

3.4.2 Cylindrical Coordinate System

For problems that 'fit' the cylindrical coordinate system, e.g., boundary-value problems for cylindrical waveguides, it makes sense to express the fields as well as potentials in this coordinate system.

⇒ Unlike rectangular coordinates, the vector Laplacian in cylindrical coordinates is NOT simply related to the scalar Laplacian,

$$\text{i.e., } \vec{\nabla}^2 \vec{E}(r, \phi, z) \neq \hat{a}_r \vec{\nabla}^2 E_r + \hat{a}_\phi \vec{\nabla}^2 E_\phi + \hat{a}_z \vec{\nabla}^2 E_z$$

$$\text{Specifically, } \vec{\nabla}^2(\hat{a}_r E_r) \neq \hat{a}_r \vec{\nabla}^2 E_r$$

$$\vec{\nabla}^2(\hat{a}_\phi E_\phi) \neq \hat{a}_\phi \vec{\nabla}^2 E_\phi$$

However, since 'z' is the same variable as in rectangular coordinates

$$\vec{\nabla}^2(\hat{a}_z E_z) = \hat{a}_z \vec{\nabla}^2 E_z$$

In cylindrical coordinates, the scalar Laplacian of some function ψ is

$$\begin{aligned} \nabla^2 \psi &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \\ &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \end{aligned} \quad \begin{array}{l} \text{use} \\ \text{product} \\ \text{rule} \end{array} \quad (3-55)$$

3.4.2 cont.

Returning to the time-harmonic wave equations

$$\bar{\nabla}^2 \bar{E} = -\beta^2 \bar{E} \quad (3-18a)$$

$$\bar{\nabla}^2 \bar{H} = -\beta^2 \bar{H} \quad (3-18b)$$

where $\beta^2 = \omega_{de}^2$ $(3-18c)$,

we see that we'll need to evaluate/express the vector Laplacian in cylindrical coordinates

$$\bar{\nabla}^2 \bar{E} = \bar{\nabla}(\bar{\nabla} \cdot \bar{E}) - \bar{\nabla} \times \bar{\nabla} \times \bar{E} = -\beta^2 \bar{E}. \quad (3-53)$$

When evaluated and split into 3 scalar partial differential equations based on the vector components, we get

$$\rightarrow \nabla^2 E_\rho + \left(\frac{-E_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} \right) = -\beta^2 E_\rho \quad (3-54a)$$

coupled!

$$\rightarrow \nabla^2 E_\phi + \left(\frac{-E_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} \right) = -\beta^2 E_\phi \quad (3-54b)$$

$$\nabla^2 E_z = -\beta^2 E_z \quad (3-54c)$$

As seen, the " ρ " + " ϕ " equations are coupled 2nd order partial differential equations \Rightarrow far more difficult to solve. However, the " z " equation is still uncoupled, i.e., E_z only depends on E_z \Rightarrow TE² + TM² mode solutions favorable.

3.4.2 cont.

Using a generic variable Ψ , our wave equation in cylindrical coordinates is then

$$\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{\partial^2 \Psi}{\partial z^2} = -\beta^2 \Psi \quad (3-56)$$

Again, we will assume a separable solution

$$\Psi = f(\rho) g(\phi) h(z) \quad (3-57)$$

which yields

$$\begin{aligned} & \div fgh \left(gh \frac{\partial^2 f}{\partial \rho^2} + gh \frac{1}{\rho} \frac{\partial f}{\partial \rho} + fh \frac{1}{\rho^2} \frac{\partial^2 g}{\partial \phi^2} + fg \frac{\partial^2 h}{\partial z^2} \right) = -\beta^2 fgh \\ & \frac{1}{f} \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{g} \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{h} \frac{1}{\rho^2} \frac{\partial^2 g}{\partial \phi^2} + \frac{1}{h} \frac{\partial^2 h}{\partial z^2} = -\beta^2 \end{aligned} \quad (3-59)$$

The last term on the LHS of (3-59) only depends on variable z . Therefore, it must be equal to a constant in order for (3-59) to hold true, i.e., let

$$\frac{1}{h} \frac{\partial^2 h}{\partial z^2} = -\beta_z^2 \stackrel{\text{w/constant}}{\Rightarrow} \boxed{\frac{\partial^2 h}{\partial z^2} = -\beta_z^2 h} \quad (3-66c)$$

Substitute into (3-59) to get

$$\begin{aligned} & \frac{1}{f} \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{g} \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{h} \frac{1}{\rho^2} \frac{\partial^2 g}{\partial \phi^2} + \frac{1}{h} (-\beta_z^2 h) = -\beta^2 \\ & \times \rho^2 \left(\frac{\rho^2}{f} \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{f} \rho \frac{\partial f}{\partial \rho} + \frac{1}{g} \frac{1}{\rho^2} \frac{\partial^2 g}{\partial \phi^2} + (\beta^2 - \beta_z^2) \rho^2 \right) = 0 \end{aligned} \quad (3-61)$$

3.4.2 cont.

In (3-61), the third term on the LHS depends only on ϕ while the remaining terms depend only on ρ . This can only hold true if

$$\frac{1}{g} \frac{d^2 g}{d\phi^2} = -m^2 \stackrel{\text{some constant}}{\Rightarrow} \boxed{\frac{d^2 g}{d\phi^2} = -m^2 g} \quad (3-66b)$$

Put into (3-61), this yields

$$\frac{\rho^2}{f} \frac{d^2 f}{d\rho^2} + \frac{\rho}{f} \frac{df}{d\rho} - m^2 + (\beta^2 - \beta_2^2) \rho^2 = 0$$

an equation that is only dependent on ρ !

$$\text{Define } \beta_p^2 = \beta^2 - \beta_2^2 \text{ or } \boxed{\beta_p^2 + \beta_2^2 = \beta^2} \quad (3-63) \text{ || } (3-66d)$$

Then, we have, after some algebra, ^{constraint/dispersion} equation!

$$\boxed{\rho^2 \frac{d^2 f}{d\rho^2} + \rho \frac{df}{d\rho} + [(\beta_p \rho)^2 - m^2] f = 0} \quad (3-64) \text{ || } (3-66a)$$

which is the classic bessel differential equation.

Equations (3-66b) + (3-66c) are of the same form as was encountered for the wave equation in rectangular coordinates. Therefore, the solution forms/functions are the same, only using $\phi + z$ as the variables.

3.4.2 cont.

on $g_1(\phi) = A_2 e^{-jm\phi} + B_2 e^{jm\phi}$ (3-68a) traveling

$$g_2(\phi) = C_2 \cos(m\phi) + D_2 \sin(m\phi) \quad (3-68b) \text{ standing/periodic}$$

on $h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$ (3-69a) traveling

$$h_2(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z) \quad (3-69b) \text{ standing}$$

However, the solutions to (3-66a) are of different forms

$$f_1(p) = A_1 J_m(\beta_p p) + B_1 Y_m(\beta_p p) \quad (3-67a) \text{ standing}$$

\uparrow \uparrow see
Appendix IV
bessel function bessel function
of first kind, of second kind,
order m order m
(AIKA: Neumann functions)

or

$$f_2(p) = C_1 H_m^{(1)}(\beta_p p) + D_1 H_m^{(2)}(\beta_p p) \quad (3-67b) \text{ traveling}$$

\uparrow \uparrow see
Appendix IV
Hankel function Hankel function
of first kind, of second kind,
order m order m

The combination of these possible solutions that is appropriate depends on the problem to which they are applied.

A summary of some possibilities as well as zeros & infinities is given in Table 3-2.

TABLE 3-2 Wave functions, zeroes, and infinities for radial wave functions in cylindrical coordinates

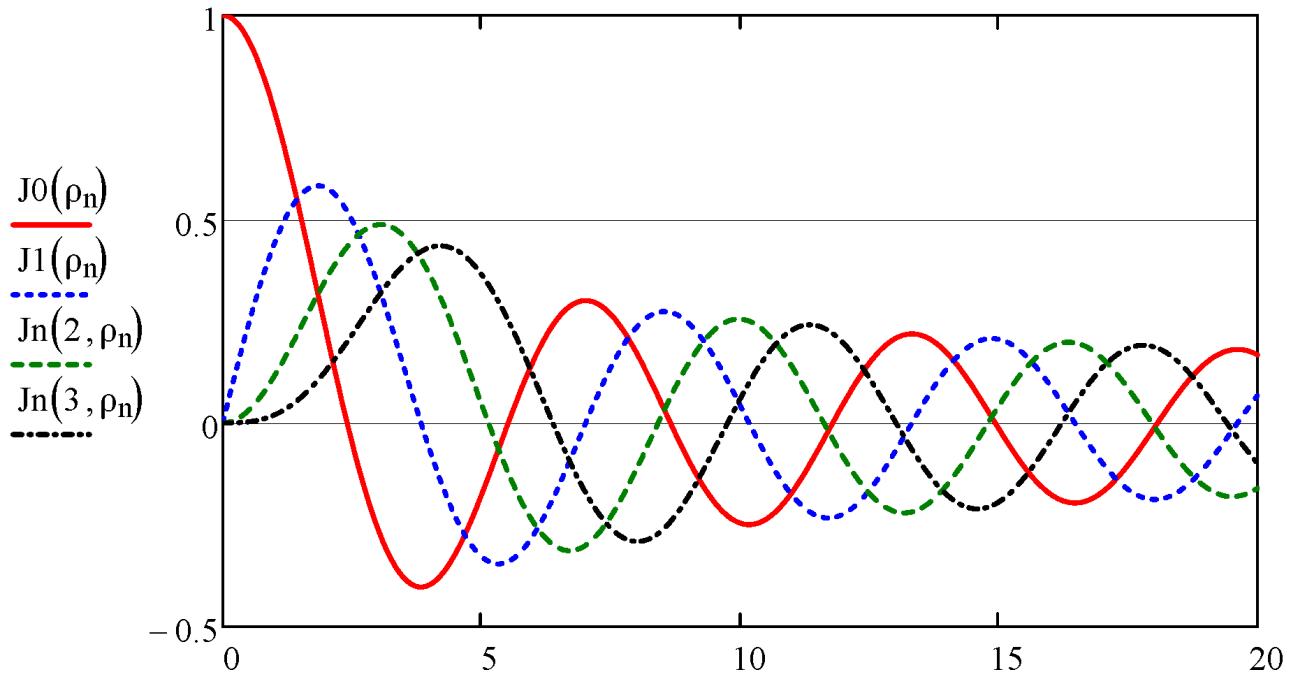
Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$H_m^{(1)}(\beta\rho) = J_m(\beta\rho) + jY_m(\beta\rho)$ for $-\rho$ travel	$\beta\rho \rightarrow +j\infty$	$\beta\rho = 0$ $\beta\rho \rightarrow -j\infty$
	$H_m^{(2)}(\beta\rho) = J_m(\beta\rho) - jY_m(\beta\rho)$ for $+\rho$ travel	$\beta\rho \rightarrow -j\infty$	$\beta\rho = 0$ $\beta\rho \rightarrow +j\infty$
Standing waves	$J_m(\beta\rho)$ for $\pm\rho$	Infinite number (see Table 9-2)	$\beta\rho \rightarrow \pm j\infty$
	$Y_m(\beta\rho)$ for $\pm\rho$	Infinite number	$\beta\rho = 0$ $\beta\rho \rightarrow \pm j\infty$
Evanescent waves	$K_m(\alpha\rho) = \frac{\pi}{2}(-j)^{m+1}H_m^{(2)}(-j\alpha\rho)$ for $+\rho$	$\alpha\rho \rightarrow +\infty$	$\alpha\rho \rightarrow 0$
	$I_m(\alpha\rho) = j^m J_m(-j\alpha\rho)$ for $-\rho$		$\alpha\rho \rightarrow +\infty$ for integer orders
Attenuating traveling waves	$H_m^{(1)}(-j\gamma\rho) = H_m^{(1)}(-j\alpha\rho + \beta\rho)$ for $-\rho$ travel	$\gamma\rho \rightarrow -\infty$	$\gamma\rho \rightarrow +\infty$
	$H_m^{(2)}(-j\gamma\rho) = H_m^{(2)}(-j\alpha\rho + \beta\rho)$ for $+\rho$ travel	$\gamma\rho \rightarrow +\infty$	$\gamma\rho \rightarrow -\infty$
Attenuating standing waves	$J_m(-j\gamma\rho) = J_m(-j\alpha\rho + \beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \rightarrow \pm j\infty$
	$Y_m(-j\gamma\rho) = Y_m(-j\alpha\rho + \beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \rightarrow \pm j\infty$

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

MathCAD example

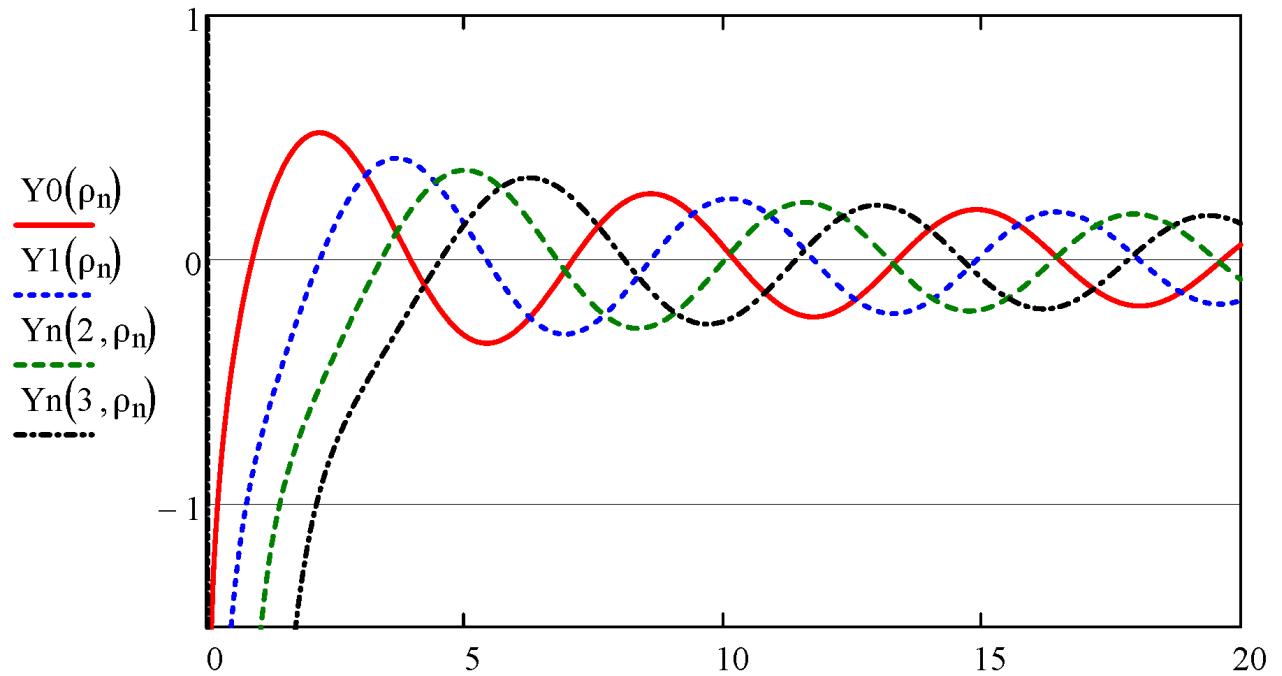
$$n := 0 .. 1000 \quad \rho_n := \frac{2n}{100}$$

Bessel functions of the first kind



Note that all Bessel functions of the first kind start at zero, except J_0 which starts at 1.

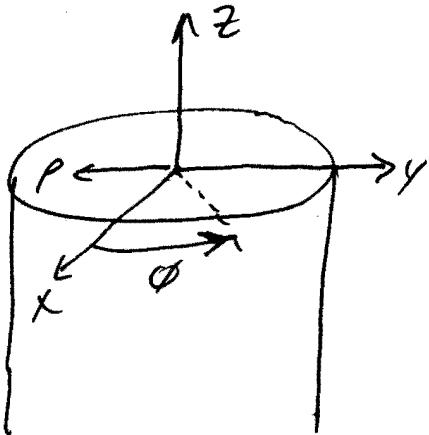
Bessel functions of the second kind



Note that all Bessel functions of the second kind head to -infinity as the argument goes to zero.

3.4.2 cont.

For example, say we want to find the fields inside a cylindrical waveguide w/ the geometry shown below



Here, the fields could propagate in the $\pm z$ -directions. However, the fields can not propagate in the ρ - or ϕ -directions. Therefore, an appropriate solution would start with the general form

$$\begin{aligned} \Psi(\rho, \phi, z) &= \underbrace{f(\rho)}_{\text{not prop.}} g(\phi) h(z) \\ &= [A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)] \\ &\quad \times [C_2 \cos(m\phi) + D_2 \sin(m\phi)] \\ &\quad \times [A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}] \end{aligned}$$

before applying physical/boundary conditions.