}\Big]$$

$$\bar{S}_{ave}^{t}\Big|_{\theta_i \geq \theta_c} = \hat{a}_x \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} \sin\theta_i \frac{|T_\perp^b|^2 |E_0|^2}{2\eta_2} e^{-2\alpha_e z}\Bigg|_{\theta_i \geq \theta_c}$$

$\Rightarrow$ No power flow in $+z$-direction across boundary!

5.3.4 cont.

A. cont.

So, when going from higher ϵ_1 to lower ϵ_2 materials @ oblique incidence, perpendicular polarization, we have 3 possibilities

1) $\theta_i < \theta_c \Rightarrow$ regular reflection + refraction
   $\Rightarrow \theta_t > \theta_i = \theta_r \Rightarrow$ power transfers
2) $\theta_i = \theta_c \Rightarrow \theta_i = \theta_r + \theta_t \rightarrow 90^\circ$
   $\Rightarrow$ surface wave in x-direction
   $\Rightarrow$ No power transfer
3) $\theta_i > \theta_c \Rightarrow \theta_i = \theta_r + \theta_t$ complex
   $\Rightarrow$ Tightly bound slow surface wave w/ power flow in x-direction + No power transfer

See Fig 5.7

B. Parallel (Vertical) Polarization

$\Rightarrow$ Similar derivation

$\Rightarrow \theta_c$ does NOT depend on polarization $\Rightarrow$ same expression

$\Rightarrow$ Expressions for $\Gamma_{\parallel}^{b}$, $T_{\parallel}^{b}$, $\bar{\zeta}_{\parallel}^{t}$ are different

$\Rightarrow$ Similar wave phenomenom

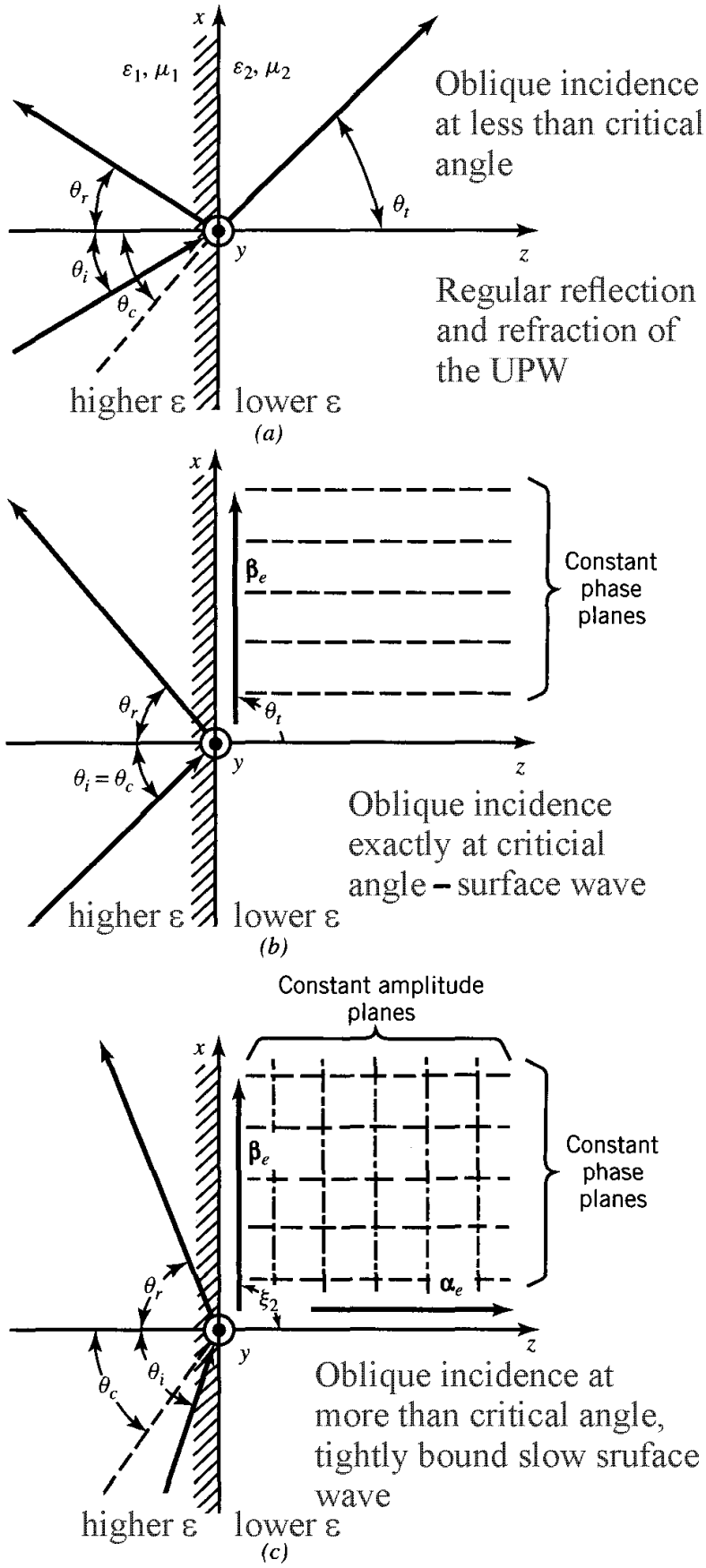


Figure 5-7 Critical angle wave propagation along an interface.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

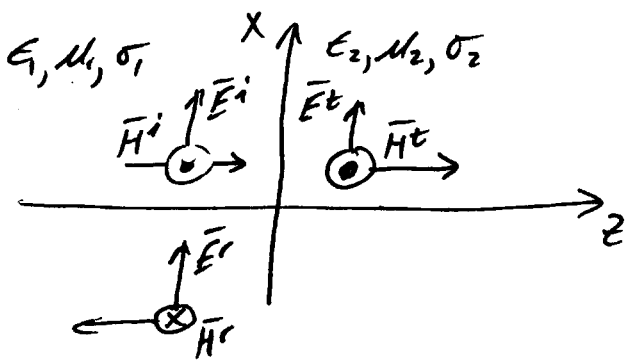
5.4 Lossy Media

5.4.1 Normal Incidence: Conductor-Conductor Interface

For lossy materials, we refer back to Chapter 4 to calculate

propagation constant $\equiv \gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$ (m^{-1})

intrinsic impedance $\equiv \eta_c = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}$ (Ω)



$$\left.\begin{aligned}
\bar{E}^i &= \hat{a}_x E_0 e^{-\alpha_1 z} e^{-j\beta_1 z} \\
\bar{H}^i &= \hat{a}_y \frac{E_0}{\eta_1} e^{-\alpha_1 z} e^{-j\beta_1 z} \\
\bar{E}^r &= \hat{a}_x \Gamma^b E_0 e^{+\alpha_1 z} e^{+j\beta_1 z} \\
\bar{H}^r &= -\hat{a}_y \frac{\Gamma^b E_0}{\eta_1} e^{+\alpha_1 z} e^{+j\beta_1 z}
\end{aligned}\right\} z \le 0$$

$$\left.\begin{aligned}
\bar{E}^t &= \hat{a}_x T^b E_0 e^{-\alpha_2 z} e^{-j\beta_2 z} \\
\bar{H}^t &= \hat{a}_y \frac{T^b E_0}{\eta_2} e^{-\alpha_2 z} e^{-j\beta_2 z}
\end{aligned}\right\} z \ge 0$$

5.4.1 cont.

where $\Gamma^b = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ & $T^b = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

$$\left.\begin{aligned} \bar{S}_{ave}^{i} &= \hat{a}_z \frac{|E_0|^2}{2} e^{-2\alpha_1 z} \mathrm{Re}\left(\frac{1}{\eta_1^*}\right) \\ \bar{S}_{ave}^{r} &= -\hat{a}_z |\Gamma^b|^2 \frac{|E_0|^2}{2} e^{+2\alpha_1 z} \mathrm{Re}\left(\frac{1}{\eta_1^*}\right) \end{aligned}\right\} z \le 0$$

$$\bar{S}_{ave}^{t} = \hat{a}_z |T^b|^2 \frac{|E_0|^2}{2} e^{-2\alpha_2 z} \mathrm{Re}\left(\frac{1}{\eta_2^*}\right) \quad z \ge 0$$

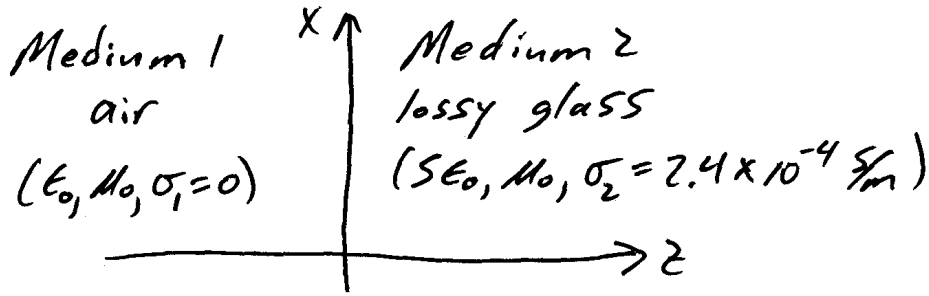
Note: η_1 & η_2 as well as Γ^b & T^b can be <u>complex</u>.

$$\bar{E}^1 = \bar{E}^i + \bar{E}^r$$
$$= \hat{a}_x E_0 e^{-\alpha_1 z} e^{-j\beta_1 z}\left(1 + \Gamma^b e^{+2\alpha_1 z} e^{+j2\beta_1 z}\right) \quad z \le 0$$

$$\bar{H}^1 = \bar{H}^i + \bar{H}^r$$
$$= \hat{a}_x \underbrace{\frac{E_0}{\eta_1} e^{-\alpha_1 z} e^{-j\beta_1 z}}_{\text{Traveling}} \underbrace{\left(1 - \Gamma^b e^{+2\alpha_1 z} e^{+j2\beta_1 z}\right)}_{\text{Standing wave}} \quad z \le 0$$

$$\Gamma(z) = \Gamma^b e^{2\alpha_1 z} e^{j2\beta_1 z} \quad z \le 0$$

Example- A UPW in air (ε_0, μ_0, $\sigma \approx 0$, $z < 0$) is normally incident on a lossy glass half-space ($5\varepsilon_0$, μ_0, $\sigma = 2.4\times10^{-4}$ S/m, $z > 0$). The 2.4 GHz incident electric field is oriented in the x-direction and has a field strength of 0.6 V/m at $z = 0$. Analyze and determine the various associated fields, related quantities, and power densities at $z = 0$ and at $z = 10$ m.



$$\gamma_1 = \sqrt{j\omega\mu_0(\sigma_1 + j\omega\varepsilon_0)} = \sqrt{j2\pi(2.4\times10^9)\,j2\pi(2.4\times10^9)8.854\times10^{-12}}$$

$$= 0 + j50.30028 \ (1/m)$$

$$\underline{\alpha_1 = 0} \qquad \underline{\beta_1 = 50.30028 \ \text{rad/m}}$$

$$\eta_1 = \sqrt{\frac{j\omega\mu_0}{\sigma_1 + j\omega\varepsilon_0}} = \sqrt{\mu_0/\varepsilon_0} = \underline{376.7303\ \Omega}$$

$$\gamma_2 = \sqrt{j\omega\mu_0(\sigma_2 + j\omega 5\varepsilon_0)}$$

$$= \sqrt{j2\pi(2.4\times10^9)(4\pi\times10^{-7})(2.4\times10^{-4} + j2\pi(2.4\times10^9)5(8.854\times10^{-12})}$$

$$= 0.0202175 + j112.474848 \ (1/m)$$

$$\underline{\alpha_2 = 0.02022 \ (\text{Np/m})} \qquad \underline{\beta_2 = 112.47485 \ \frac{\text{rad}}{\text{m}}}$$

$$\underline{\delta_2 = \frac{1}{\alpha_2} = 49.462\,\text{m}}$$

$$\eta_2 = \sqrt{\frac{j\omega\mu_0}{\sigma_2 + j\omega 5\varepsilon_0}} = \underline{168.47891 + j0.03028\ \Omega}$$

$$\underline{\eta_2 = 168.4789\angle 0.0103^\circ\ \Omega}$$

$$\Gamma^b = \frac{\eta_2 + \eta_1}{\eta_2 + \eta_1} = \frac{168.4789\angle 0.0103^\circ - 376.7303}{168.4789\angle 0.0103^\circ + 376.7303}$$

$$\Gamma^b = -0.38197 + j7.6763\times10^{-5} = 0.38197\angle 179.9885^\circ$$

$$T^b = \frac{2\eta_2}{\eta_2 + \eta_1} = 0.618034 + j7.6763\times10^{-5}$$

$$T^b = 0.618034\angle 0.00712^\circ$$

$$\bar{E}^i = \hat{a}_x\, 0.6\, e^{-j50.3z}\ (V/m) \quad z \le 0 \qquad (5\text{-}47a)$$

$$\bar{H}^i = \hat{a}_y\, 1.59265\, e^{-j50.3z}\ \left(\frac{mA}{m}\right) \quad z \le 0 \qquad (5\text{-}47b)$$

$$\bar{E}^r = \hat{a}_x\left(-0.38197 + j7.676\times10^{-5}\right)0.6\, e^{+j50.3z} \qquad (5\text{-}48a)$$

$$\bar{E}^r = -\hat{a}_x\left(0.22918 - j4.606\times10^{-5}\right)e^{+j50.3z}\ (V/m) \quad z \le 0$$

$$\bar{H}^r = -\hat{a}_y\,\frac{\left(-0.38197 + j7.676\times10^{-5}\right)0.6}{376.7303}\, e^{+j50.3z} \qquad (5\text{-}48b)$$

$$\bar{H}^r = \hat{a}_y\left(0.60834 - j1.223\times10^{-4}\right)e^{+j50.3z}\ \left(\frac{mA}{m}\right) \quad z \le 0$$

$$\boxed{\begin{aligned} \bar{E}_{air} &= \bar{E}^i + \bar{E}^r = \hat{a}_x\left[0.6\, e^{-j50.3z} - \left(0.2292 - j4.606\times10^{-5}\right)e^{j50.3z}\right]\left(\frac{V}{m}\right) \\ \bar{H}_{air} &= \hat{a}_y\left[1.59265\, e^{-j50.3z} + \left(0.60834 - j1.223\times10^{-4}\right)e^{j50.3z}\right]\left(\frac{mA}{m}\right) \\ & \qquad z \le 0 \end{aligned}}$$

$$\bar{E}^t = \hat{a}_x (0.618 + j7.676\times10^{-5})\, 0.6\, e^{-0.02z} e^{-j112.475z} \quad (5\text{-}49a)$$

$$\bar{E}_{glass} = \bar{E}^t = \hat{a}_x (0.3708 + j4.606\times10^{-5})\, e^{-0.0202z} e^{-j112.475z} \left(\frac{V}{m}\right) \quad z \geq 0$$

$$\bar{H}^t = \hat{a}_y \frac{(0.618 + j7.676\times10^{-5})\, 0.6}{168.479 + j0.0303} e^{-0.0202z} e^{-j112.475z}$$

$$\bar{H}_{glass} = \bar{H}^t = \hat{a}_y (2.201 - j1.223\times10^{-5})\, e^{-0.0202z} e^{-j112.475z} \left(\frac{mA}{m}\right) \quad z \geq 0$$

$$\bar{S}^i_{ave} = \tfrac{1}{2} Re(\bar{E}^i \times \bar{H}^{i*}) = \hat{a}_z\, 0.4778 \left(\frac{mW}{m^2}\right) \quad z \leq 0$$

$$\bar{S}^r_{ave} = \tfrac{1}{2} Re\left(-\hat{a}_x (0.22918 - j4.606\times10^{-5}) e^{j50.3z} \times \hat{a}_y (0.6083\times10^{-3} + j1.22\times10^{-7}) e^{-j50.3}\right)$$

$$= \tfrac{1}{2} Re\left(-\hat{a}_z\, 1.39419\times10^{-4}\right) = -\hat{a}_z\, 6.971\times10^{-5}$$

$$\bar{S}^r_{ave} = -\hat{a}_z\, 0.0697 \left(\frac{mW}{m^2}\right) \quad z \leq 0$$

$$(5\text{-}51c) \quad \bar{S}^t_{ave} = \hat{a}_z |T^b|^2 \frac{|E_0|^2}{2} e^{-2\alpha_2 z} Re\left(\frac{1}{\eta_2^*}\right)$$

$$= \hat{a}_z\, 0.618034^2 \frac{0.6^2}{2} e^{-2(0.0202)z} Re\left(\frac{1}{168.479 - j0.0303}\right)$$

$$\bar{S}^t_{ave} = \hat{a}_z\, 0.40809\, e^{-0.04043z} \left(\frac{mW}{m^2}\right) \quad z \geq 0$$

$$\bar{S}^t_{ave}(z=0) = \hat{a}_z\, 0.4081 \left(\frac{mW}{m^2}\right)$$

$$\bar{S}^t_{ave}(z=10m) = \hat{a}_z\, 0.2724 \left(\frac{mW}{m^2}\right) \leftarrow$$ 66.74 % of value @ $z=0$

5.4.2 Oblique Incidence: Dielectric-Conductor Interface

→ Here, a UPW will be obliquely incident on a planar boundary from a lossless medium 1 (i.e, a dielectric) onto a lossy medium 2 ($\sigma > 0$).

→ We'll express the transmitted electric field in a general enough fashion to allow for either perpendicular or parallel polarization. See Fig 5-10

Here, $\gamma_1 = j\beta_1 = j\omega\sqrt{\mu_1\epsilon_1}$ while

$$\gamma_2 = \alpha_2 + j\beta_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\epsilon_2)}$$

and

$$\eta_1 = \sqrt{\mu_1/\epsilon_1} \quad \text{and} \quad \eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}}$$

$$\bar{E}^t = \bar{E}_2 e^{-\gamma_2(x\sin\theta_t + z\cos\theta_t)} = \bar{E}_2 e^{-(\alpha_2 + j\beta_2)(x\sin\theta_t + z\cos\theta_t)}$$

Now, Snell's Law of Refraction is modified to

$$\gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

$$j\beta_1 \sin\theta_i = (\alpha_2 + j\beta_2)\sin\theta_t$$

$$\hookrightarrow \sin\theta_t = \frac{j\beta_1}{\alpha_2 + j\beta_2}\sin\theta_i$$

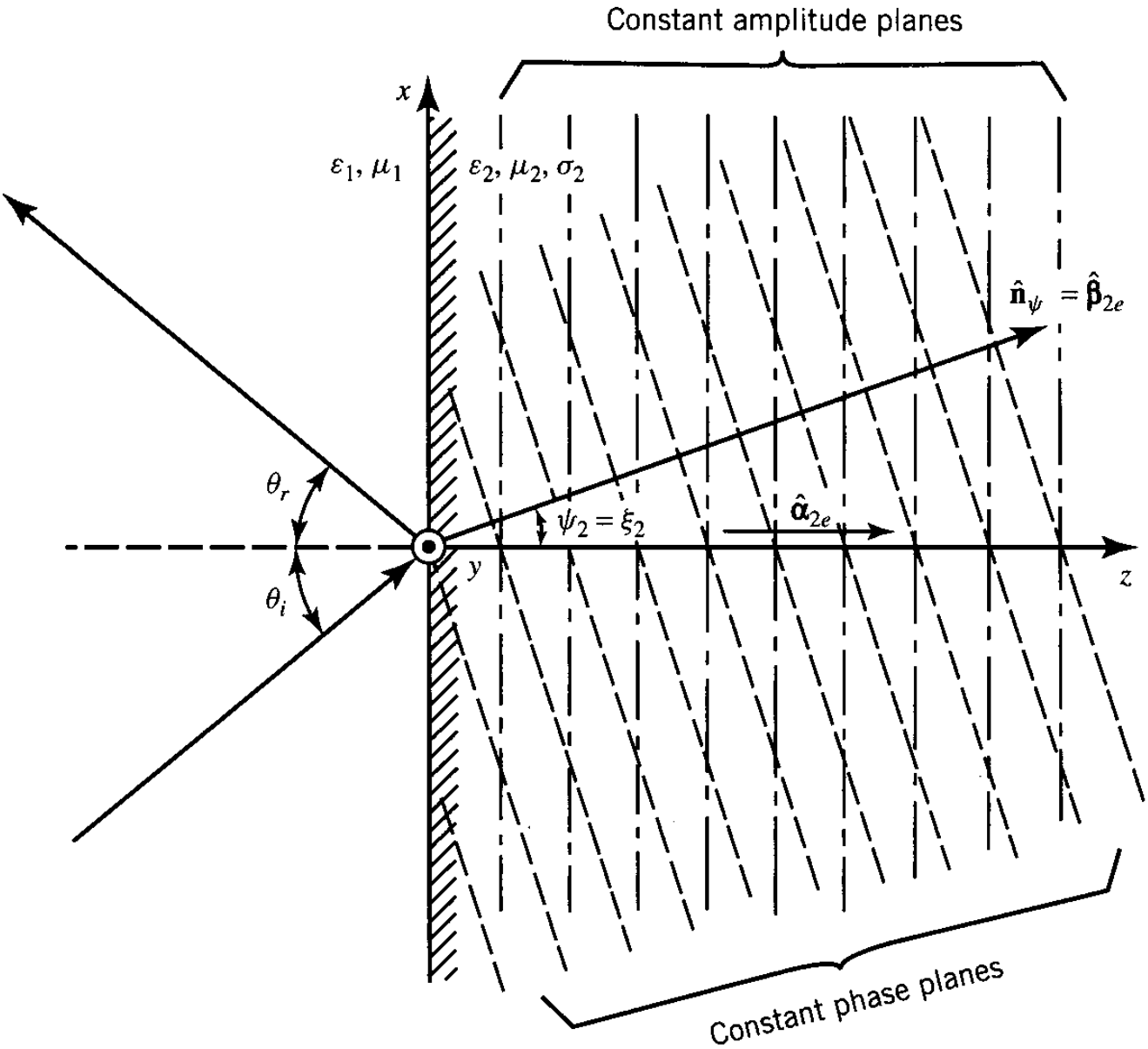


Figure 5-10 Oblique wave incidence upon a dielectric–conductor interface.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.4.2 cont.

and $\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{j\beta_1}{\alpha_2+j\beta_2}\right)^2\sin^2\theta_i}$

$$= S e^{j\zeta} = S(\cos\zeta + j\sin\zeta)$$ ← some complex #

zeta

w/ these we can re-write $\bar{E}^t$ as

$$\bar{E}^t = \bar{E}_2\, e^{-(\alpha_2+j\beta_2)\left[x\frac{j\beta_1}{\alpha_2+j\beta_2}\sin\theta_i + zS(\cos\zeta+j\sin\zeta)\right]}$$

$$= \bar{E}_2\, e^{-zS(\alpha_2\cos\zeta-\beta_2\sin\zeta)}\, e^{-j\left[\beta_1 x\sin\theta_i + zS(\alpha_2\sin\zeta+\beta_2\cos\zeta)\right]}$$

Define $p = S(\alpha_2\cos\zeta - \beta_2\sin\zeta) = \alpha_{ze}$

$q = S(\alpha_2\sin\zeta + \beta_2\cos\zeta)$

$$\underline{\bar{E}^t = \bar{E}_2\, e^{-zp}\, e^{-j(\beta_1 x\sin\theta_i + zq)}}$$ ← Non-uniform wave!

Assuming $\bar{E}_2$ is real, we get a time-domain sol'n

$$\bar{\mathcal{E}}^t = \bar{E}_2\, e^{-zp}\cos\left[\omega t - (\beta_1 x\sin\theta_i + zq)\right]$$

$$= \bar{E}_2\, e^{-\alpha_{ze} z}\cos\left[\omega t - (\beta_1 x\sin\theta_i + zq)\right]$$

↑ constant <u>amplitude</u> planes are parallel to $z=0$ interface

→ constant <u>phase</u> planes are at some angle ψ_2 that is NOT θ_t (complex #)

(See Fig 5-10)

5.4.2 cont.

Looking at the phase argument

$$\omega t - (\beta_1 x \sin\theta_i + z q) = \omega t - \sqrt{(\beta_1 \sin\theta_i)^2 + q^2} \times \left[\frac{\beta_1 \sin\theta_i \, x}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} + \frac{q z}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} \right]$$

factor this out

Now, define

$$u = \beta_1 \sin\theta_i$$

$$\sin\psi_2 = \frac{\beta_1 \sin\theta_i}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} = \frac{u}{\sqrt{u^2 + q^2}}$$

$$\cos\psi_2 = \frac{q}{\sqrt{(\beta_1 \sin\theta_i)^2 + q^2}} = \frac{q}{\sqrt{u^2 + q^2}}$$

+

$$\psi_2 = \tan^{-1}\left(\frac{\beta_1 \sin\theta_i}{q}\right) = \tan^{-1}\left(\frac{u}{q}\right)$$

Now

$$\bar{\mathcal{E}}^t = \bar{E}_2 e^{-z\rho} \, \mathrm{Re}\left\{ e^{j\left[\omega t - \sqrt{u^2+q^2}\left(\frac{u x}{\sqrt{u^2+q^2}} + \frac{q z}{\sqrt{u^2+q^2}}\right)\right]} \right\}$$

$$= \bar{E}_2 e^{-z\rho} \, \mathrm{Re}\left\{ e^{j\left[\omega t - \beta_{2e}(x \sin\psi_2 + z \cos\psi_2)\right]} \right\}$$

$$= \bar{E}_2 e^{-z\rho} \, \mathrm{Re}\left\{ e^{j\left[\omega t - \beta_{2e}(\hat{n}_\psi \cdot \bar{r})\right]} \right\}$$

where $\hat{n}_\psi = \hat{a}_x \sin\psi_2 + \hat{a}_z \cos\psi_2$

$$\beta_{2e} = \sqrt{u^2 + q^2}$$

5.4.2 cont.

After all of this, we can note:

1) Angle of refraction is really ψ_2
   (NOT θ_t which has become complex)

2) Refracted wave travels in $\hat{n}_\psi$ direction

3) constant phase planes are orthogonal to $\hat{n}_\psi$

To find phase velocity in medium 2

$$\frac{d}{dt}\left(\omega t - \beta_{2e}(\hat{n}_\psi \cdot \bar{r})\right) = \frac{d}{dt}(\text{constant})$$

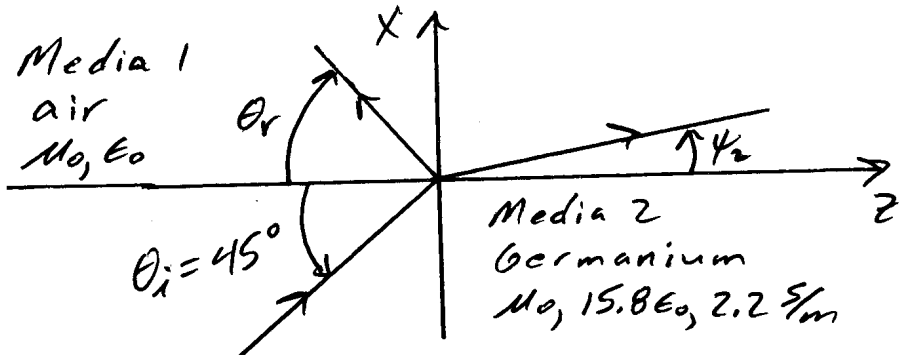
$$\omega - \beta_{2e}\left(\hat{n}_\psi \cdot \frac{d\bar{r}}{dt}\right) = 0$$

$$v_{pr} = \hat{n}_\psi \cdot \frac{d\bar{r}}{dt} = \frac{\omega}{\beta_{2e}} = \frac{\omega}{\sqrt{u^2+q^2}}$$

$$\underline{v_{pr} = \frac{\omega}{\sqrt{(\beta_1 \sin\theta_i)^2+q^2}}}$$

← function of θ_i as well as material properties $\mu_1, \epsilon_1, \mu_2, \epsilon_2, + \sigma_2$

Example- A 6 GHz UPW in air (ε_0, μ_0, $z < 0$) is obliquely incident at $\theta_i = 45°$ on a germanium half-space ($15.8\varepsilon_0$, μ_0, σ = 2.2 S/m, $z > 0$). Find the intrinsic impedance and propagation constants in the two materials, reflected θ_r & transmitted θ_t angles, reflection & transmission coefficients for both parallel and perpendicular polarizations, effective attenuation α_{2e} & phase β_{2e} constants, true refracted angle ψ_2, and phase velocity v_{pr} in the germanium.



Propagation constants

Per (4-28), $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$

$$\gamma_1 = \sqrt{j2\pi 6\times10^9(4\pi\times10^{-7})(0 + j2\pi 6\times10^9(8.8542\times10^{-12})}$$

$$\gamma_1 = j125.750701 \ m^{-1}$$

$$\gamma_2 = \sqrt{j2\pi 6\times10^9(4\pi\times10^{-7})(2.2 + j2\pi 6\times10^9(15.8)8.8542\times10^{-12}}$$

$$\gamma_2 = 102.143607 + j510.178881 \ m^{-1}$$

Intrinsic impedances

Per (4-30), $z_w = \eta_c = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}$

$$\eta_1 = \sqrt{\frac{j\omega 4\pi\times10^{-7}}{0 + j\omega 8.8542\times10^{-12}}} \Rightarrow \eta_1 = \eta_0 = 376.7303 \ \Omega$$

$$\eta_2 = \sqrt{\frac{j2\pi 6\times10^9(4\pi\times10^{-7})}{2.2 + j2\pi 6\times10^9(15.8)8.8542\times10^{-12}}}$$

$$\eta_2 = 89.27911 + j17.87469 = 91.051\angle 11.3216° \ \Omega$$

Using Snell's law of refraction for lossy media (5-54) $\gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$

(5-55a) $\sin\theta_t = \frac{\gamma_1}{\gamma_2}\sin\theta_i = \frac{j125.75}{102.14+j510.18}\sin 45^\circ$

$= 0.16757 + j0.03355$

(5-55b) $\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-(0.168+j0.034)^2}$

$= 0.986447 - j0.005699 = S e^{j\xi}$

$S = |\cos\theta_t| = |0.98645 - j0.0057| = 0.986463$

$\xi = \measuredangle \cos\theta_t = \measuredangle(0.98645 - j0.0057) = -0.005778 \text{ rad}$

$= -0.33103^\circ$

Per (5-55a) $\theta_t = \sin^{-1}(\sin\theta_t) = \sin^{-1}(0.1676 + j0.03355)$

$\underline{\theta_t = 0.16827 + j0.03402 \text{ rad}}$

or $\underline{\theta_t = 9.641 + j1.949^\circ}$

Snell's law of reflection

Per (5-15a), $\underline{\theta_r = \theta_i = \pi/4 = 45^\circ}$

<u>Perpendicular Polarization Γ & T</u>

(5-17a) $\Gamma_\perp^b = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$

$= \frac{(89.28+j17.87)\cos 45^\circ - 376.73(0.986 - j0.0057)}{(89.28+j17.87)\cos 45^\circ + 376.73(0.986 - j0.0057)}$

$\underline{\Gamma_\perp^b = -0.70835 + j0.051106 = 0.71019\angle 175.8734^\circ}$

(5-17b) $$T_\perp^b = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$= \frac{2(89.28 + j17.87)\cos 45^\circ}{(89.28 + j17.87)\cos 45^\circ + 376.73(0.986 - j0.0057)}$$

$$\underline{T_\perp^b = 0.29165 + j0.051106 = 0.296093\angle 9.93909^\circ}$$

Parallel Polarization Γ & T

(5-24c) $$\Gamma_{\parallel}^b = \frac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

$$= \frac{-376.73\cos 45^\circ + (89.28 + j17.87)(0.986 - j0.0057)}{376.73\cos 45^\circ + (89.28 + j17.87)(0.986 - j0.0057)}$$

$$\underline{\Gamma_{\parallel}^b = -0.499148 + j0.072402 = 0.29609\angle 9.93909^\circ}$$

(5-24d) $$T_{\parallel}^b = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

$$= \frac{2(89.28 + j17.87)\cos 45^\circ}{376.73\cos 45^\circ + (89.28 + j17.87)(0.986 - j0.0057)}$$

$$\underline{T_{\parallel}^b = 0.35871 + j0.053972 = 0.36275\angle 8.5566^\circ}$$

Effective attenuation in media 2

(5-57a) $$\alpha_{2e} = \rho = s(\alpha_2 \cos\zeta - \beta_2 \sin\zeta)$$

$$= 0.98645(102.144\cos(-0.331^\circ) - 510.1789\sin(-0.331^\circ))$$

$$\underline{\alpha_{2e} = 103.6669 \ \mathrm{Np/m}}$$

Effective phase constant in media 2

(5-60a) $u = \beta_1 \sin\theta_i = 125.75 \sin 45^\circ$

$u = 88.919174 \ m^{-1}$

(5-57b) $q = s(\alpha_2 \sin\xi + \beta_2 \cos\xi)$

$= 0.98645(102.144 \sin(-0.331^\circ) + 510.1709 \cos(-0.331^\circ)$

$= 502.682189 \ m^{-1}$

(5-61b) $\beta_{2e} = \sqrt{u^2 + q^2} = \sqrt{88.919^2 + 502.682^2}$

$\underline{\beta_{2e} = 510.48605 \ rad/m}$

True refracted angle in media 2

(5-60d) $\psi_2 = \tan^{-1}(u/q) = \tan^{-1}\left(\frac{88.919}{502.682}\right)$

$\underline{\underline{\psi_2 = 10.03125^\circ}}$ ← Not too far from $\mathrm{Re}(\theta_t) = 9.641^\circ$ or $|\theta_t| = 9.836^\circ$

Phase velocity in media 2

(5-62a) $v_{pr} = \frac{\omega}{\beta_{2e}} = \frac{2\pi(6\times10^9)}{510.486}$

$\underline{v_{pr} = 0.7384945 \times 10^8 \ m/s}$

(Note: $c/\sqrt{15.8} = 0.7542 \times 10^8 \ m/s$)

$\Rightarrow$ about 2.1% difference

5.4.3 Oblique Incidence: Conductor-Conductor Interface

→ Section 5.3.4 Critical Angle ($\epsilon_1 > \epsilon_2$) showed that for $\theta_i \geq \theta_c$ the transmitted field was a non-uniform plane wave (tightly bound surface wave) where the constant amplitude planes (z-direction) α_{ze} were orthogonal to constant phase planes β_{ze} (x-direction)

→ Section 5.4.2 lossless to loss/conductor
@ oblique incidence, we also got a non-uniform plane waves, but the $\psi_2 = \delta_2$ angle between $\underset{\hat{a}_z}{\overline{\alpha}_{ze}}$ + $\underset{\hat{n}_y}{\overline{\beta}_{ze}}$ is between 0° and 90°

where $\psi_2 \to 0$ as $\sigma_2 \to \infty$ (good/perfect conductors)

$\psi_2 \to 90°$ as $\sigma_2 \to 0$ (toward being a dielectric)

→ Math for a conductor-conductor boundary quite complicated, even by this chapters standards!

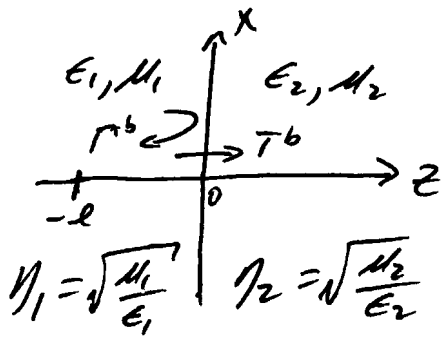
5.5 Reflection and Transmission of Multiple Interfaces

→ limit ourselves to normal incidence

5.5.1 Reflection Coefficient of a Single Slab Layer

Back in section 5.2

$$\Gamma^b = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$



$$\Gamma_{in}(z=-\ell) = \Gamma^b e^{-j2\beta_1\ell}$$

Now, like TLs, let's consider this in terms of input impedances

$$@\ z=0^+,\ Z_{in}(z=0^+) = \eta_2 = \frac{E^t|_{z=0^+}}{H^t|_{z=0^+}}$$

$$@\ z=-\ell,\ Z_{in}(z=-\ell) = \frac{E_1|_{z=-\ell}}{H_1|_{z=-\ell}} = \frac{E^{tot}|_{z=-\ell}}{H^{tot}|_{z=-\ell}}$$

from section 5.2, the magnitude of the total electric field + magnetic field in material 1 is

$$E^{tot}\Big|_{z=-\ell} = (E^i + E^r)\Big|_{z=-\ell} = E_0 e^{+j\beta_1\ell}\left[1 + \Gamma^b e^{-j2\beta_1\ell}\right] = E_0 e^{j\beta_1\ell}\left[1 + \Gamma_{in}(-\ell)\right]$$

$$H^{tot}\Big|_{z=-\ell} = (H^i + H^r)\Big|_{z=-\ell} = \frac{E_0}{\eta_1} e^{+j\beta_1\ell}\left[1 - \Gamma^b e^{-j2\beta_1\ell}\right] = \frac{E_0}{\eta_1} e^{j\beta_1\ell}\left[1 - \Gamma_{in}(-\ell)\right]$$

5.5.2 cont.

So

$$Z_{in}(z=-\ell) = \frac{E_0 e^{j\beta_1 \ell}\left[1+\Gamma^b e^{-j2\beta_1\ell}\right]}{\frac{E_0}{\eta_1} e^{j\beta_1\ell}\left[1-\Gamma^b e^{-j2\beta_1\ell}\right]} = \eta_1\left(\frac{1+\Gamma^b e^{-j2\beta_1\ell}}{1-\Gamma^b e^{-j2\beta_1\ell}}\right)$$

$$= \eta_1\left(\frac{1+\Gamma_{in}(-\ell)}{1-\Gamma_{in}(-\ell)}\right) = \eta_1\left[\frac{\eta_2 + j\eta_1\tan(\beta_1\ell)}{\eta_1 + j\eta_2\tan(\beta_1\ell)}\right] \quad (5\text{-}66d)$$

$\Rightarrow$ All of these expressions are analogous to TL theory w/ η_i swapped w/ $Z_{c,i}$

Next, as shown in Fig 5-11, we'll extend this to the case of a slab of thickness d and material 2 between two semi-infinite materials 1 ($z \le -d, \epsilon_1, \mu_1$) and 3 ($z \ge 0, \epsilon_3, \mu_3$).

Starting @ $z=0^+$

$$Z_{in}(z=0^+) = \eta_3$$

The reflection coefficient @ $z=0^-$ is then

$$\Gamma_{in}(z=0^-) = \frac{Z_{in}(z=0^+) - \eta_2}{Z_{in}(z=0^+) + \eta_2} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

just inside material 2 by 2-3 interface.

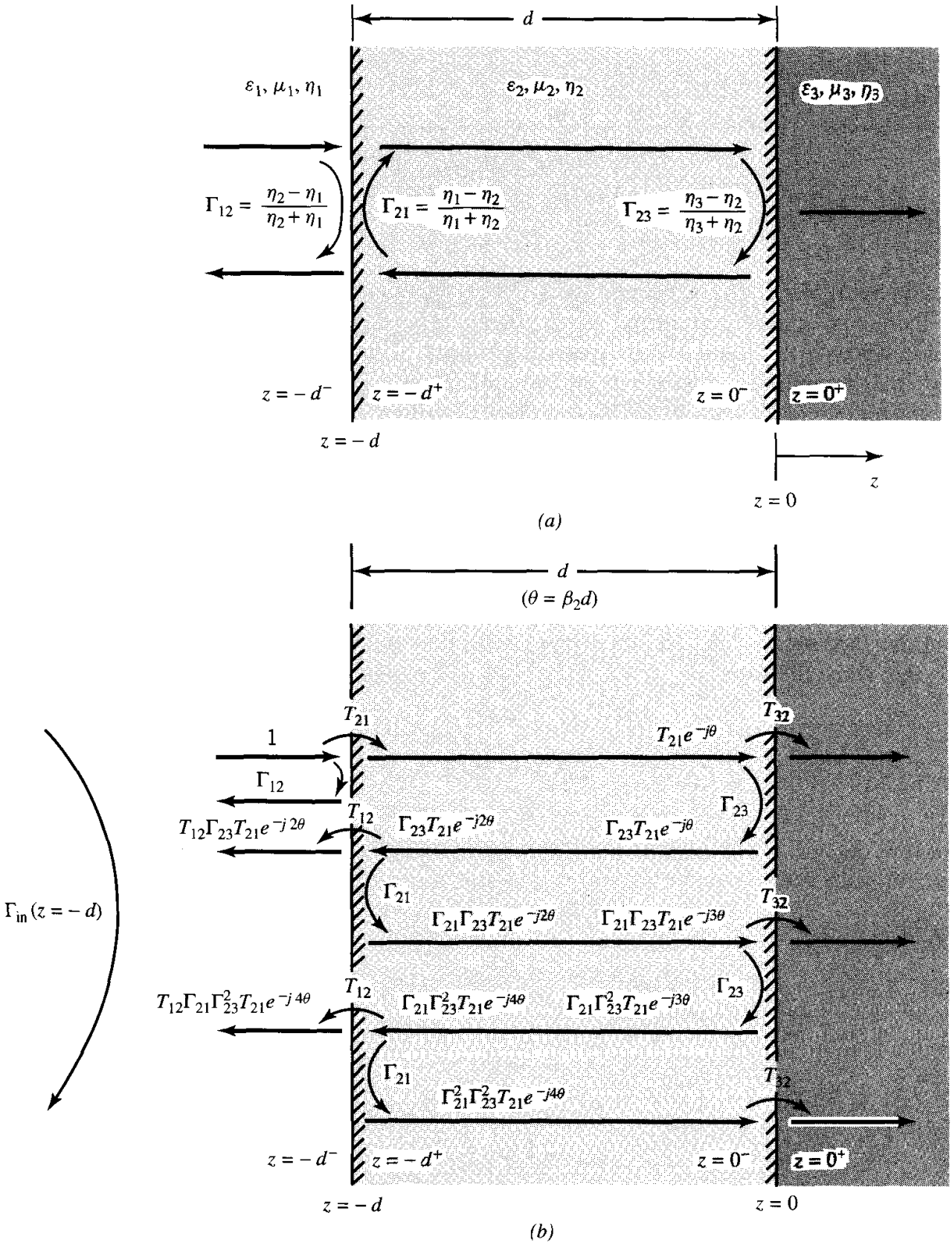


Figure 5-11 Impedances and reflection and transmission coefficients for wave propagation in dielectric slab. (*a*) Dielectric slab. (*b*) Reflection and transmission coefficients.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

5.5.2 cont.

Translating to the other side of the material 2 slab, @ $z = -d^+$ (following (5-66d))

$$Z_{in}(z=-d^+) = \eta_2 \frac{1+\Gamma_{in}(z=0^-)e^{-j\beta_2 d}}{1-\Gamma_{in}(z=0^-)e^{-j\beta_2 d}}$$

use $\Gamma_{in}(0^-) = \frac{\eta_3-\eta_2}{\eta_3+\eta_2}$

$$= \eta_2 \left[\frac{(\eta_3+\eta_2)+(\eta_3-\eta_2)e^{-j2\beta_2 d}}{(\eta_3+\eta_2)-(\eta_3-\eta_2)e^{-j2\beta_2 d}} \right]$$

Then, going across the material 1-2 boundary, @ $z = -d^-$, the reflection coefficient is

$$\Gamma_{in}(z=-d^-) = \frac{Z_{in}(z=-d^+)-\eta_1}{Z_{in}(z=-d^+)+\eta_1}$$

$$= \frac{\eta_2\left[(\eta_3+\eta_2)+(\eta_3-\eta_2)e^{-j2\beta_2 d}\right]-\eta_1\left[(\eta_3+\eta_2)-(\eta_3-\eta_2)e^{-j2\beta_2 d}\right]}{\eta_2\left[(\eta_3+\eta_2)+(\eta_3-\eta_2)e^{-j2\beta_1 d}\right]+\eta_1\left[(\eta_3+\eta_2)-(\eta_3-\eta_2)e^{-j2\beta_2 d}\right]}$$

This expression can be simplified by defining individual/intrinsic coefficients @ each boundary as shown in Fig 5-11a

$$\Gamma_{12} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}, \quad \Gamma_{21} = \frac{\eta_1-\eta_2}{\eta_1+\eta_2}, \text{ + } \Gamma_{23} = \frac{\eta_3-\eta_2}{\eta_3+\eta_2}$$

which can be used to get

$$\Gamma_{in}(z=-d^-) = \frac{\Gamma_{12}+\Gamma_{23}e^{-j2\beta_2 d}}{1+\Gamma_{12}\Gamma_{23}e^{-j2\beta_2 d}}$$

5.5.2 cont.

An alterate derivation is to consider all the multiple reflections/transmissions at the material 1-2 and material 2-3 interfaces as shown in Fig 5-11b. We'll skip this as it leads to the same result.

$$\Gamma_{in}(z=-d^-) = \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_2 d}}$$

If $|\Gamma_{12}|$ & $|\Gamma_{23}|$ are 'small'

$$\underline{\Gamma_{in}(z=-d^-) \simeq \Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}$$ ← Much simpler

What is 'small'? $|\Gamma_{12}| \leq 0.2$ & $|\Gamma_{23}| \leq 0.2$

$\Rightarrow |\Gamma_{12}||\Gamma_{23}| < 0.2^2 = 0.04$

What if some or all of the materials are lossy? Per 'Basic Electromagnetic Theory' by Paris & Hurd (1969) –

$$\Gamma_{in} = \frac{E^r}{E^i} = \frac{(1-z_{12})(1+z_{23}) + (1+z_{12})(1-z_{23})e^{-2\gamma_2 d}}{(1+z_{12})(1+z_{23}) + (1-z_{12})(1-z_{23})e^{-2\gamma_2 d}}$$

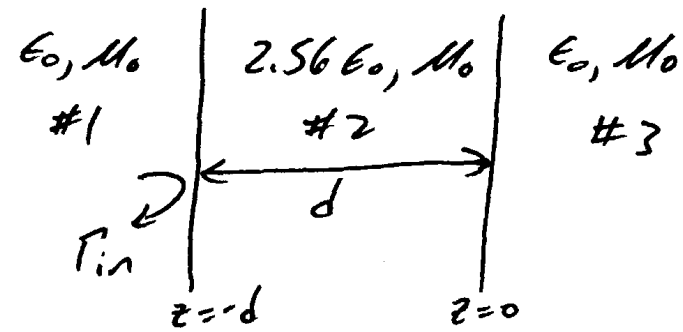
$$T = \frac{E^t}{E^i} = \frac{4}{(1-z_{12})(1-z_{23})e^{-\gamma_2 d} + (1+z_{12})(1+z_{23})e^{+\gamma_2 d}}$$

where $z_{ij} = \dfrac{\mu_i \gamma_j}{\mu_j \gamma_i}$ $\quad i,j = 1,2,3$

$$\gamma_k = \pm\sqrt{j\omega\mu_k(\sigma_k + j\omega\epsilon_k)} \quad k = 1,2,3$$

5.5.2 cont.

ex. Dielectric slab bounded by air



At 10 GHz, how thick should the slab be to ensure $\Gamma_{in} = 0$?

$$\eta_1 = \eta_3 = \eta_0 = 376.7303\,\Omega$$

$$\eta_2 = \sqrt{\frac{\mu_0}{2.56\epsilon_0}} = \frac{\eta_0}{\sqrt{2.56}} = 235.45645\,\Omega$$

$$\Gamma_{12} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{235.46 - 376.73}{235.46 + 376.73} = -0.23077$$

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} = \frac{376.73 - 235.46}{376.73 + 235.46} = +0.23077$$

$$\beta_2 = \omega\sqrt{\mu_0 2.56\epsilon_0} = 2\pi(10\times10^9)\sqrt{4\pi\times10^{-7}(2.56)8.8542\times10^{-12}}$$

$$\beta_2 = 335.3352 \text{ rad/m}$$

For $|\Gamma_{in}| = 0$, set the magnitude of the numerator of (5-67d) to zero

$$\left|\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}\right| = 0$$

5.5.2 cont.

ex. cont.

Since $\Gamma_{12} = -\Gamma_{23} = -0.23077$, we can write

$$|\Gamma_{12}|\,|1 - e^{-j2\beta_2 d}| = 0$$

$$\hookrightarrow \quad |1 - e^{-j2\beta_2 d}| = 0 \Rightarrow e^{-j2\beta_2 d} = 1$$

which is true when $2\beta_2 d = 0, 2\pi, 4\pi, \ldots$

We'll ignore the 0 sol'n since then $d = 0$

to get $\quad 2\beta_2 d = 2\pi n \qquad n = 1, 2, 3, \ldots$

$$\hookrightarrow \quad \beta_2 d = \frac{2\pi}{\lambda_2} d = \pi n$$

$$\hookrightarrow \quad d = \frac{n\pi}{\beta_2} = n\frac{\lambda_2}{2} \qquad n = 1, 2, 3, \ldots$$

$\Rightarrow$ The slab must be an integer multiple of a <u>half wavelength thick</u> (wavelength in material 2)!

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{335.3352} = 0.018737\,\text{m} = 1.8737\,\text{cm}$$

Then, $\underline{d = (0.93685\,\text{cm})\,n} \qquad n = 1, 2, \ldots$

What happens to Γ_{in} @ $f = 8$ & 12 GHz?

$$\beta_{2,8} = 2\pi(8\times10^9)\sqrt{\mu_0 2.56\epsilon_0} = 268.26816 \text{ rad/m}$$

$$\beta_{2,12} = 2\pi(12\times10^9)\sqrt{\mu_0 2.56\epsilon_0} = 402.40224 \text{ rad/m}$$

5.5.2 cont.

ex. cont.

$$\Gamma_{in,8GHz} = \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_{2,8} d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_{2,8} d}}$$

$$= \frac{-0.23077 + 0.23077 e^{-j2(268.268)(0.93685\times10^{-2})}}{1 + (-0.23077)(0.23077) e^{-j2(268.268)(0.93685\times10^{-2})}}$$

$$= -0.1732 + j0.2142 = \underline{0.27546\angle 128.95^\circ}$$

$$\Gamma_{in,12GHz} = \frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_{2,12} d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_{2,12} d}}$$

$$= \frac{-0.23077 + 0.23077 e^{-j2(402.4)(0.93685\times10^{-2})}}{1 + (-0.23077)(0.23077) e^{-j2(402.4)(0.93685\times10^{-2})}}$$

$$= -0.1732 - j0.2142 = \underline{0.27546\angle -128.95^\circ}$$

In terms of SWR, we get

$$SWR_{8GHz} = \frac{1 + |\Gamma_{in,8GHz}|}{1 - |\Gamma_{in,8GHz}|} = \frac{1 + 0.2755}{1 - 0.2755} = \underline{1.76}$$

$$SWR_{12GHz} = \frac{1 + |\Gamma_{in,12GHz}|}{1 - |\Gamma_{in,12GHz}|} = \frac{1 + 0.2755}{1 - 0.2755} = \underline{1.76}$$

5.5.2 cont.

$\lambda/4$ wave transformer

What intrinsic impedance η_2 is required for a $\frac{\lambda_2}{4}$ thick slab between two semi-infinite materials 1 + 3 for $\Gamma_{in} = 0$?

$$\Gamma_{in}(z=-d) = \left.\frac{\Gamma_{12} + \Gamma_{23} e^{-j2\beta_2 d}}{1 + \Gamma_{12}\Gamma_{23} e^{-j2\beta_2 d}}\right|_{d=\lambda_2/4} = 0$$

$$\hookrightarrow \left|\Gamma_{12} + \Gamma_{23} e^{-j2\left(\frac{2\pi}{\lambda_2}\right)\left(\frac{\lambda_2}{4}\right)}\right| = 0$$

$$\left|\Gamma_{12} + \Gamma_{23} \underbrace{e^{-j\pi}}_{-1}\right| = 0$$

$$|\Gamma_{12} - \Gamma_{23}| = 0$$

$$\left|\frac{\eta_2-\eta_1}{\eta_2+\eta_1} - \frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right| = \left|\frac{(\eta_2-\eta_1)(\eta_3+\eta_2)-(\eta_3-\eta_2)(\eta_2+\eta_1)}{(\eta_3+\eta_1)(\eta_3+\eta_2)}\right| = 0$$

$$\hookrightarrow \left|(\cancel{\eta_2\eta_3} + \eta_2^2 - \eta_1\eta_3 - \cancel{\eta_1\eta_2}) - (\cancel{\eta_3\eta_2} + \eta_1\eta_3 - \eta_2^2 - \cancel{\eta_1\eta_2})\right| = 0$$

$$|2\eta_2^2 - 2\eta_1\eta_3| = 2|\eta_2^2 - \eta_1\eta_3| = 0$$

$$\Rightarrow \underline{\underline{\eta_2 = \sqrt{\eta_1\eta_3}}} \quad !!$$