Chapter 4 Wave Propagation + Polarization

4.1 Introduction

→ time-harmonic waves in infinite media

→ transverse electromagnetic (TEM) waves

4.2 TEM Modes

→ Modes are particular field configurations that satisfy the wave equations and boundary conditions

→ TEM modes have both $\bar{E}$ and $\bar{H}$ @ every point in space contained in a local <u>equiphase</u> plane that's independent of time

→ Equiphase planes for a TEM wave are not necessarily parallel @ differing points in space. However, if they are, the $\bar{E}$ + $\bar{H}$ fields form a TEM <u>plane wave</u>

→ If a TEM plane wave have constant amplitudes of $\bar{E}$ + $\bar{H}$ over each planar surface, it is a <u>uniform plane wave</u> → $\bar{E}$ + $\bar{H}$ are not functions of the spatial coordinates w/in the plane

4.2.1 Uniform Plane Waves (UPW) in Unbounded Lossless Medium – Principal Axis

→ assume medium described by magnetic permeability μ and electric permittivity ϵ

→ assume time-harmonic UPW propagates in the $\pm \hat{a}_z$ directions and that $\bar{E} = \hat{a}_x E_x$
↳ reasonable as we are selecting the coordinate system

Going to the Cartesian coordinate wave eqns of Chapter 3 (eqn 3-20 a,b,c), we chose

$$\nabla^2 E_x(x,y,z) + \beta^2 E_x(x,y,z) = 0 \qquad (3\text{-}20a)$$

→ A UPW propagating in the $\pm \hat{a}_z$ directions implies that $\bar{E}$ + $\bar{H}$ are in x-y planes and therefore can NOT be functions of x or y

$$\nabla^2 E_x(z) + \beta^2 E_x(z) = 0$$

Following the separation of variables solution in section 3.4.1, $E_x(z) = h(z)$

where $h(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}$ (3-30a)

or $\qquad = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)$ (3-30b)

4.2.1 cont.

So

$$E_x(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$$

Now, let $A_3 = E_0^+$ (mag. of fwd or $+\hat{a}_z$ traveling wave)

$B_3 = E_0^-$ (mag. of bwd or $-\hat{a}_z$ wave)

& Note $\beta = \beta_z$ in this case

$$(4\text{-}2) \quad \underline{E_x(z) = E_x^+(z) + E_x^-(z) = E_0^+ e^{-j\beta z} + E_0^- e^{j\beta z}}$$

What about the corresponding $\bar{H}$?

→ Could use another set of wave eqns, but we know that we must satisfy Maxwell's Eqns

Using Ampere's Law, $\bar{\nabla} \times \bar{E} = -j\omega\mu \bar{H}$

$$\bar{H} = \frac{1}{-j\omega\mu}\left(\bar{\nabla} \times \hat{a}_x E_x(z)\right) = -\hat{a}_y \frac{1}{j\omega\mu} \frac{\partial E_x(z)}{\partial z}$$

$$= \hat{a}_y \frac{\beta}{\omega\mu}\left(E_0^+ e^{-j\beta z} - E_0^- e^{j\beta z}\right)$$

Now $\beta = \omega\sqrt{\mu\epsilon}$, so we get

$$\underline{\bar{H} = \hat{a}_y \frac{1}{\sqrt{\mu/\epsilon}}\left\{E_0^+ e^{-j\beta z} - E_0^- e^{j\beta z}\right\} = \hat{a}_y\left\{H_y^+(z) + H_y^-(z)\right\}}$$

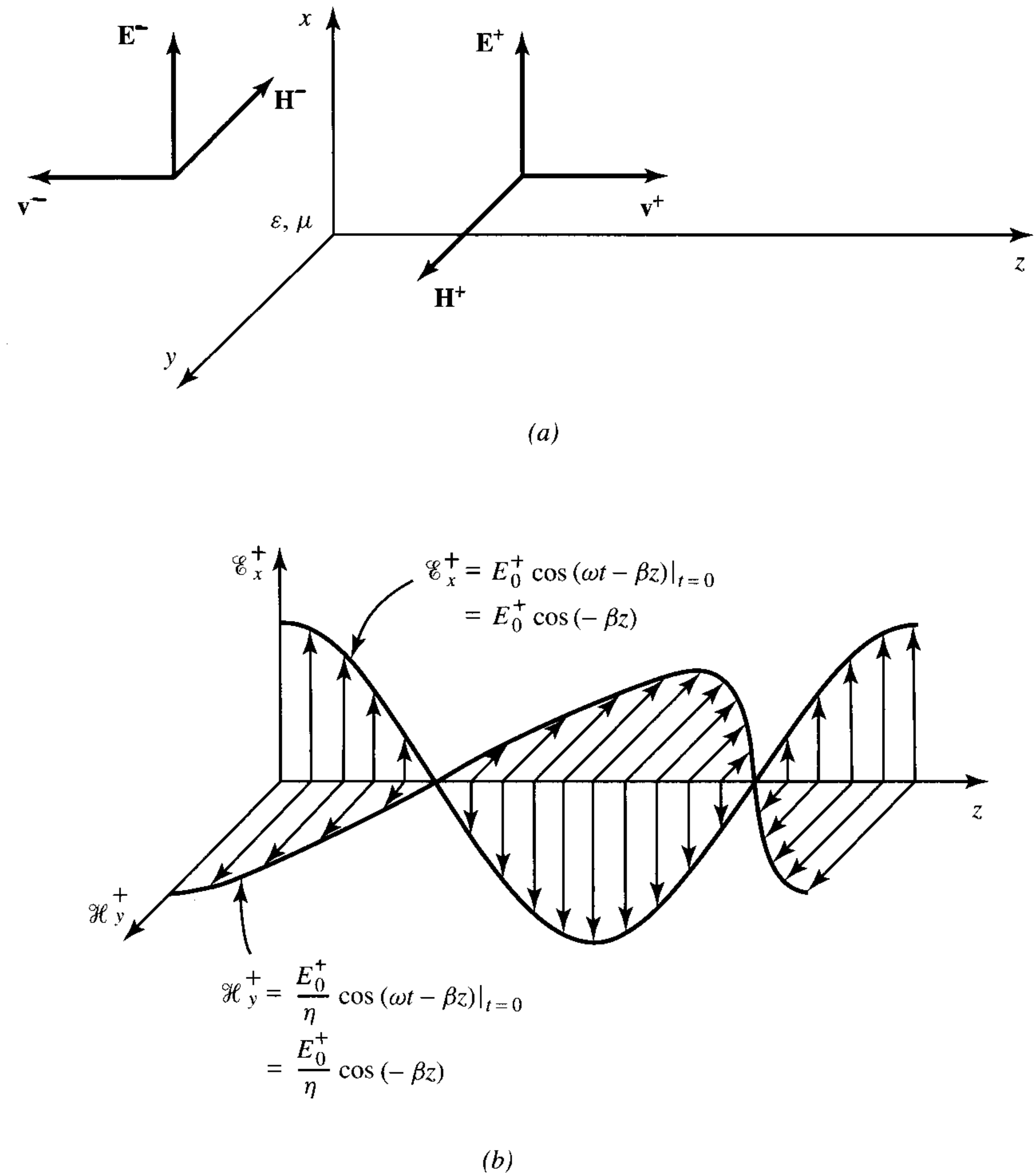


Figure 4-2 Uniform plane wave fields. (*a*) Complex. (*b*) Instantaneous.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

4.2.1 cont.

→ The factor $\sqrt{\mu/\varepsilon}$ must have units of V/A or Ω in order for $\bar{H}$ to have units of A/m

→ Define wave impedance as

$$\boxed{Z_w = \frac{E_x^+(z)}{H_y^+(z)} = -\frac{E_x^-(z)}{H_y^-(z)} = \sqrt{\mu/\varepsilon}}$$

Note: $Z_w = \sqrt{\mu/\varepsilon} = \eta$ ← intrinsic impedance

↳ True for UPW, plane waves, + TEM waves, but not true for TE + TM modes

→ To determine the magnetic field components related to a given electric field components, use the RHR and Z_w

1) RH fingers in direction of $\bar{E}$ component
2) RH thumb in direction of wave prop.
3) curl fingers of RH 90° to point in direction of $\bar{H}$ component
4) obtain magnitude of $\bar{H}$ component by dividing magnitude of $\bar{E}$ component by Z_w

4.2.1 cont.

Alternate approach (not in this text section)

$$\boxed{\bar{H} = \frac{\hat{a}_n \times \bar{E}}{Z_w}} \qquad \boxed{\bar{E} = Z_w \left(\bar{H} \times \hat{a}_n \right)}$$

where $\hat{a}_n$ is unit vector in direction of wave propagation

ex. A UPW is propagating through teflon ($\epsilon = 2.1\epsilon_o$, $\mu = \mu_o$) and has an electric field $\bar{E} = \hat{a}_z \left(10 e^{-j5x} + 6 e^{+j5x} \right)$ V/m.

Find the corresponding $\bar{H}$.

→ wave is propagating in $\pm\hat{a}_x$ directions (from exponential terms)

$$\rightarrow \quad Z_w = \sqrt{\mu/\epsilon} = \sqrt{\frac{4\pi \times 10^{-7}}{2.1(8.854\times10^{-12})}} = \underline{260\,\Omega}$$

$$+\hat{a}_x \qquad \bar{H}^+ = \frac{\hat{a}_x \times \hat{a}_z\, 10 e^{-j5x}}{260} = -\hat{a}_y \frac{1}{26} e^{-j5x} \quad (\text{A/m})$$

$$-\hat{a}_x \qquad \bar{H}^- = \frac{-\hat{a}_x \times \hat{a}_z\, 6 e^{-j5x}}{260} = +\hat{a}_y \frac{3}{130} e^{+j5x} \quad (\text{A/m})$$

$$\bar{H} = \bar{H}^+ + \bar{H}^- = \underline{\underline{\hat{a}_y \left[-38.46 e^{-j5x} + 23.08 e^{+j5x} \right] \text{mA/m}}}$$

4.2.1 cont.

To get time expressions for the electric and magnetic fields, follow the usual phasor → time-domain procedure, i.e.

$$\bar{\mathcal{E}} = \mathrm{Re}\{\bar{E}e^{j\omega t}\} \quad \text{and} \quad \bar{\mathcal{H}} = \mathrm{Re}\{\bar{H}e^{j\omega t}\}$$

ex. $\bar{E} = \hat{a}_x\{E_o^+ e^{-j\beta z} + E_o^- e^{j\beta z}\}$

$$\bar{\mathcal{E}} = \mathrm{Re}\{E_o^+ e^{-j\beta z}e^{j\omega t} + E_o^- e^{j\beta z}e^{j\omega t}\}$$

$$\boxed{\bar{\mathcal{E}} = \hat{a}_x\left[|E_o^+|\cos(\omega t - \beta z + \theta^+) + |E_o^-|\cos(\omega t + \beta z + \theta^-)\right]}$$

→ assumes $E_o^+ = |E_o^+|\angle\theta^+$ and $E_o^- = |E_o^-|\angle\theta^-$, omit θ^+ and θ^- if $E_o^+ + E_o^-$ are real #s

Similarly

$$\boxed{\bar{\mathcal{H}} = \hat{a}_y\frac{\left[|E_o^+|\cos(\omega t - \beta z + \theta^+) - |E_o^-|\cos(\omega t + \beta z + \theta^-)\right]}{\sqrt{\mu/\epsilon}}}$$

from (3-35) the phase velocity v_p is found when the arguments of the fwd or bwd propagating waves are constant

fwd $\boxed{v_p^+ = \frac{dz}{dt} = \omega/\beta = \frac{1}{\sqrt{\mu\epsilon}}}$ bwd $\boxed{v_p^- = -\frac{1}{\sqrt{\mu\epsilon}}}$

4.2.1 cont.

$$\bar{v}_p^{\,+} = \hat{a}_z \frac{1}{\sqrt{\mu\epsilon}} \quad \text{and} \quad \bar{v}_p^{\,-} = -\hat{a}_z \frac{1}{\sqrt{\mu\epsilon}}$$

in vector form

phase velocities can be > c for UPW going at oblique angles

The electric + magnetic energy densities are found using

(1-58e) $w_m = \frac{1}{2} \mu |\bar{\mathcal{H}}|^2 \quad (J/m^3)$

(1-58f) $w_e = \frac{1}{2} \epsilon |\bar{\mathcal{E}}|^2 \quad (J/m^3)$

and the power density (AKA instantaneous Poynting vector) is found using

(1-56) $\bar{\mathcal{S}} = \bar{\mathcal{E}} \times \bar{\mathcal{H}} \quad (W/m^2)$

ex. for the fwd ($+\hat{a}_z$) wave

$$w_m^+ = \frac{1}{2} \mu \left[\frac{\hat{a}_y |E_o^+| \cos(\omega t - \beta z + \theta^+)}{\sqrt{\mu/\epsilon}} \cdot \frac{\hat{a}_y |E_o^+| \cos(\omega t - \beta z + \theta^+)}{\sqrt{\mu/\epsilon}} \right]$$

$$w_m^+ = \frac{1}{2} \epsilon |E_o^+|^2 \cos^2(\omega t - \beta z + \theta^+)$$

Same! $w_e^+ = \frac{1}{2} \epsilon \left[\hat{a}_x |E_o^+| \cos(\omega t - \beta z + \theta^+) \cdot \hat{a}_x |E_o^+| \cos(\omega t - \beta z + \theta^+) \right]$

$$w_e^+ = \frac{1}{2} \epsilon |E_o^+|^2 \cos^2(\omega t - \beta z + \theta^+)$$

4.2.1 cont.

ex. cont.

$$\bar{\mathcal{S}}^+ = \hat{a}_x |E_o^+| \cos(\omega t - \beta z + \theta^+) \times \hat{a}_y \frac{|E_o^+| \cos(\omega t - \beta z + \theta^+)}{\sqrt{\mu/\epsilon}}$$

$$\bar{\mathcal{S}}^+ = \hat{a}_z \frac{|E_o^+|^2}{\sqrt{\mu/\epsilon}} \cos^2(\omega t - \beta z + \theta^+)$$

The energy or group velocity V_e is defined as the ratio of $\bar{\mathcal{S}}$ to $W = W_e + W_m$ (total energy density)

ex. $$\bar{V}_e^+ = \frac{\bar{\mathcal{S}}^+}{\omega_e^+ + \omega_m^+} = \frac{\hat{a}_z \frac{|E_o^+|^2}{\sqrt{\mu/\epsilon}} \cos^2(\omega t - \beta z + \theta^+)}{2\left(\frac{1}{2}\right)\epsilon |E_o^+|^2 \cos^2(\omega t - \beta z + \theta^+)}$$

$$\bar{V}_e^+ = \hat{a}_z \frac{1}{\sqrt{\mu\epsilon}}$$ ← same as $\bar{V}_p^+$ for UPW

→ unlike V_p, V_e is always less than or equal to the speed of light.

The product of the magnitude of the phase velocity and group velocity is always the square of the speed of light in the medium

ex. $$V_e^+ V_p^+ = V^{+2} = \frac{1}{\mu\epsilon}$$

4.2.1 cont.

The time-average power density (AKA Poynting vector) is found using

(1-70) $\bar{\mathcal{S}}_{ave} = \bar{S} = \frac{1}{2}\,\mathrm{Re}\{\bar{E} \times \bar{H}^*\}$

conjugate

ex. $\bar{\mathcal{S}}^+_{ave} = \bar{S}^+ = \frac{1}{2}\,\mathrm{Re}\left\{\hat{a}_x E_0^+ e^{-j\beta z} \times \hat{a}_y \frac{E_0^{+*}}{\sqrt{\mu/\varepsilon}} e^{+j\beta z}\right\}$

$$= \frac{1}{2}\,\hat{a}_z \frac{|E_0^+|^2}{\sqrt{\mu/\varepsilon}}$$

$$\underline{\underline{\bar{S}^+ = \hat{a}_z \frac{|E_0^+|^2}{2\sqrt{\mu/\varepsilon}}}}$$

Since we have both forward and backward propagating UPWs, we can expect <u>standing waves</u> to be formed as they constructively / destructively interfere. To illustrate

$$E_x(z) = E_0^+ e^{-j\beta z} + E_0^- e^{j\beta z}$$

$\downarrow$ Apply Euler's ID + complex arith.
(E_0^+ + E_0^- are <u>assumed real</u>)

$$\boxed{E_x(z) = \sqrt{(E_0^+)^2 + (E_0^-)^2 + 2E_0^+E_0^-\cos(2\beta z)} \times e^{-j\tan^{-1}\left[\frac{E_0^+ - E_0^-}{E_0^+ + E_0^-}\tan(\beta z)\right]}}$$

4.2.1 cont.

So $|E_x(z)| = \sqrt{(E_0^+)^2 + (E_0^-)^2 + 2E_0^+E_0^- \cos(2\beta z)}$

Find $|E_x(z)|_{max}$ and $|E_x(z)|_{min}$

$$|E_x(z)|_{max} = \sqrt{(E_0^+)^2 + (E_0^-)^2 + 2E_0^+E_0^-} = \underline{|E_0^+| + |E_0^-|}$$

occurs when $\cos(2\beta z) = 1$

$$\Rightarrow \underline{\beta z = m\pi \quad m = 0, 1, 2, \ldots}$$

$$\hookrightarrow z = m\pi(\lambda/2\pi) = m\lambda/2$$

Assume $|E_0^+| > |E_0^-|$

$$|E_x(z)|_{min} = \sqrt{(E_0^+)^2 + (E_0^-)^2 - 2E_0^+E_0^-}$$

$$= \underline{|E_0^+| - |E_0^-|}$$

occurs when $\cos(2\beta z) = -1 \Rightarrow \underline{\beta z = \frac{(2m+1)\pi}{2}}$

$$\underline{m = 0, 1, 2, \ldots}$$

$\Rightarrow$ Like voltage + current standing waves on a lossless transmission line

$\Rightarrow$ maxima + minima are separated by $\lambda/2$ intervals, respectively.

$\Rightarrow$ adj. maxima + minima are $\lambda/4$ apart

4.2.1 cont.

We define a standing wave ratio (SWR)

as

$$SWR = \frac{|E_x(z)|_{max}}{|E_x(z)|_{min}} = \frac{|E_0^+| + |E_0^-|}{|E_0^+| - |E_0^-|}$$

If we define a reflection coefficient (Γ)

as $\boxed{\Gamma = \frac{E_0^-}{E_0^+}}$ $\quad 0 \le |\Gamma| \le 1$

Then

$$\boxed{SWR = \frac{1 + \frac{|E_0^-|}{|E_0^+|}}{1 - \frac{|E_0^-|}{|E_0^+|}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}} \quad 1 \le SWR < \infty$$

when $|\Gamma| = \left|\frac{E_0^-}{E_0^+}\right| = 1$, we can get complete destructive interference (see Fig 4-3), and

$$\underbrace{|E_x(z)| = 2E_0^+ |\cos(\beta z)| = 2E_0^- |\cos(\beta z)|}_{\text{full-wave rectified cosine function}}$$

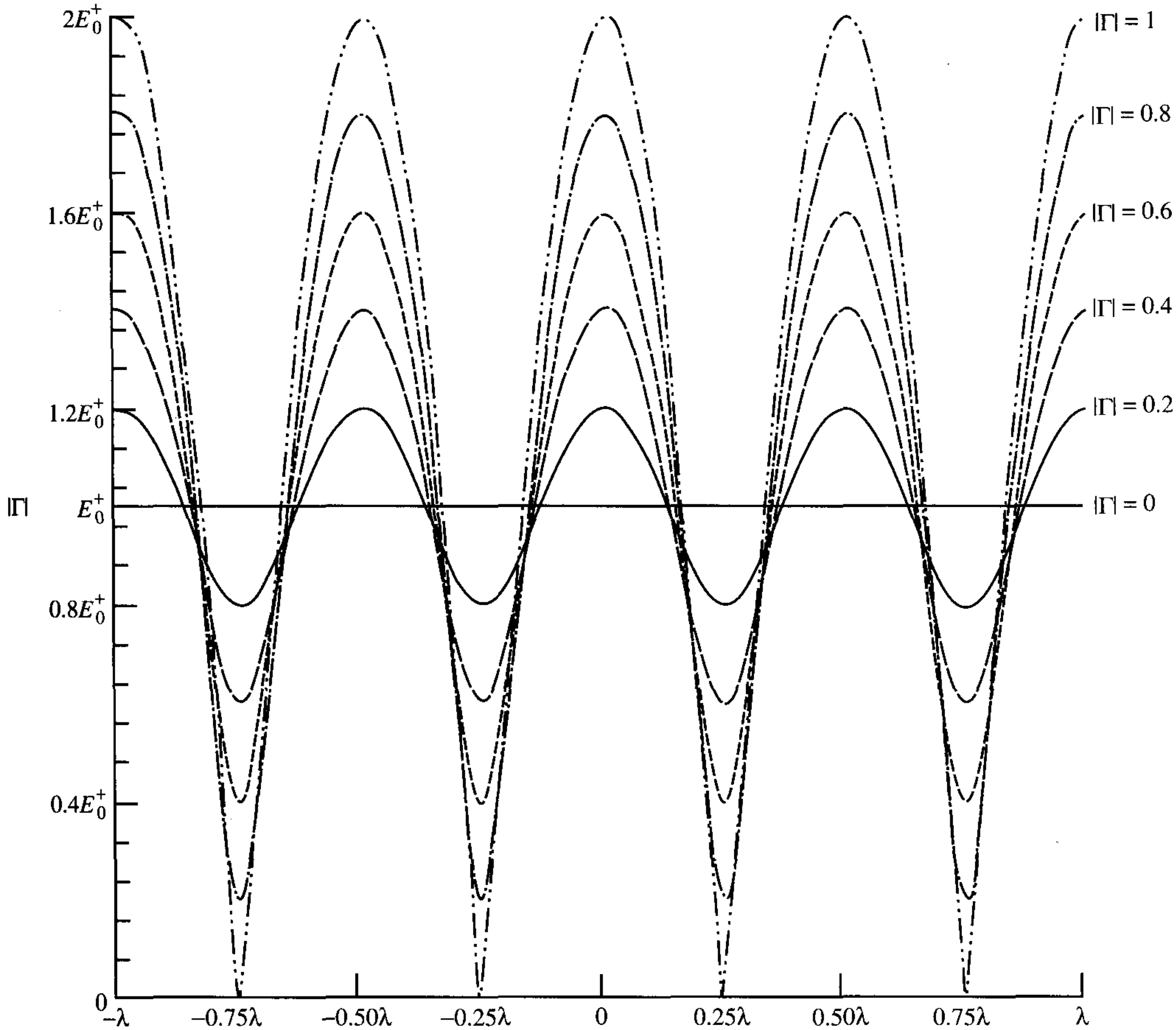


Figure 4-3 Standing wave pattern as a function of distance for a uniform plane wave with different reflection coefficients.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

4.2.2 UPW in an Unbounded Lossless Medium- Oblique Angle

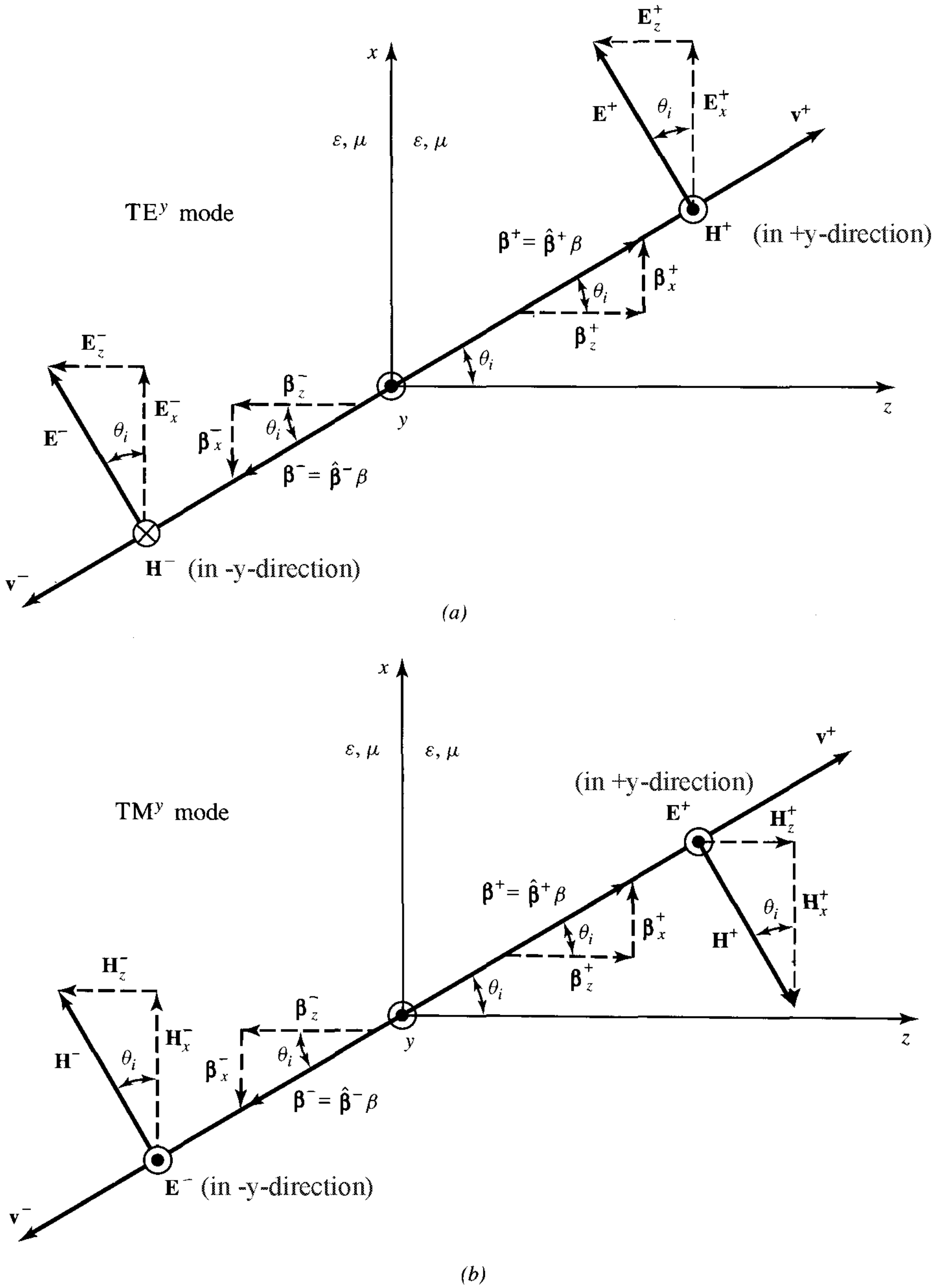


Figure 4-4 Transverse electric and magnetic uniform plane waves in an unbounded medium at an oblique angle. (*a*) TE^y mode. (*b*) TM^y mode.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

4.2.2 cont.

TE^y mode → see Fig 4-4a

→ Note that both E^- and E^+ electric fields are confined to the x-z plane(s) and have NO y-component → transverse electric to y (TE^y)

→ Note that H^- & H^+ oriented in $\pm\hat{a}_y$ directions

→ Still a UPW to direction of propagation

TM^y mode → see Fig 4-4b

→ Note that both H^- & H^+ magnetic fields are confined to the x-z plane and have NO y-component → transverse magnetic to y (TM^y)

→ Note that E^- & E^+ oriented in $\pm\hat{a}_y$ directions

→ still a UPW to direction of propagation

4.2.2 cont.

Notation - for mathematical convenience, the phase constants will now be expressed as

$$\bar{\beta}^{+} = \hat{\beta}^{+}\beta = \hat{a}_x \beta_x^{+} + \hat{a}_y \beta_y^{+} + \hat{a}_z \beta_z^{+} \qquad (4\text{-}5a)$$

$$\bar{\beta}^{-} = \hat{\beta}^{-}\beta = \hat{a}_x \beta_x^{-} + \hat{a}_y \beta_y^{-} + \hat{a}_z \beta_z^{-} \qquad (4\text{-}5b)$$

and position will be indicated with a position vector

$$\bar{r} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z \qquad (4\text{-}5c)$$

→ allows exponential terms of sol'n to wave eq'n to be written in form $e^{-j\bar{\beta}^{\pm}\cdot\bar{r}}$

→ $\hat{\beta}^{\pm}$ are unit vectors in the ± directions of propagation

→ Again, the magnitudes of the forward and backward propagating electric fields are E_o^{+} and E_o^{-}, respectively

→ $\beta^2 = \omega^2 \mu \epsilon$ → $\beta = \omega\sqrt{\mu\epsilon} = \omega/u = 2\pi/\lambda$

4.2.2 cont.

Apply this notation to the TE^y mode (Fig 4-4a) where we define the angle θ_i wrt the x and z axes and decompose E^- + E^+ into $\hat{a}_x$ + $\hat{a}_z$ components yielding

(4-17) $$\bar{E} = \bar{E}^+ + \bar{E}^- = E_0^+(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)e^{-j\bar{\beta}^+\cdot\bar{r}} + E_0^-(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)e^{-j\bar{\beta}^-\cdot\bar{r}}$$

Note that the phase constants $\bar{\beta}^-$ and $\bar{\beta}^+$ are also confined to the x-z plane. So, we can write

(4-17a) $$\bar{\beta}^+ = \hat{\beta}^+\beta = \hat{a}_x\beta_x^+ + \hat{a}_z\beta_z^+ = \beta(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i)$$

(4-17b) $$\bar{\beta}^- = \hat{\beta}^-\beta = \hat{a}_x\beta_x^- + \hat{a}_z\beta_z^- = -\beta(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i)$$

(point in opposite directions (as expected)

Before we re-write $\bar{E}$ for the TE^y mode, look at Figure 4-5 which shows how the position vector $\bar{r}$ points to a location in the constant phase plane of the UPW (either TE^y or TM^y mode)

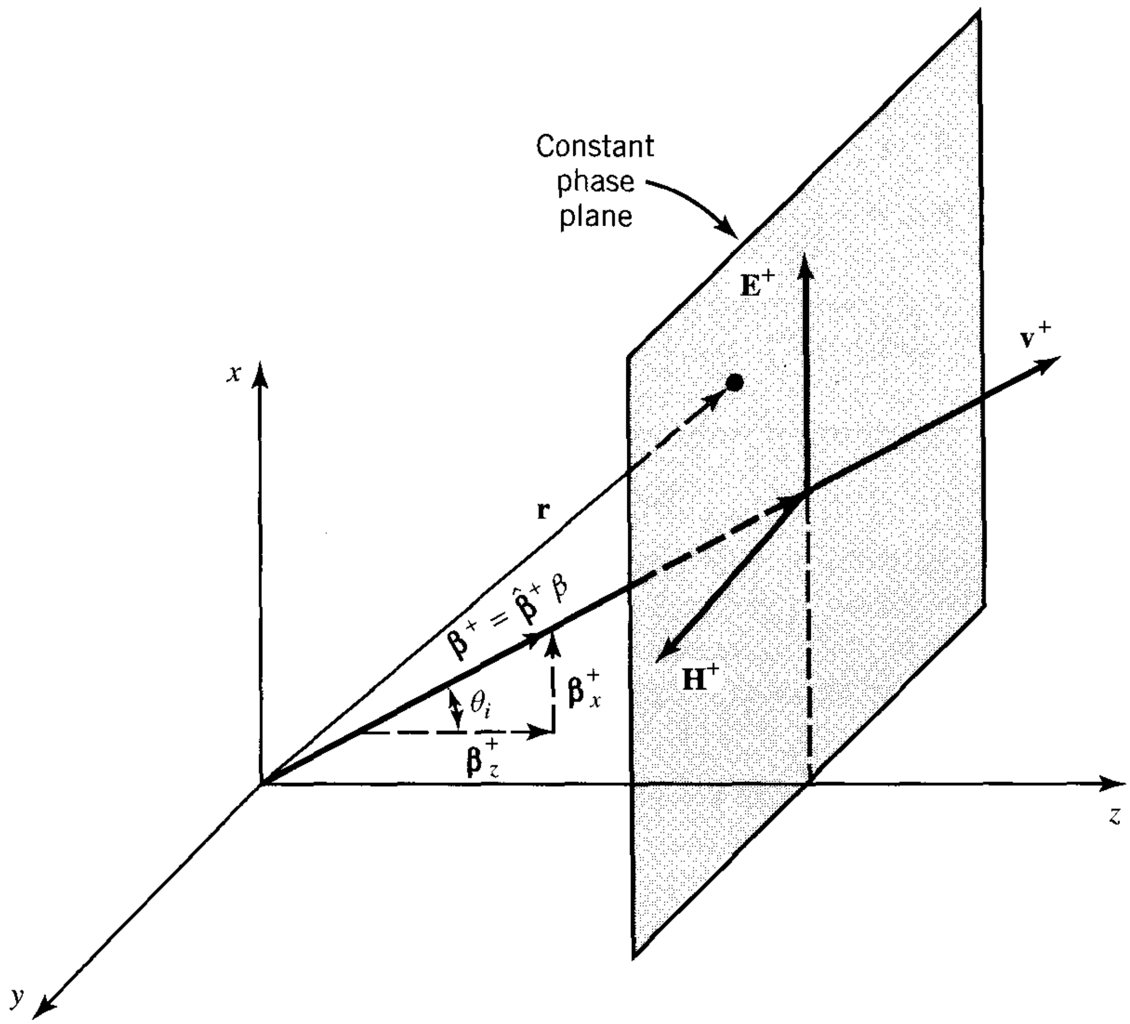


Figure 4-5 Phase front of a TEM wave traveling in a general direction.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

Now, $\bar{E}$ for the TE^y mode can be written

$$\text{(4-18a)} \quad \boxed{\begin{aligned}\bar{E} &= E_0^+(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)\, e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} \\ &\quad + E_0^-(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)\, e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\end{aligned}}$$

→ Since we have a UPW, we can again either use Maxwell's Eqns <u>or</u> the procedures using the wave impedance ($Z_w = \eta = \sqrt{\mu/\epsilon}$) from section 4.2.1 to find $\bar{H}$

4.2.2 cont.

$$\bar{H} = \bar{H}^+ + \bar{H}^-$$

$$\text{(4-18b)}\quad \boxed{\bar{H} = \hat{a}_y\left[\frac{E_o^+}{\eta} e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} - \frac{E_o^-}{\eta} e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right]}$$

$$\text{(4-18c)}\quad \bar{H} = \frac{1}{\eta}\left[\hat{\beta}^+ \times \bar{E}^+ + \hat{\beta}^- \times \bar{E}^-\right]$$

re-cast the $\frac{\hat{a}_n \times \bar{E}}{\eta}$ eqn of previous section

→ Note that these results converge to those of section 4.2.1 when $\theta_i \to 0$

What are the planes of constant phase for the TE^y mode? Set the arguments of the exponential terms equal to constants

$$\text{(4-19a)}\quad \overbrace{\bar{\beta}^+ \cdot \bar{r} = \beta_x^+ x + \beta_y^+ y + \beta_z^+ z}^{\text{general}} = \overbrace{\beta(x\sin\theta_i + z\cos\theta_i)}^{TE^y\ \text{mode}} = \underset{\text{constant}}{C^+}$$

$$\text{(4-19b)}\quad \bar{\beta}^- \cdot \bar{r} = \beta_x^- x + \beta_y^- y + \beta_z^- z = -\beta(x\sin\theta_i + z\cos\theta_i) = C^-$$

To find phase velocities, we need to introduce the $e^{j\omega t}$ term to the exponential terms

$$\text{(4-19c)}\quad \bar{\beta}^+ \cdot \bar{r} - \omega t = \beta(x\sin\theta_i + z\cos\theta_i) - \omega t = \underset{\text{constant}}{C_o^+}$$

$$\text{(4-19d)}\quad \underbrace{\bar{\beta}^- \cdot \bar{r} - \omega t}_{\text{General}} = -\beta(x\sin\theta_i + z\cos\theta_i) - \omega t = C_o^-$$

4.2.2 cont.

* The phase velocity is found by talking the derivative wrt time. Note, we can get phase velocities in different directions (won't need to be equal) such as the r, x, or z directions (save derivations for HW!)

Before proceeding, the TM^y mode has fields (see Fig 4-4b) described by

$$\boxed{\begin{aligned}\bar{E} &= \bar{E}^+ + \bar{E}^- \\ \bar{E} &= \hat{a}_y\left[E_o^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} + E_o^- e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right]\end{aligned}}$$

and

$$\boxed{\begin{aligned}\bar{H} &= \bar{H}^+ + \bar{H}^- \\ \bar{H} &= \frac{E_o^+}{\eta}\left(-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} \\ &\quad + \frac{E_o^-}{\eta}\left(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i\right)e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\end{aligned}}$$

→ Here $\bar{E}$ is oriented in the $+\hat{a}_y$ direction.

& $\bar{H}$ is in the x–z plane(s)

4.2.2 cont.

Wave Impedances

→ In the direction(s) of propagation, the wave is TEM and $Z_w = \eta = \sqrt{\mu/\epsilon}$

→ what about in other directions (e.g. x & z directions)?

Again, we will divide the electric field for the desired direction by the corresponding magnetic field component (direction is found by cross product of E component w/ H component, i.e., direction for Z_w in direction of power flow)

TE^y mode (see Fig 4-4a)

x-direction $\quad Z_x^+ = Z_x^- = -\dfrac{E_z^+}{H_y^+} = \dfrac{E_z^-}{H_y^-} = \eta \sin\theta_i \quad$ (4-20a)

z-direction $\quad Z_z^+ = Z_z^- = \dfrac{E_x^+}{H_y^+} = -\dfrac{E_x^-}{H_y^-} = \eta \cos\theta_i \quad$ (4-20b)

→ $0 \le Z_{x,z}^{\pm} \le \eta$!

TM^y mode (see Fig 4-4b)

x-direction $\quad Z_x^+ = Z_x^- = \dfrac{E_y^+}{H_z^+} = -\dfrac{E_y^-}{H_z^-} = \dfrac{\eta}{\sin\theta_i} \quad$ (4-21a)

z-direction $\quad Z_z^+ = Z_z^- = -\dfrac{E_y^+}{H_x^+} = \dfrac{E_y^-}{H_x^-} = \dfrac{\eta}{\cos\theta_i} \quad$ (4-21b)

→ $\eta \le Z_{x,z}^{\pm} \le \infty$!

4.2.2 cont.

Phase + Energy/Group Velocities

→ call the wave velocity in the direction(s) of $\bar{\beta}$ (wave propagation) v_r and it's equal to the speed of light in the medium $v = \frac{1}{\sqrt{\mu\epsilon}}$

→ remember that equiphase planes are orthogonal to $\bar{\beta}$ (see Fig 4-6)

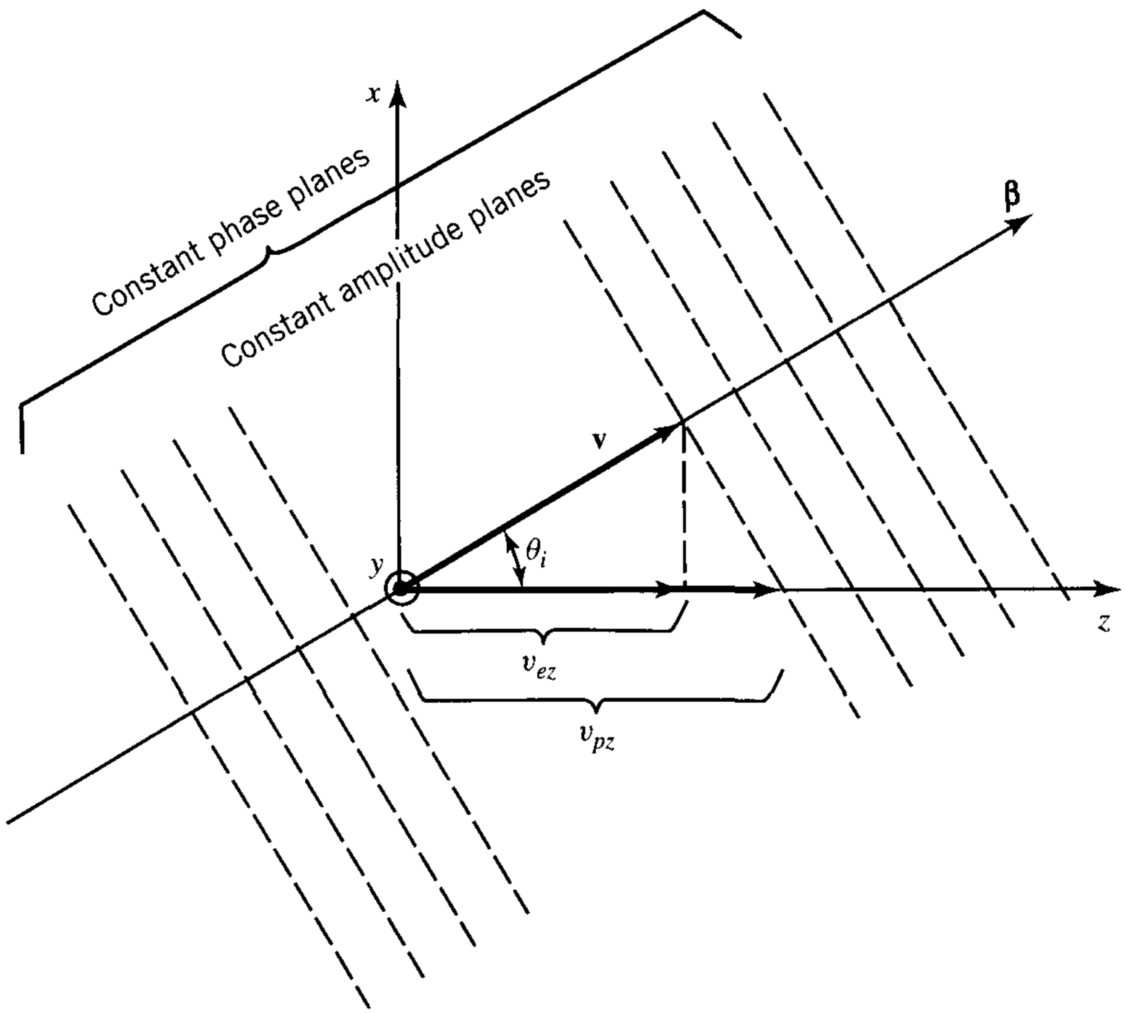


Figure 4-6 Phase and energy (group) velocities of a uniform plane wave.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

4.2.2 cont.

→ The phase v_{pr} + energy/group v_{er} velocities in the direction of propagation ($\bar{\beta}$) are also equal to the speed of light v in the material

$$v_r = v_{pr} = v_{er} = v = \frac{1}{\sqrt{\mu\epsilon}}$$

→ However, at an oblique angle (θ_i), the phase velocities in the x & z directions can NOT be equal to v if the oblique equiphase planes are to be maintained

e.g.

Looking @ Fig 4-6 + the velocity vectors/triangles

$$v_{pz} = \frac{v}{\cos\theta_i} \geq v \,! \qquad (4\text{-}23)$$

← Not real velocity of wave

Notice how $\bar{v}_{pz}$ goes from origin to equiphase plane (Simple trigonometry/geometry)

4.2.2 cont.

→ How about the energy/group velocities?
Again, look at Fig 4-6, and note

$$v_{ez} = v\cos\theta_i \le v \qquad (4\text{-}24)$$

→ Also, for any wave, the product of phase and group velocities in a particular direction is equal to v^2

$$v_{pr} v_{er} = v_{pz} v_{ez} = v^2 = \frac{1}{\mu\epsilon} \qquad (4\text{-}25)$$

↳ way to save time (only need to find the phase or group/energy velocity)

Power Density

For the $\bar{\beta}^+$ direction the power density is

$$(\bar{S}^+_{ave})_r = \tfrac{1}{2}\,\mathrm{Re}\left[\bar{E}^+ \times \bar{H}^{+*}\right]$$

For the TE^y mode, $\bar{E}^+$ (1st part of 4-18a) & $\bar{H}^+$ (1st part of 4-18b), yield:

$$(\bar{S}^+_{ave})_r = \tfrac{1}{2}\,\mathrm{Re}\Big[E_0^+(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} \times \hat{a}_y\frac{E_0^{+*}}{\eta}e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\Big\}$$

4.2.2 cont.

yielding power density in direction of wave prop.

$$\left(\bar{S}_{ave}^{+}\right)_r = \hat{a}_x \frac{|E_0^+|^2}{2\eta} \sin\theta_i + \hat{a}_z \frac{|E_0^+|^2}{2\eta} \cos\theta_i$$

$$= \hat{a}_r \frac{|E_0^+|^2}{2\eta} = \hat{a}_x \left(S_{ave}^{+}\right)_x + \hat{a}_z \left(S_{ave}^{+}\right)_z \quad (4\text{-}26)$$

Notice

$$\left(S_{ave}^{+}\right)_x = \sin\theta_i \left(S_{ave}^{+}\right)_r \quad (4\text{-}26a)$$

$$\left(S_{ave}^{+}\right)_z = \cos\theta_i \left(S_{ave}^{+}\right)_r \quad (4\text{-}26b)$$

directional power densities

4.3 Transverse Electromagnetic Modes in Lossy Media

4.3.1 UPW in an Unbounded Lossy Medium – Principal Axis

→ lossy media ($\epsilon, \mu, + \sigma$)

→ assume electric field oriented in $\hat{a}_x$ direction $\bar{E} = \hat{a}_x E_x$ and that wave propagates in $\pm \hat{a}_z$ directions

→ from the wave eq'n for lossy media (see section 3.4.1B)

$$\boxed{\begin{aligned}\bar{E}(z) &= \hat{a}_x E_x(z) = \hat{a}_x\left(E_0^+ e^{-\gamma z} + E_0^- e^{+\gamma z}\right) \\ &= \hat{a}_x\left(E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{\alpha z} e^{j\beta z}\right)\end{aligned} \quad (4\text{-}27)}$$

where γ the propagation constant is

$$\boxed{\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \sqrt{-\omega^2\mu\epsilon + j\omega\mu\sigma} \quad (4\text{-}28)}$$

and $\gamma_x = \gamma_y = 0$ and let $\gamma_z = \gamma$

attenuation constant $\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\sigma/\omega\epsilon\right)^2} - 1\right]\right\}^{1/2}$ Np/m (4-28c)

phase constant $\beta = \omega\sqrt{\mu\epsilon}\left\{1/2\left[\sqrt{1 + \left(\sigma/\omega\epsilon\right)^2} + 1\right]\right\}^{1/2}$ rad/m (4-28d)

→ phase constant β and wave number k are same thing

→ conversion factor for α Np/m to α dB/m is: 1 Np/m $= 20\log_{10} e = 8.686$ dB/m

4.3.1 cont.

Using Faraday's Law, $\bar{\nabla} \times \bar{E} = -j\omega\mu \bar{H}$, we can find

$$\boxed{\begin{aligned}\bar{H} &= \hat{a}_y \frac{\gamma}{j\omega\mu}\left(E_o^+ e^{-\gamma z} - E_o^- e^{\gamma z}\right) \\ &= \hat{a}_y \sqrt{\frac{\sigma + j\omega\epsilon}{j\omega\mu}}\left(E_o^+ e^{-\gamma z} - E_o^- e^{\gamma z}\right) \\ &= \hat{a}_y \frac{1}{Z_w}\left(E_o^+ e^{-\gamma z} - E_o^- e^{\gamma z}\right) \qquad (4\text{-}29a)\end{aligned}}$$

where $\boxed{Z_w \equiv \text{wave impedance} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \; (\Omega) \quad (4\text{-}30)}$

Note: $Z_w = \eta_c \leftarrow$ intrinsic impedance of media for TEM waves for both lossless + lossy media

Time-ave power density $\bar{S} = \frac{1}{2}\mathrm{Re}\{\bar{E} \times \bar{H}^*\}$ W/m²

So for the fwd traveling wave

$$\bar{S}^+ = \frac{1}{2}\mathrm{Re}\{\bar{E}^+ \times \bar{H}^{+*}\} = \frac{1}{2}\mathrm{Re}\left\{\hat{a}_x E_o^+ e^{-\alpha z} e^{-j\beta z} \times \hat{a}_y \frac{E_o^{+*}}{\eta_c^*} e^{-\alpha z} e^{+j\beta z}\right\}$$

$$\boxed{\bar{S}^+ = \hat{a}_z \frac{|E_o^+|^2}{2} e^{-2\alpha z}\, \mathrm{Re}\left\{\frac{1}{\eta_c^*}\right\} \quad (4\text{-}31)}$$

(Note how power density falls exponentially as wave propagates in the $+\hat{a}_z$ direction

→ Since we again have both fwd + bwd propagating components to $\bar{E}$ and $\bar{H}$, standing waves are possible. However, they are not as dramatic (in general) due to the $e^{\pm\alpha z}$ terms (See Fig 4.7)

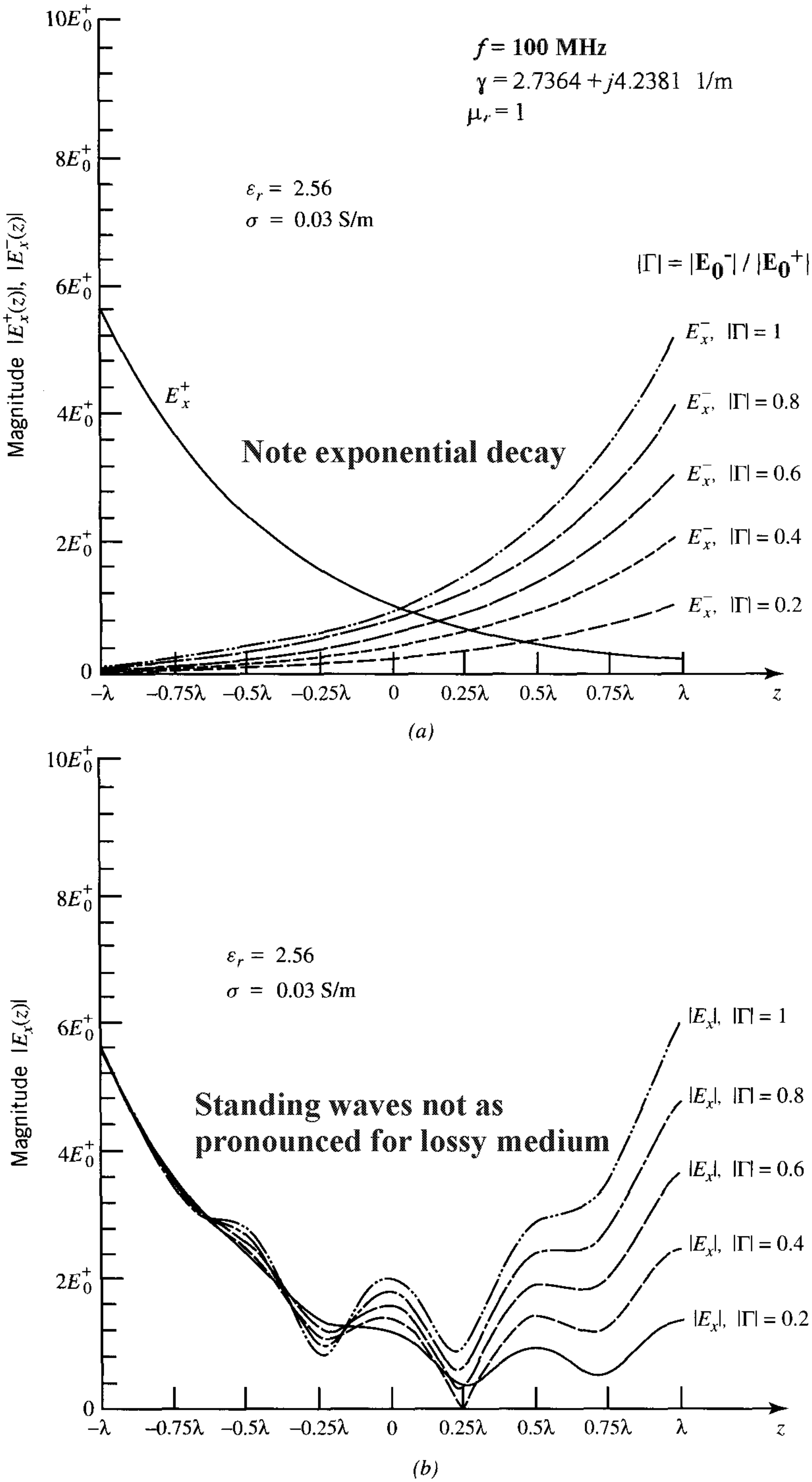


Figure 4-7 Wave patterns of uniform plane waves in a lossy medium. (*a*) Traveling. (*b*) Standing.

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

4.3.1 cont.

Define skin depth δ as the distance the wave must propagate thru lossy media for its magnitude to be reduced to $e^{-1} = 0.36788$ or 36.788% of its starting value. For an e^{-1} term $z = \delta = 1/\alpha$ (e.g. $e^{-\alpha z}\big|_{z=\delta} = e^{-\alpha\delta} = e^{-\alpha(1/\alpha)} = e^{-1}$), so the skin depth is:

$$\boxed{(4\text{-}34) \quad \delta = 1/\alpha = \frac{1}{Re\{\gamma\}} = \frac{1}{\omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]\right\}^{1/2}}}$$

units of meters

Some approximations can be made when we have good dielectrics $\left(\frac{\sigma}{\omega\epsilon}\right)^2 \ll 1$ and good conductors $\left(\frac{\sigma}{\omega\epsilon}\right)^2 \gg 1$.

Good Dielectrics $\left(\frac{\sigma}{\omega\epsilon}\right)^2 \ll 1$

→ implies that displacement current $\bar{J}_d = j\omega\epsilon\bar{E}$ is much larger than conduction current $\bar{J}_c = \sigma\bar{E}$

e.g., $\dfrac{|\bar{J}_c|}{|\bar{J}_d|} = \dfrac{\sigma}{\omega\epsilon} \ll 1$

→ will use binomial expansion on $\sqrt{1+(\sigma/\omega\epsilon)^2}$ terms & then only keep 1st $\sigma/\omega\epsilon$ term

Binomial Expansion $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2(4)}x^2 + \frac{1(3)}{2(4)(6)}x^3 - \ldots$

4.3.1 cont.

For example

$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1+(\sigma/\omega\epsilon)^2} - 1\right]\right\}^{1/2}$$

$$= \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\left(1+\frac{1}{2}(\sigma/\omega\epsilon)^2 - \cancel{\frac{1}{8}(\sigma/\omega\epsilon)^4} \ldots\right) - 1\right]\right\}^{1/2}$$

$$= \omega\sqrt{\mu\epsilon}\left\{\frac{1}{4}(\sigma/\omega\epsilon)^2\right\}^{1/2}$$

$$\boxed{\alpha \cong \frac{\sigma}{2}\sqrt{\mu/\epsilon} \quad (4\text{-}36a)}$$

Similarly $\boxed{\beta \cong \omega\sqrt{\mu\epsilon} \quad (4\text{-}37)}$,

$$\boxed{Z_w \cong \sqrt{\mu/\epsilon} \quad (4\text{-}38)},$$

Good dielectrics $\left(\frac{\sigma}{\omega\epsilon}\right)^2 << 1$

and

$$\boxed{\delta = \frac{1}{\alpha} \cong \frac{2}{\sigma}\sqrt{\epsilon/\mu} \quad (4\text{-}39)}$$

Good Conductors $(\sigma/\omega\epsilon)^2 >> 1$

$|\bar{J}_c| >> |\bar{J}_d| \rightarrow$ Again use binomial expansion

For example

$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\sqrt{1+(\sigma/\omega\epsilon)^2} - 1\right]\right\}^{1/2}$$

$$= \omega\sqrt{\mu\epsilon}\left\{\frac{1}{2}\left[\frac{\sigma}{\omega\epsilon}\sqrt{1+\frac{1}{(\sigma/\omega\epsilon)^2}} - 1\right]\right\}^{1/2}$$

4.3.1 cont.

$$\alpha = \omega\sqrt{\mu\epsilon}\left\{\tfrac{1}{2}\left[\frac{\sigma}{\omega\epsilon}\left(1 + \frac{1}{2}\frac{1}{(\sigma/\omega\epsilon)^2} \;\cdots\right) - 1\right]\right\}^{1/2}$$

$$= \omega\sqrt{\mu\epsilon}\left\{\tfrac{1}{2}\left[\overset{\text{big}}{\frac{\sigma}{\omega\epsilon}} + \cancel{\frac{1}{2}\frac{1}{(\sigma/\omega\epsilon)}} - \cdots - 1\right]\right\}^{1/2}$$

$$\cong \omega\sqrt{\mu\epsilon}\left\{\tfrac{1}{2}\frac{\sigma}{\omega\epsilon}\right\}^{1/2}$$

$$\boxed{\alpha \cong \sqrt{\frac{\omega\mu\sigma}{2}} \quad (4\text{-}40a)}$$

Similarly

$$\boxed{\beta = \sqrt{\frac{\omega\mu\sigma}{2}} \quad (4\text{-}41)},$$

$$\boxed{Z_w = \eta_c \cong \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) \quad (4\text{-}42)},$$

and

$$\boxed{\delta = \frac{1}{\alpha} \cong \sqrt{\frac{2}{\omega\mu\sigma}} \quad (4\text{-}43)}$$

(Most commonly used in texts for good conductors where skin depth is important

→ All of these expressions are summarized in Table 4-1

TABLE 4-1 Propagation constant, wave impedance, wavelength, velocity, and skin depth of TEM wave in lossy media

	Exact	Good dielectric $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \ll 1$	Good conductor $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \gg 1$
Attenuation constant α	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2}$	$\simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
Phase constant β	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2}$	$\simeq \omega\sqrt{\mu\varepsilon}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
Wave Z_w intrinsic η_c impedances $Z_w = \eta_c$	$= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\simeq \sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$
Wavelength λ	$= \frac{2\pi}{\beta}$	$\simeq \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$	$\simeq 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$
Velocity v	$= \frac{\omega}{\beta}$	$\simeq \frac{1}{\sqrt{\mu\varepsilon}}$	$\simeq \sqrt{\frac{2\omega}{\mu\sigma}}$
Skin depth δ	$= \frac{1}{\alpha}$	$\simeq \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$	$\simeq \sqrt{\frac{2}{\omega\mu\sigma}}$

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

Example- For a material where $\varepsilon = 2.2\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 2*10^{-4}$ S/m at 1 GHz, find the propagation, attenuation, & phase constants, wave impedance, and skin depth. Is this material a good dielectric or conductor at this frequency?

Define constants

$\varepsilon 0 := 8.8541878\cdot 10^{-12}$ $\varepsilon := 2.2\cdot\varepsilon 0$ $\varepsilon = 1.9479\times 10^{-11}$ F/m

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$ $\mu := \mu 0$ $\mu = 1.25664\times 10^{-6}$ H/m

$\sigma := 2\cdot 10^{-4}$ S/m

$f := 1\cdot 10^{9}$ Hz $\omega := 2\cdot\pi\cdot f$ $\omega = 6.2832\times 10^{9}$ rad/s

Calculations

Per (4-28), the phase constant $\gamma = \alpha + j\beta$ is:

$\gamma := \sqrt{j\cdot\omega\cdot\mu\cdot(\sigma + j\cdot\omega\cdot\varepsilon)}$ $\boxed{\gamma = 0.025399 + 31.086416i}$ 1/m

attenuation constant α

$\alpha := \mathrm{Re}(\gamma)$ $\boxed{\alpha = 0.0254}$ Np/m or, using (4-28c), α is:

$$\alpha 2 := \omega\cdot\sqrt{\mu\cdot\varepsilon}\cdot\left[0.5\cdot\left[\sqrt{1 + \left(\frac{\sigma}{\omega\cdot\varepsilon}\right)^2} - 1\right]\right]^{0.5}$$ $\boxed{\alpha 2 = 0.025399}$ Np/m

In dB/m, α is: $\alpha dB := \alpha\cdot 20\cdot\log(e)$ $\boxed{\alpha dB = 0.2206}$ dB/m

phase constant β

$\beta := \mathrm{Im}(\gamma)$ $\boxed{\beta = 31.08642}$ rad/m or, (4-28d), β is:

$$\beta 2 := \omega\cdot\sqrt{\mu\cdot\varepsilon}\cdot\left[0.5\cdot\left[\sqrt{1 + \left(\frac{\sigma}{\omega\cdot\varepsilon}\right)^2} + 1\right]\right]^{0.5}$$ $\boxed{\beta 2 = 31.086}$ rad/m

skin depth δ

$$\delta := \frac{1}{\alpha} \qquad \boxed{\delta = 39.37141} \ \text{m}$$

or, using (4-34), δ is:

$$\delta 2 := \frac{1}{\left[\omega\cdot\sqrt{\mu\cdot\varepsilon}\cdot\left[0.5\cdot\left[\sqrt{1+\left(\frac{\sigma}{\omega\cdot\varepsilon}\right)^2}-1\right]\right]^{0.5}\right]} \qquad \boxed{\delta 2 = 39.37141} \ \text{m}$$

wave impedance Z_W

Per, (4-30), Z_W is:

$$ZW := \sqrt{\frac{j\cdot\omega\cdot\mu}{\sigma + j\cdot\omega\cdot\varepsilon}} \qquad \boxed{ZW = 253.9913 + 0.2075i} \ \Omega$$

Is material a good dielectric or conductor at this frequency?

$\left(\frac{\sigma}{\omega\cdot\varepsilon}\right)^2 = 2.67\times 10^{-6}$ $\ll 1$. This implies material is a **good dielectric**.

Now, we can use (4-36a) for α and (4-37) for β, (4-38) for Z_W, and (4-39) for δ.

$$\alpha\_approx := \frac{\sigma}{2}\cdot\sqrt{\frac{\mu}{\varepsilon}} \qquad \boxed{\alpha\_approx = 0.025399}$$

Np/m, excellent agreement

$$\beta\_approx := \omega\sqrt{\mu\cdot\varepsilon} \qquad \boxed{\beta\_approx = 31.08641}$$

rad/m, excellent agreement

$$ZW\_approx := \sqrt{\frac{\mu}{\varepsilon}} \qquad \boxed{ZW\_approx = 253.9915}$$

Ω, excellent agreement for real part

$$\delta\_approx := \frac{2}{\sigma}\cdot\sqrt{\frac{\varepsilon}{\mu}} \qquad \boxed{\delta\_approx = 39.37139}$$

m, excellent agreement

4.3.2 UPW in an Unbounded Lossy Medium – Oblique Angle

→ replace β_r w/ γ_r along direction of propagation and must decompose the propagation constant into directional components (replace $\bar{\beta}$ w/ $\bar{\gamma}$)

→ same pictures as shown in section 4.2.2

Choose TE^y mode UPW (See Fig 4-4a)

Define:

$$\bar{\gamma}^+ = \gamma\left(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right) = (\alpha + j\beta)\left(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right) \quad (4\text{-}44a)$$

$$\bar{\gamma}^- = -\gamma\left(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right) = -(\alpha + j\beta)\left(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right) \quad (4\text{-}44b)$$

Now we can write the electric + magnetic fields as

$$\bar{E} = E_0^+\left(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right) e^{-\bar{\gamma}^+ \cdot \bar{r}} + E_0^-\left(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right) e^{-\bar{\gamma}^- \cdot \bar{r}}$$

$$\bar{E} = E_0^+\left(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right) e^{-(\alpha + j\beta)(x \sin\theta_i + z\cos\theta_i)}$$
$$+ E_0^-\left(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right) e^{+(\alpha + j\beta)(x \sin\theta_i + z\cos\theta_i)} \quad (4\text{-}45a)$$

and

$$\bar{H} = \hat{a}_y\left[\frac{E_0^+}{\eta_c} e^{-(\alpha + j\beta)(x \sin\theta_i + z\cos\theta_i)} - \frac{E_0^-}{\eta_c} e^{+(\alpha + j\beta)(x \sin\theta_i + z\cos\theta_i)}\right] \quad (4\text{-}45b)$$

4.3.2 cont.

What about the wave impedance?

In direction of wave travel

$$Z_{w_r} = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \quad (4\text{-}46)$$

However the directional wave impedances are

$$\boxed{\begin{aligned} Z_x^+ = Z_x^- &= \frac{-E_z^+}{H_y^+} = \frac{E_z^-}{H_y^-} = \eta_c \sin\theta_i \quad (4\text{-}47a) \\ Z_z^+ = Z_z^- &= \frac{E_x^+}{H_y^+} = \frac{-E_x^-}{H_y^-} = \eta_c \cos\theta_i \quad (4\text{-}47b) \end{aligned}}$$

Note: $\left(0 \le Z_{x/z}^{+/-} \le \eta_c \text{ for } TE^y \text{ mode}\right)$

Also

$$\boxed{\begin{aligned} v_r = v_{pr} = v_{er} = v = \omega/\beta \quad (4\text{-}48a) \\ v_{pz} = \frac{v}{\cos\theta_i} = \frac{\omega}{\beta\cos\theta_i} \quad (4\text{-}48b) \\ v_{ez} = v\cos\theta_i = \frac{\omega}{\beta}\cos\theta_i \quad (4\text{-}48c) \end{aligned}}$$

Note! $v_{pr} v_{er} = v_{pz} v_{ez} = v^2$ still

4.3.2 cont.

What about time-ave power density?

For the fwd prop. wave

$$\boxed{\begin{aligned}\left(\bar{S}^{+}_{ave}\right)_r &= \tfrac{1}{2}\,\mathrm{Re}\left\{\bar{E}^{+}\times\bar{H}^{+*}\right\}\\ &= \left(\hat{a}_x\sin\theta_i+\hat{a}_z\cos\theta_i\right)\frac{|E_0^{+}|^2}{2}\,e^{-2\alpha(x\sin\theta_i+z\cos\theta_i)}\,\mathrm{Re}\left\{\frac{1}{\eta_c^{*}}\right\}\\ &= \hat{a}_r\frac{|E_0^{+}|^2}{2}\,e^{-2\alpha r}\,\mathrm{Re}\left\{\frac{1}{\eta_c^{*}}\right\} \qquad (4\text{-}49a)\end{aligned}}$$

Picking off the x + z components

$$\boxed{\begin{aligned}\left(S^{+}_{ave}\right)_x &= \sin\theta_i\,\frac{|E_0^{+}|^2}{2}\,e^{-2\alpha(x\sin\theta_i+z\cos\theta_i)}\,\mathrm{Re}\left[\frac{1}{\eta_c^{*}}\right] \qquad (4\text{-}49b)\\ \left(S_{ave}^{+}\right)_z &= \cos\theta_i\,\frac{|E_0^{+}|^2}{2}\,e^{-2\alpha(x\sin\theta_i+z\cos\theta_i)}\,\mathrm{Re}\left[\frac{1}{\eta_c^{*}}\right] \qquad (4\text{-}49c)\end{aligned}}$$

→ Can follow similar process for TM^y mode

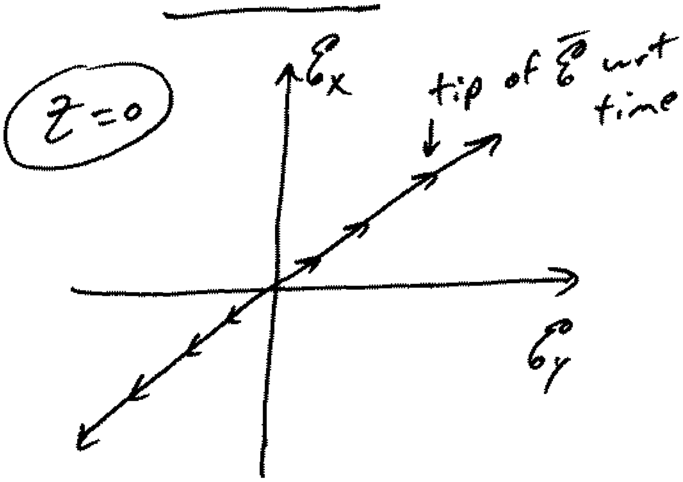
4.4 Polarization

→ Polarization is defined by the curve/locus traced out by the tip of the electric field vector wrt time at a fixed location observed in the direction of propagation (both shape and sense (direction of rotation)

→ For a wave propagating in the $\pm z$ directions, we would usually choose $z=0$ for observation location

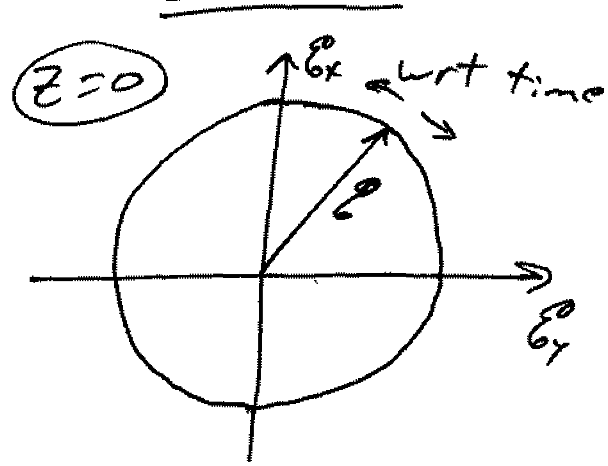
→ Three categories of polarization (assume $+z$-direction for wave propagation)

Linear



→ No sense of rotation applicable

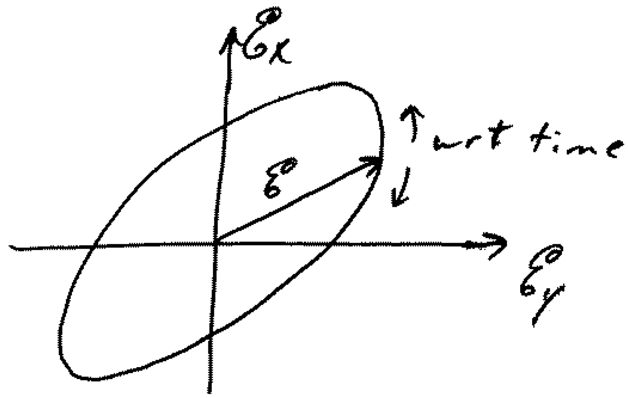
Circular



→ Sense of rotation from RHR (thumb of RH has fingers curling w/ $\mathcal{E}$) <u>or</u> LHR (thumb of LH has fingers curling w/ $\mathcal{E}$)

4.4 cont.

Elliptical



→ Most general case (linear + circular are subsets)

→ traces / loci sometimes called polarization ellipse

4.4.1 Linear Polarization

→ Far-field $\mathscr{E}$ for many antennas are linearly polarized

e.g., dipoles, loop, horns, ...

Assume that our time-harmonic plane wave is propagating in the +z-direction and has the following fields

$$\boxed{\begin{aligned} \bar{\mathscr{E}}(z,t) &= \hat{a}_x E_{x0}^{+}\cos(\omega t-\beta z+\phi_x)+\hat{a}_y E_{y0}^{+}\cos(\omega t-\beta z+\phi_y) \quad (4\text{-}50a) \\ \bar{\mathscr{H}}(z,t) &= \hat{a}_y \frac{E_{x0}^{+}}{\eta}\cos(\omega t-\beta z+\phi_x)-\hat{a}_x \frac{E_{y0}^{+}}{\eta}\cos(\omega t-\beta z+\phi_y) \quad (4\text{-}50b) \end{aligned}}$$

General case (don't know polarization yet)

E_x^+ + E_y^+ can be complex, but E_{x0}^+ + E_{y0}^+ are real #

4.4.1 cont.

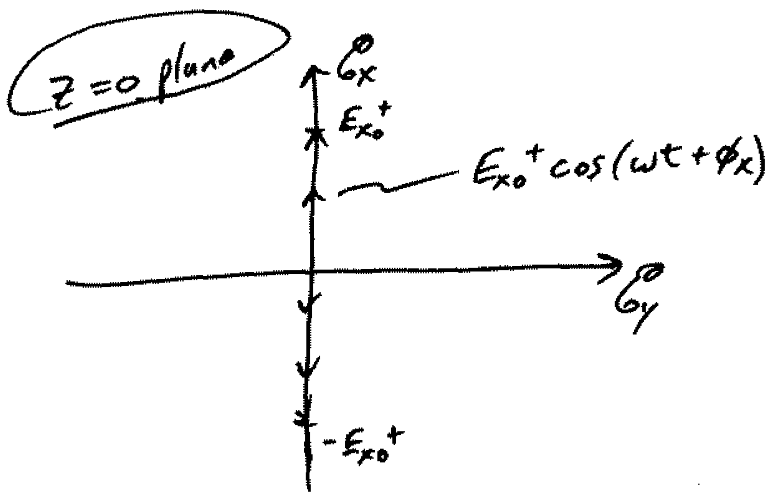
Case 1 (Single vector component)

→ the wave will be linearly polarized if $\bar{\mathcal{E}}^o$ has only a single vector component (i.e. either $\mathcal{E}_x^o$ or $\mathcal{E}_y^o$ but <u>NOT</u> both)

e.g.

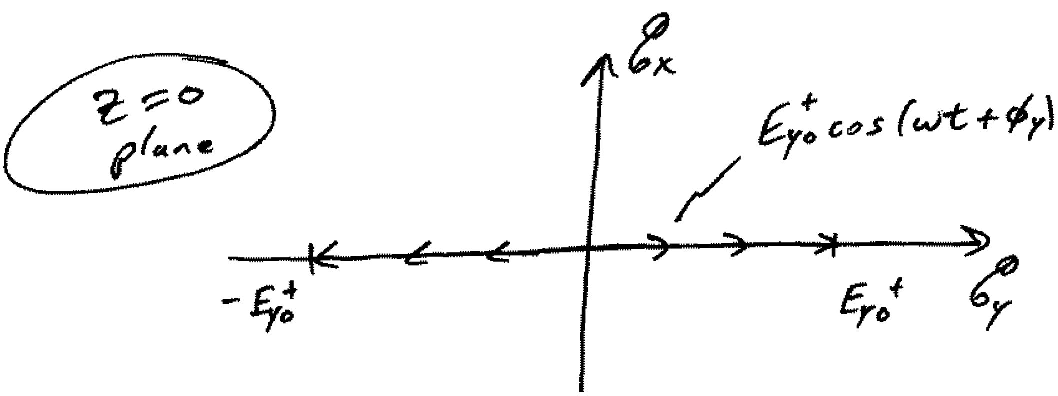
a) $\bar{\mathcal{E}}^o = \hat{a}_x E_{x0}^+ \cos(\omega t - \beta z + \phi_x)$ V/m ϕ_x can be any ∡

b) $\bar{\mathcal{E}}^o = \hat{a}_y E_{y0}^+ \cos(\omega t - \beta z + \phi_y)$ V/m ϕ_y can be any ∡



← usually evaluate this expression as ωt varies from 0 to 2π

(a)



(b)

4.4.1 cont.

Case 2 ($\mathcal{E}_x$ & $\mathcal{E}_y$ are in phase)

→ wave will be linearly polarized if $\phi_x = \phi_y = \phi$

e.g. $\bar{\mathcal{E}} = \hat{a}_x E_{x0}^+ \cos(\omega t - \beta z + \phi) + \hat{a}_y E_{y0}^+ \cos(\omega t - \beta z + \phi)$

Here $|\bar{\mathcal{E}}| = \sqrt{\bar{\mathcal{E}} \cdot \bar{\mathcal{E}}} = \sqrt{E_{x0}^{+2} + E_{y0}^{+2}} \cos(\omega t - \beta z + \phi)$

and the polarization ellipse (line in this case) is tilted wrt the axes

→ E_{x0}^+ & E_{y0}^+ can be equal or NOT equal

→ define a tilt angle ψ wrt positive x-axis

$$\boxed{\psi = \tan^{-1}\left[\frac{\mathcal{E}_y}{\mathcal{E}_x}\right] = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right]}$$

ex. Say $\bar{\mathcal{E}} = \hat{a}_x 5 \cos(\omega t - \beta z + \pi/3) - \hat{a}_y 3 \cos(\omega t - \beta z + \pi/3)$ (V/m)

Choose $z = 0$

$$\bar{\mathcal{E}} = \hat{a}_x \overbrace{5 \cos(\omega t + \pi/3)}^{\mathcal{E}_x} - \hat{a}_y \overbrace{3 \cos(\omega t + \pi/3)}^{\mathcal{E}_y} \text{ V/m}$$

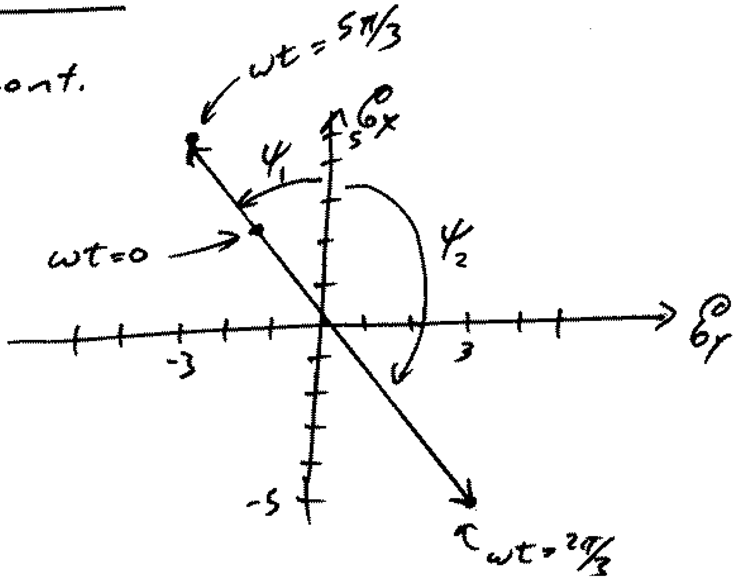
ωt	$\mathcal{E}_x$	$\mathcal{E}_y$
0	2.5	-1.5
$\pi/4$	-1.294	0.776
$\pi/2$	-4.33	2.598
$3\pi/4$	-4.83	2.898
π	-2.5	1.5

ωt	$\mathcal{E}_x$	$\mathcal{E}_y$
$5\pi/4$	1.294	-0.776
$3\pi/2$	4.33	-2.598
$\frac{7\pi}{4}$	4.83	-2.898

ωt	$\mathcal{E}_x$	$\mathcal{E}_y$
$\frac{2\pi}{3}$	-5	3
$\frac{5\pi}{3}$	5	-3

4.4.1 cont.

ex. cont.



$\psi_1 = \tan^{-1}\left(\frac{5}{-3}\right) = -59°$

or

$\psi_2 = \tan^{-1}\left(\frac{5}{-3}\right) = 121°$

Case 3 ($\mathscr{E}_x$ + $\mathscr{E}_y$ are $\pm\pi = \pm 180°$ out of phase)

→ wave will be linearly polarized if $\phi_y - \phi_x = \pm 180°$

→ E_{xo}^+ + E_{yo}^+ can be equal or NOT equal

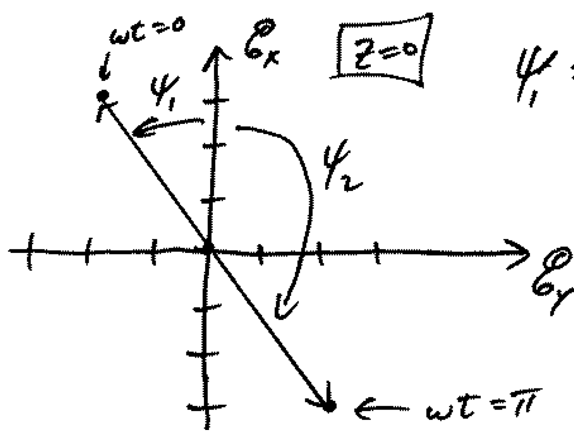
e.g. $\bar{\mathscr{E}} = \hat{a}_x E_{xo}^+ \cos(\omega t - \beta z + \phi_x) + \hat{a}_y E_{yo}^+ \cos(\omega t - \beta z + \phi_x \pm 180°)$

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right]$$

ex. Say $\bar{\mathscr{E}} = \hat{a}_x 3\cos(\omega t - \beta z) + \hat{a}_y 2\cos(\omega t - \beta z + 180°)$ (V/m)

choosing $z = 0$, $\bar{\mathscr{E}} = \hat{a}_x 3\cos(\omega t) + \hat{a}_y 2\cos(\omega t + 180°)$ V/m

ωt	$\mathscr{E}_x$	$\mathscr{E}_y$
0	3	-2
$\pi/2$	0	0
π	-3	2
$\frac{3\pi}{2}$	0	0



$\psi_1 = \tan^{-1}\left[\frac{-2}{3}\right] = -33.69°$

or

$\psi_2 = 146.3°$

4.4.2 Circular Polarization

→ $\bar{\mathcal{E}}$ traces out a circular locus in space wrt time

→ $|\bar{\mathcal{E}}| = \text{constant} = E_{xo}^{+} = E_{yo}^{+} = E_{L/R}$

→ $\phi_y - \phi_x = (2n+1)(\pm 90^\circ) = (2n+1)(\pm \pi/2) \quad n = 0, 1, \ldots$

odd multiple of 90° or $\pi/2$

For RH (cw) $\phi_y - \phi_x = -\pi/2 = -90^\circ$

LH (ccw) $\phi_y - \phi_x = +\pi/2 = 90^\circ$

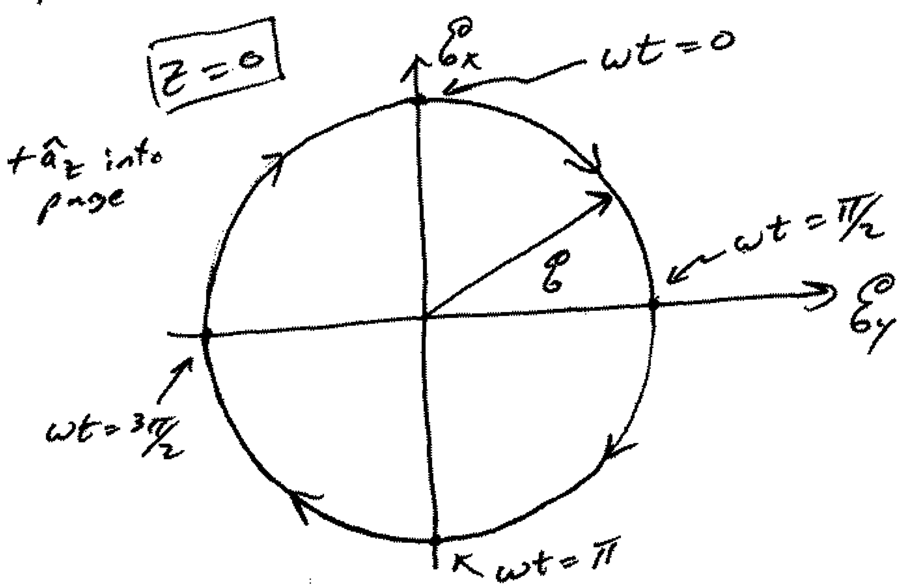
ex. Let $E_{xo}^{+} = E_{yo}^{+} = 4 \frac{V}{m}$ and $\phi_y = 3\pi/2$ & $\phi_x = 0$

$$\bar{\mathcal{E}} = \hat{a}_x 4 \cos(\omega t - \beta z) + \hat{a}_y 4 \cos(\omega t - \beta z + 3\pi/2) \ \text{V/m}$$

select <u>z = 0 plane</u>

$$\bar{\mathcal{E}} = \hat{a}_x 4 \cos(\omega t) + \hat{a}_y 4 \cos(\omega t + 3\pi/2) \ \text{V/m}$$

ωt	$\mathcal{E}_x$	$\mathcal{E}_y$
0	4	0
$\pi/2$	0	4
π	-4	0
$3\pi/2$	0	-4



<u>RH Circular Polarization</u>

4.4.3 Elliptical Polarization

Case 1 ($E_{x0}^+ \neq E_{y0}^+$ and $\phi_y - \phi_x = \pm(2n+1)\pi/2$)

ex. Let $E_{x0}^+ = -4$ V/m, $E_{y0}^+ = 3$ V/m, $\phi_y = \frac{-3\pi}{2}$, + $\phi_x = 0$

$$\bar{\mathcal{E}} = -\hat{a}_x 4\cos(\omega t - \beta z) + \hat{a}_y 3\cos(\omega t - \beta z - 3\pi/2) \text{ (V/m)}$$

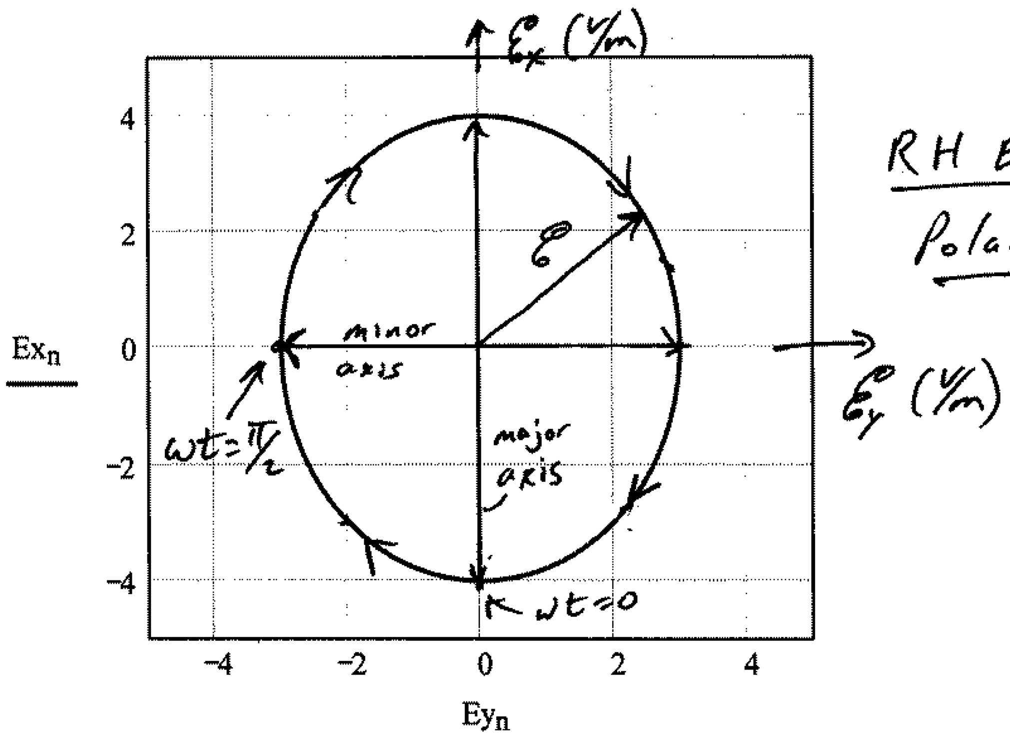
Case 1 Elliptical Polarization at z = 0

$Ex0 := -4$ $Ey0 := 3$ $\phi x := 0$ $\phi y := \frac{-3\cdot\pi}{2}$ $n := 0..180$ $wt_n := \frac{2\cdot\pi\cdot n}{180}$

$Ex_n := Ex0\cdot\cos(wt_n + \phi x)$ $Ey_n := Ey0\cdot\cos(wt_n + \phi y)$

$Ex_0 = -4$ $Ey_0 = 0$ $wt_0 = 0$

$Ex_{45} = 0$ $Ey_{45} = -3$ $wt_{45} = 1.571$



RH Elliptical Polarization!

$magE_n := \sqrt{(Ex_n)^2 + (Ey_n)^2}$ $AR := \frac{\max(magE)}{\min(magE)}$ $AR = 1.333$

$\max(magE) = 4$ $\min(magE) = 3$ $magE_0 = 4$

$Ex_0 = -4$

$tilt := \text{atan}\left(\frac{Ey_0}{Ex_0}\right)\cdot\frac{180}{\pi}$ $tilt = 0$ deg

$Ey_0 = 0$

4.4.3 cont.

Case 2 ($\phi_y - \phi_x \neq \pm(2n+1)\pi/2$, E_{xo}^+ + E_{yo}^+ anything $\neq 0$)

ex. Choose $E_{xo}^+ = 4$ V/m, $E_{yo}^+ = 5$ V/m, $\phi_y = \pi/3$, $\phi_x = \pi/6$

$$\bar{\mathcal{E}} = \hat{a}_x 4\cos(\omega t - \beta z + \pi/6) + \hat{a}_y 5\cos(\omega t - \beta z + \pi/3) \text{ V/m}$$

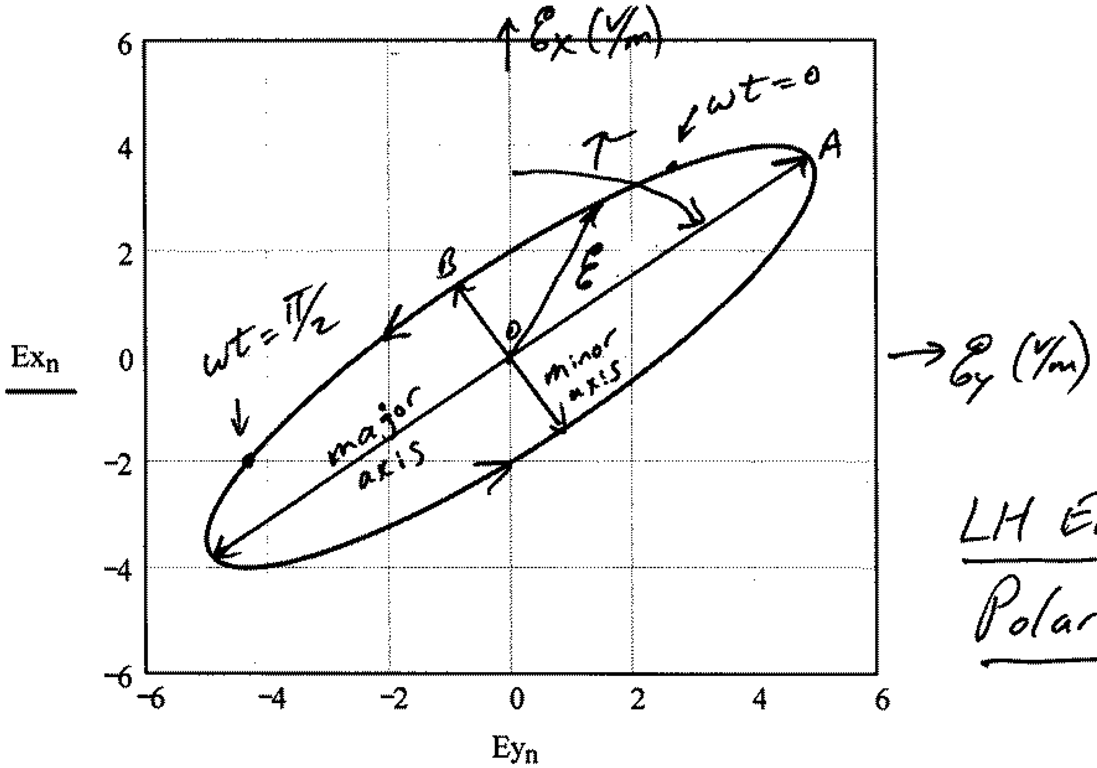
Case 2 Elliptical Polarization at z = 0

$Ex0 := 4 \quad Ey0 := 5 \quad \phi x := \frac{\pi}{6} \quad \phi y := \frac{\pi}{3} \quad n := 0..180 \quad wt_n := \frac{2\cdot\pi\cdot n}{180}$

$Ex_n := Ex0\cdot\cos(wt_n + \phi x) \qquad Ey_n := Ey0\cdot\cos(wt_n + \phi y)$

$Ex_0 = 3.464 \qquad Ey_0 = 2.5 \qquad wt_0 = 0$

$Ex_{45} = -2 \qquad Ey_{45} = -4.33 \qquad wt_{45} = 1.571$



LH Elliptical Polarization!

$magE_n := \sqrt{(Ex_n)^2 + (Ey_n)^2} \qquad AR := \frac{\max(magE)}{\min(magE)} \qquad AR = 3.836$

$\max(magE) = 6.196 = OA \qquad \min(magE) = 1.615 = OB \qquad magE_{66} = 6.196$

$Ex_{66} = -3.804$

$Ey_{66} = -4.891 \qquad tilt := \operatorname{atan}\left(\frac{Ey_{66}}{Ex_{66}}\right)\cdot\frac{180}{\pi} \qquad tilt = 52.123 \text{ deg} = \tau$

4.4.3 cont.

Some parameters used to describe elliptically polarized waves are the axial ratio (AR) and tilt angle (τ) wrt positive x-axis

$$AR = \frac{\text{length of major axis of ellipse}}{\text{length of minor axis of ellipse}} \qquad 1 \leq AR < \infty$$

→ can get these quantities off plot of polarization ellipse or develop the equations [e.g. (4-54g), (4-57), + (4-57c)]

4.4.4 Poincaré Sphere

→ Interesting but not in widespread use