Chapter 3 Wave Equation and its Solution

3.1 Introduction

→ Use Maxwell's equations to get uncoupled, second-order partial differential equations, i.e., wave equations for the electric + magnetic fields

3.2 Time-Varying Electromagnetic Fields

Start w/ Faraday's + Ampere's Law

$$\bar{\nabla} \times \bar{\mathcal{E}} = -\bar{\mathcal{M}}_i - \mu \frac{\partial \bar{\mathcal{H}}}{\partial t}$$

$$\bar{\nabla} \times \bar{\mathcal{H}} = \bar{\mathcal{J}}_i + \sigma \bar{\mathcal{E}} + \epsilon \frac{\partial \bar{\mathcal{E}}}{\partial t}$$

Note: $\sigma = \sigma_e \equiv$ eff. conductivity and $\epsilon = \epsilon'$

Take curl of both equations

$$\bar{\nabla} \times \bar{\nabla} \times \bar{\mathcal{E}} = -\bar{\nabla} \times \bar{\mathcal{M}}_i - \mu \bar{\nabla} \times \frac{\partial \bar{\mathcal{H}}}{\partial t} = -\bar{\nabla} \times \bar{\mathcal{M}}_i - \mu \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{\mathcal{H}})$$

$$\bar{\nabla} \times \bar{\nabla} \times \bar{\mathcal{H}} = \bar{\nabla} \times \bar{\mathcal{J}}_i + \sigma \bar{\nabla} \times \bar{\mathcal{E}} + \epsilon \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{\mathcal{E}})$$

Next, use Faraday's + Ampere's Law to cross-substitute for $\bar{\nabla} \times \bar{\mathcal{H}}$ + $\bar{\nabla} \times \bar{\mathcal{E}}$ terms on RHS.

Use vector identity $\bar{\nabla} \times \bar{\nabla} \times \bar{F} = \bar{\nabla}(\bar{\nabla} \cdot \bar{F}) - \bar{\nabla}^2 \bar{F}$ on LHS.

3.2 cont.

$\boxed{\bar{\mathcal{E}}}$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{\mathcal{E}})-\bar{\nabla}^2\bar{\mathcal{E}}=-\bar{\nabla}\times\bar{\mathcal{M}}_i-\mu\frac{\partial}{\partial t}\left[\bar{\mathcal{J}}_i+\sigma\bar{\mathcal{E}}+\epsilon\frac{\partial\bar{\mathcal{E}}}{\partial t}\right]$$

$$=-\bar{\nabla}\times\bar{\mathcal{M}}_i-\mu\frac{\partial\bar{\mathcal{J}}_i}{\partial t}-\mu\sigma\frac{\partial\bar{\mathcal{E}}}{\partial t}-\mu\epsilon\frac{\partial^2\bar{\mathcal{E}}}{\partial t^2}$$

use Gauss' Law $\bar{\nabla}\cdot\bar{\mathcal{D}}=\epsilon\bar{\nabla}\cdot\bar{\mathcal{E}}=q_{ev}\Rightarrow\bar{\nabla}\cdot\bar{\mathcal{E}}=\frac{q_{ev}}{\epsilon}$

$$\bigstar\quad\boxed{\bar{\nabla}^2\bar{\mathcal{E}}=\bar{\nabla}\times\bar{\mathcal{M}}_i+\mu\frac{\partial\bar{\mathcal{J}}_i}{\partial t}+\frac{1}{\epsilon}\bar{\nabla}q_{ev}+\mu\sigma\frac{\partial\bar{\mathcal{E}}}{\partial t}+\mu\epsilon\frac{\partial^2\bar{\mathcal{E}}}{\partial t^2}}$$

$\boxed{\bar{\mathcal{H}}}$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{\mathcal{H}})-\bar{\nabla}^2\bar{\mathcal{H}}=\bar{\nabla}\times\bar{\mathcal{J}}_i+\sigma\left[-\bar{\mathcal{M}}_i-\mu\frac{\partial\bar{\mathcal{H}}}{\partial t}\right]+\epsilon\frac{\partial}{\partial t}\left[-\bar{\mathcal{M}}_i-\mu\frac{\partial\bar{\mathcal{H}}}{\partial t}\right]$$

$$=\bar{\nabla}\times\bar{\mathcal{J}}_i-\sigma\bar{\mathcal{M}}_i-\mu\sigma\frac{\partial\bar{\mathcal{H}}}{\partial t}-\epsilon\frac{\partial\bar{\mathcal{M}}_i}{\partial t}-\mu\epsilon\frac{\partial^2\bar{\mathcal{H}}}{\partial t^2}$$

use Maxwell eq'n $\bar{\nabla}\cdot\bar{\mathcal{B}}=\mu\bar{\nabla}\cdot\bar{\mathcal{H}}=q_{mv}\Rightarrow\bar{\nabla}\cdot\bar{\mathcal{H}}=\frac{q_{mv}}{\mu}$
to get

$$\bigstar\quad\boxed{\bar{\nabla}^2\bar{\mathcal{H}}=-\bar{\nabla}\times\bar{\mathcal{J}}_i+\sigma\bar{\mathcal{M}}_i+\frac{1}{\mu}\bar{\nabla}q_{mv}+\epsilon\frac{\partial\bar{\mathcal{M}}_i}{\partial t}+\mu\sigma\frac{\partial\bar{\mathcal{H}}}{\partial t}+\mu\epsilon\frac{\partial^2\bar{\mathcal{H}}}{\partial t}}$$

$\bigstar\rightarrow$ uncoupled 2nd order differential eq'n for $\bar{\mathcal{H}}$ + $\bar{\mathcal{E}}$

i.e., <u>vector wave equations</u>

3.2 cont.

Source-free regions $\Rightarrow \bar{\mathcal{J}}_i = q_{ev} = 0$ & $\bar{\mathcal{M}}_i = q_{mv} = 0$ allow the equations to be simplified to

$$\boxed{\begin{aligned} \bar{\nabla}^2 \bar{\mathcal{E}} &= \mu\sigma \frac{\partial \bar{\mathcal{E}}}{\partial t} + \mu\epsilon \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2} \\ \bar{\nabla}^2 \bar{\mathcal{H}} &= \mu\sigma \frac{\partial \bar{\mathcal{H}}}{\partial t} + \mu\epsilon \frac{\partial^2 \bar{\mathcal{H}}}{\partial t^2} \end{aligned}}$$

If, in addition, we assume a lossless medium where $\sigma = 0$

$$\boxed{\begin{aligned} \nabla^2 \bar{\mathcal{E}} &= \mu\epsilon \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2} \\ \nabla^2 \bar{\mathcal{H}} &= \mu\epsilon \frac{\partial^2 \bar{\mathcal{H}}}{\partial t^2} \end{aligned}}$$

3.3 Time-Harmonic Electromagnetic Fields

⇒ Adapt time-domain wave eqns by substituting phasors, $\frac{\partial}{\partial t} \to j\omega$, and $\frac{\partial^2}{\partial t^2} \to (j\omega)^2 = -\omega^2$

Full wave equations

$$\boxed{\begin{aligned} \bar{\nabla}^2 \bar{E} &= \bar{\nabla} \times \bar{M}_i + j\omega\mu \bar{J}_i + \frac{1}{\epsilon} \bar{\nabla} q_{ev} + j\omega\mu\sigma \bar{E} - \omega^2 \mu\epsilon \bar{E} \\ \bar{\nabla}^2 \bar{H} &= -\bar{\nabla} \times \bar{J}_i + \sigma \bar{M}_i + j\omega\epsilon \bar{M}_i + \frac{1}{\mu} \bar{\nabla} q_{mv} + j\omega\mu\sigma \bar{H} - \omega^2 \mu\epsilon \bar{H} \end{aligned}}$$

Source-free wave equations

$$\boxed{\begin{aligned} \bar{\nabla}^2 \bar{E} &= j\omega\mu\sigma \bar{E} - \omega^2 \mu\epsilon \bar{E} = \gamma^2 \bar{E} \\ \bar{\nabla}^2 \bar{H} &= j\omega\mu\sigma \bar{H} - \omega^2 \mu\epsilon \bar{H} = \gamma^2 \bar{H} \end{aligned}}$$

Define $\gamma^2 = j\omega\mu\sigma - \omega^2\mu\epsilon$

propagation constant $\equiv \gamma = \alpha + j\beta = \sqrt{j\omega\mu\sigma - \omega^2\mu\epsilon}$ $(1/m)$

Attenuation constant $\equiv \alpha = Re(\gamma)$ (Np/m)

phase constant $\equiv \beta = Im(\gamma)$ $\left(\frac{rad}{m}\right)$

⇒ Note similarities to lossy transmission line equations!

3.3 cont.

For the lossless & source-free case:

$$\boxed{\begin{aligned} \bar{\nabla}^2 \bar{E} &= -\omega^2 \mu \epsilon \bar{E} = (\pm j\beta)^2 \bar{E} = -\beta^2 \bar{E} \\ \bar{\nabla}^2 \bar{H} &= -\omega^2 \mu \epsilon \bar{H} = (\pm j\beta)^2 \bar{H} = -\beta^2 \bar{H} \end{aligned}}$$

Here, $\beta^2 = \omega^2 \mu \epsilon \Rightarrow \beta = \omega\sqrt{\mu\epsilon}$

Note: Sometimes the variable k will be used for β (check context).

$\Rightarrow$ Note similarities to lossless transmission line equations!

3.4 Solution to the Wave Equation

⇒ Can often break the vector wave equations into multiple scalar Helmholtz/wave equations which can be solved and put back into the vector wave eqns.

⇒ Will consider time-harmonic equations

3.4.1 Rectangular Coordinate System

A. Source-Free and Lossless Media

⇒ $\bar{J}_i = \bar{M}_i = q_{ve} = q_{vm} = \sigma = 0$

⇒ Since the wave eqns for $\bar{E}$ & $\bar{H}$ are identical in form, we'll look at the sol'n to the electric field w/ the understanding that the solution for the magnetic field is identical w/ $\bar{H}$ swapped in for $\bar{E}$.

$$\bar{E} = \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z$$

← all can be functions of $x, y, \& z$

So

$$\bar{\nabla}^2 \bar{E} + \beta^2 \bar{E} = \bar{\nabla}^2 (\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) + \beta^2 (\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = 0$$

In rectangular coordinates, the <u>vector Laplacian</u> is

$$\bar{\nabla}^2 \bar{E} = \hat{a}_x \nabla^2 E_x + \hat{a}_y \nabla^2 E_y + \hat{a}_z \nabla^2 E_z$$

3.4.1 A. cont.

In turn, the scalar Laplacian is defined in rectangular coordinates as

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$$

Using this and noting that each vector component must also equal zero, we get

$$\nabla^2 E_x + \beta^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

$$\nabla^2 E_y + \beta^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + \beta^2 E_y = 0$$

$$\nabla^2 E_z + \beta^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \beta^2 E_z = 0$$

Next, apply separation-of-variables and assume

$$E_x = f(x)\, g(y)\, h(z) \leftarrow \text{single variable functions}$$

This makes

$$\nabla^2 E_x + \beta^2 E_x = gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} + \beta^2 fgh = 0$$

Next, we can replace the partial derivatives w/ ordinary derivatives and divide through by fgh

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} + \beta^2 = 0$$

or

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2$$

3.4.1 A. cont.

Since each term on the LHS is a function of an independent variable, this can only be true IFF each term is constant, i.e.,

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\beta_x^2 \Rightarrow \frac{d^2 f}{dx^2} = -\beta_x^2 f$$

$$\frac{1}{g}\frac{d^2 g}{dy^2} = -\beta_y^2 \Rightarrow \frac{d^2 g}{dy^2} = -\beta_y g$$

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\beta_z^2 \Rightarrow \frac{d^2 h}{dz^2} = -\beta_z h$$

2nd order ODEs!

$$\Downarrow$$

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \epsilon$$

Constraint/dispersion equation.

The constants β_x, β_y, + β_z are called the <u>wave constants/numbers</u> for their respective directions.

In EE (and many other areas), we have seen solutions to similar 2nd order ODEs many times. E.g., current + voltage along a lossless transmission line.

3.4.1 A. Cont.

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x} = C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)$$

$$g(y) = A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y} = C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)$$

$$h(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z} = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)$$

use for Traveling waves or Standing waves

See first two rows of Table 3-1.

TABLE 3-1 Wave functions, zeroes, and infinities of plane wave functions in rectangular coordinates

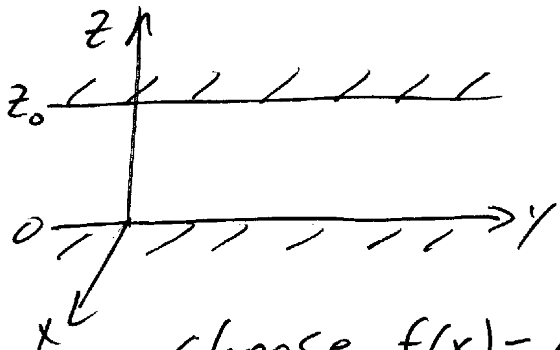
Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$e^{-j\beta x}$ for $+x$ travel $e^{+j\beta x}$ for $-x$ travel	$\beta x \to -j\infty$ $\beta x \to +j\infty$	$\beta x \to +j\infty$ $\beta x \to -j\infty$
Standing waves	$\cos(\beta x)$ for $\pm x$ $\sin(\beta x)$ for $\pm x$	$\beta x = \pm(n+\frac{1}{2})\pi$ $\beta x = \pm n\pi$ $n = 0, 1, 2, \ldots$	$\beta x \to \pm j\infty$ $\beta x \to \pm j\infty$
Evanescent waves	$e^{-\alpha x}$ for $+x$ $e^{+\alpha x}$ for $-x$ $\cosh(\alpha x)$ for $\pm x$ $\sinh(\alpha x)$ for $\pm x$	$\alpha x \to +\infty$ $\alpha x \to -\infty$ $\alpha x = \pm j(n+\frac{1}{2})\pi$ $\alpha x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\alpha x \to -\infty$ $\alpha x \to +\infty$ $\alpha x \to \pm\infty$ $\alpha x \to \pm\infty$
Attenuating traveling waves	$e^{-\gamma x} = e^{-\alpha x}e^{-j\beta x}$ for $+x$ travel $e^{+\gamma x} = e^{+\alpha x}e^{+j\beta x}$ for $-x$ travel	$\gamma x \to +\infty$ $\gamma x \to -\infty$	$\gamma x \to -\infty$ $\gamma x \to +\infty$
Attenuating standing waves	$\cos(\gamma x) = \cos(\alpha x)\cosh(\beta x) - j\sin(\alpha x)\sinh(\beta x)$ for $\pm x$ $\sin(\gamma x) = \sin(\alpha x)\cosh(\beta x) + j\cos(\alpha x)\sinh(\beta x)$ for $\pm x$	$\gamma x = \pm j(n+\frac{1}{2})\pi$ $\gamma x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\gamma x \to \pm j\infty$ $\gamma x \to \pm j\infty$

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

3.4.1 A. cont.

Which forms to put f, g, & h into depends on the physics of the physical problem.

For example, say we have parallel PEC planes at $z = 0$ and $z = z_0$ (parallel plate waveguide)



Here a wave could travel or propagate in the x & y directions but <u>not</u> the z.

choose
$$\left.\begin{aligned} f(x) &= A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x} \\ g(y) &= A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y} \end{aligned}\right\} \text{Traveling}$$

but
$$h(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z) \leftarrow \text{standing}$$

and
$$E_x = \left[A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}\right]\left[A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y}\right]$$
$$* \left[C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)\right]$$

For the traveling waves w/ $e^{j\omega t}$ time dependence, the

e^{-ju} terms $\rightarrow$ $+u$-direction propagating waves

&

e^{+ju} terms $\rightarrow$ $-u$-direction prop. waves

3.4.1 A. cont.

Example - let's examine E_x & $\mathcal{E}_x$ for my parallel-plate waveguide on some fixed line $y = y_0$ & $z = z_1$

Then $A_2 e^{-j\beta_y y_0} + B_2 e^{j\beta_y y_0} = K_2 \angle\theta_y$

$C_3 \cos(\beta_z z_1) + D_3 (\beta_z z_1) = K_3 \angle\theta_z$

Assume $\theta_y = \theta_z = 0$ for simplicity

and

$$\mathcal{E}_x = R_e\left[E_x e^{j\omega t}\right]$$

$$= R_e\left[\left(A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}\right) K_2 K_3 e^{j\omega t}\right]$$

$$= \underbrace{K_2 K_3 A_1}_{E\ \text{fwd}} \cos(\omega t - \beta_x x) + \underbrace{K_2 K_3 B_1}_{E\ \text{bwd}} \cos(\omega t + \beta_x x)$$

positive/forward or +x-dir wave

neg./backward or −x-dir wave

As will be demonstrated, the first term represents a positive (+x-dir) traveling wave and the second term represents a negative (−x-dir) traveling wave

MathCAD example

Illustrate forward and backward wave propagation of $\mathcal{E}_x$

Define some constants

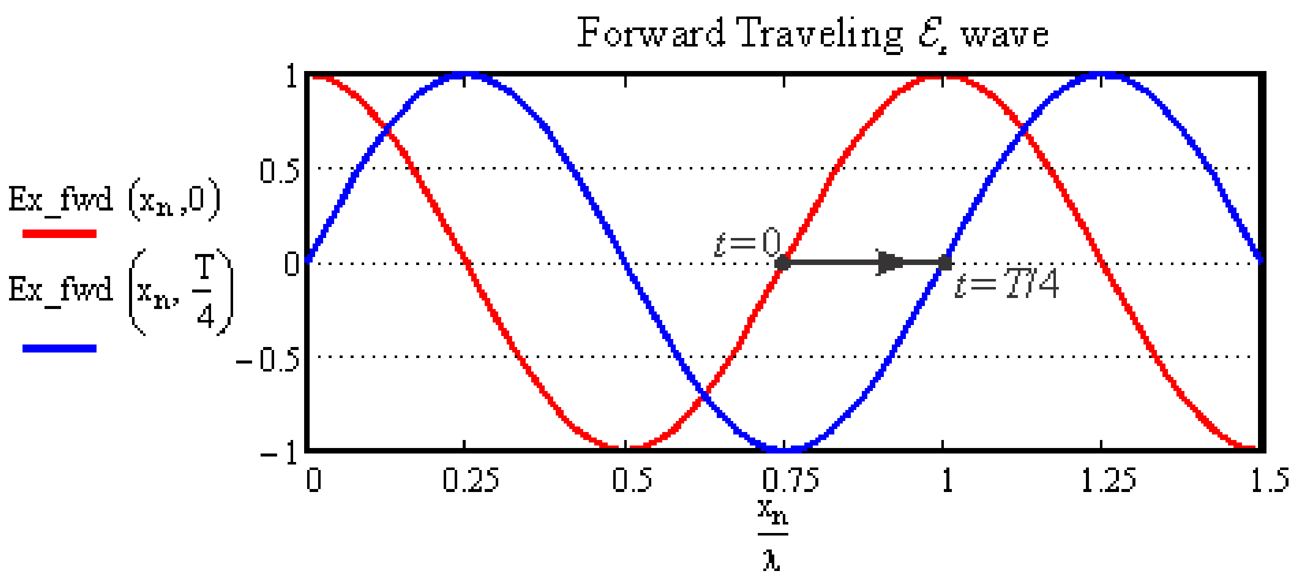
$u := 3 \cdot 10^8$ (m/s) Efwd := 1 (V/m) Ebwd := 1 (V/m)

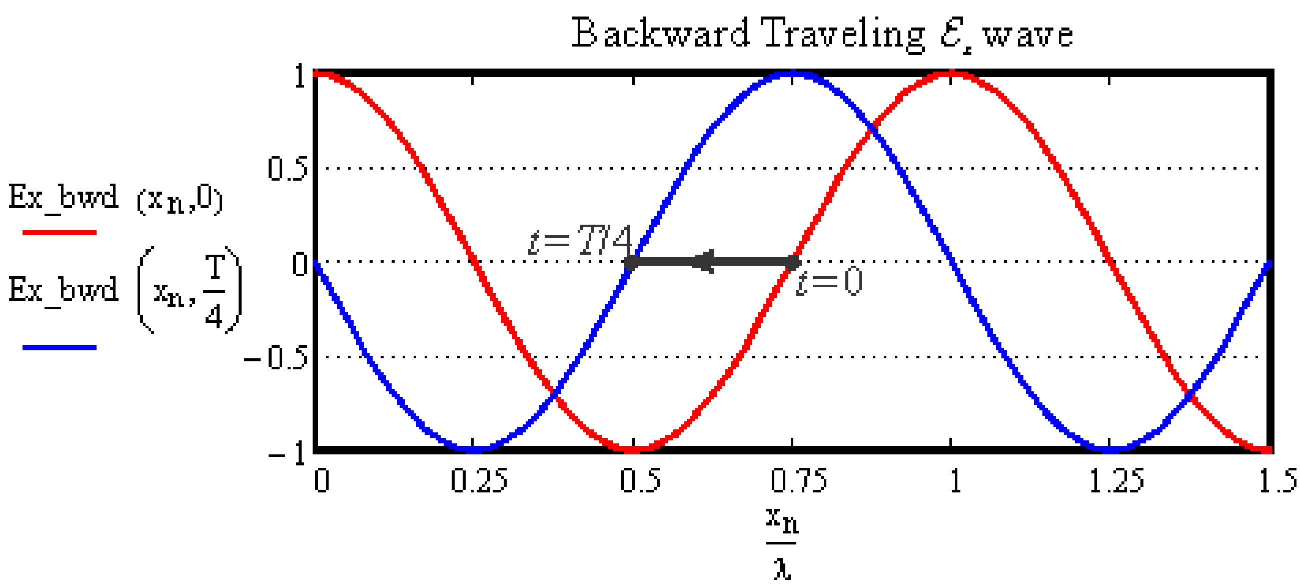
$\omega := 2 \cdot \pi \cdot 60$ (rad/s) $\lambda := 2 \cdot \pi \cdot \frac{u}{\omega}$ (m) $\beta x := \frac{\omega}{u}$ (rad/m) $T := \frac{2\pi}{\omega}$ (s)

Define functions for forward and backward components of the $\mathcal{E}_x$ wave

$Ex\_fwd(x,t) := Efwd \cdot \cos(\omega \cdot t - \beta x \cdot x)$ $Ex\_bwd(x,t) := Ebwd \cdot \cos(\omega \cdot t + \beta x \cdot x)$

$n := 0..60$ $x_n := \frac{1.5 \cdot \lambda \cdot n}{60}$ Show 1.5 wavelengths





3.4.1 A. cont.

As seen in the example, as we let time t advance, like points on the waves forward & backward components 'travel'. These points are called <u>equiphase points</u> and the velocity at which they move is called the <u>phase velocity</u> v_p.

equiphase points $\omega t - \beta_x x = \text{constant}$ forward traveling wave

$\frac{d}{dt}$

$$\omega(1) - \beta_x \frac{dx}{dt} = 0$$

$$v_{px} = \frac{dx}{dt} = \frac{\omega}{\beta_x} \quad \text{or} \quad \beta_x = \frac{\omega}{v_{px}}$$

3.4.1 B. Source-Free and Lossy Media

Here, $\bar{J}_i = \bar{M}_i = q_{ve} = q_{mv} = 0$ but $\sigma \neq 0$ which means we start w/

$$\bar{\nabla}^2 \bar{E} = j\omega\mu\sigma \bar{E} - \omega^2 \mu\epsilon \bar{E} = \gamma^2 \bar{E}$$

$$\bar{\nabla}^2 \bar{H} = j\omega\mu\sigma \bar{H} - \omega^2 \mu\epsilon \bar{H} = \gamma^2 \bar{H}$$

where $\gamma^2 = j\omega\mu\sigma - \omega^2\mu\epsilon = j\omega\mu(\sigma + j\omega\epsilon) = (\pm\gamma)^2$

$$\gamma = \pm\sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \begin{cases} \pm(\alpha + j\beta) & \sigma > 0 \\ \pm(\alpha - j\beta) & \sigma < 0 \text{ active media} \end{cases}$$

$\equiv$ prop. constant $(\frac{1}{m})$

where $\alpha \equiv$ atten. constant (Np/m) + $\beta \equiv$ phase constant $(\frac{\text{rad}}{\text{m}})$

Looking at the electric field vector wave eq'n, we see it reduces to 3 scalar wave eq'ns:

$$\nabla^2 E_x - \gamma^2 E_x = 0 \quad \rightarrow \text{assume } E_x = f(x)\, g(y)\, h(z)$$

$$\nabla^2 E_y - \gamma^2 E_y = 0$$

$$\nabla^2 E_z - \gamma^2 E_z = 0$$

which have sol'n forms (shown for E_x) of

$$f(x) = A_1 e^{-\gamma_x x} + B_1 e^{\gamma_x x} = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x)$$

$$g(y) = A_2 e^{-\gamma_y y} + B_2 e^{\gamma_y y} = C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y)$$

$$h(z) = A_3 e^{-\gamma_z z} + B_3 e^{\gamma_z z} = C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z)$$

↑ attenuating traveling waves   ↑ attenuating standing waves

3.4.1 B. cont.

Also, now

$$\nabla^2 E_x - \gamma^2 E_x = gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} - \gamma^2 fgh = 0$$

$$\div fgh \Rightarrow \quad \frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2} + \frac{1}{h}\frac{\partial^2 h}{\partial z^2} = \gamma^2$$

which is true IFF each term on LHS is a constant

$$\underline{\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2} \quad \text{constraint/dispersion equation}$$

Now, we'll revisit our possible form for γ as well as γ_z (similar possibilities for γ_x + γ_y)

$$\gamma = \begin{cases} +(\alpha + j\beta) \\ -(\alpha + j\beta) \end{cases} \sigma > 0 \qquad \alpha + \beta > 0$$
$$\qquad \begin{cases} +(\alpha - j\beta) \\ -(\alpha - j\beta) \end{cases} \sigma < 0$$

$$\gamma_z = \begin{cases} +(\alpha_z + j\beta_z) \\ -(\alpha_z + j\beta_z) \\ +(\alpha_z - j\beta_z) \\ -(\alpha_z - j\beta_z) \end{cases} \qquad \alpha + \beta > 0$$

Looking at $h(z) = A_3 e^{-\gamma_z z} + B_3 e^{\gamma_z z}$ and if we wish the $A_3 e^{-\gamma_z z}$ term to represent a <u>forward</u> (+z) traveling (w/ attenuation) wave, let's see how our various choices of γ_z work out w/ $e^{j\omega t}$ time variation.

3.4.1 B. cont

Calling this term $h_1^+(z)$ and substituting in our γ_z choices -

$$h_1^+(z) = A_3 e^{-\gamma_z z} = \begin{cases} A_3 e^{-\alpha_z z} e^{-j\beta_z z} & \leftarrow \text{decays + travels in } +z\text{-direction} \\ A_3 e^{+\alpha_z z} e^{+j\beta_z z} & \leftarrow \text{decays + travels in } -z\text{-direction} \\ A_3 e^{-\alpha_z z} e^{+j\beta_z z} & \leftarrow \text{grows + travels in } -z\text{-direction} \\ A_3 e^{+\alpha_z z} e^{-j\beta_z z} & \leftarrow \text{grows + travels in } +z\text{-direction} \end{cases}$$

$\Rightarrow$ Choose $\gamma_z = +(\alpha_z + j\beta_z)$ and first sol'n!

Don't want wrong direction or growing waves (violates conservation of energy for passive media)

$\Rightarrow$ In general, $\gamma_i = +(\alpha_i + j\beta_i)$

Then, $h_1^-(z) = B_3 e^{+\gamma_z z} = B_3 e^{+\alpha_z z} e^{+j\beta_z z}$

will be a wave traveling in the $-z$-direction and attenuating.