2.9 AC Variations in Materials

→ ϵ, μ, σ all vary w/ frequency.

2.9.1 Complex Permittivity

→ Make analogy between polarizing an atom w/ a mechanical model of a mass, attached to a spring, being moved back-n-forth

$\bar{E}_a = 0$

→ X
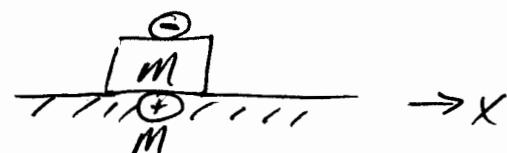
$\bar{E}_a > 0$
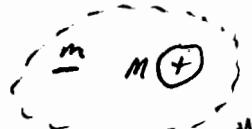
$\leftarrow \ell \rightarrow$ Nucleus stays put $\leftarrow \ell \rightarrow$
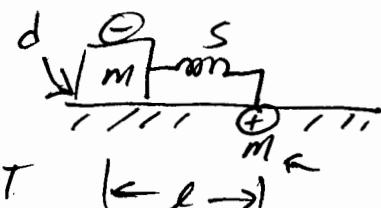
$\bar{E}_a < 0$

$\leftarrow \ell \rightarrow$ $\leftarrow \ell \rightarrow$
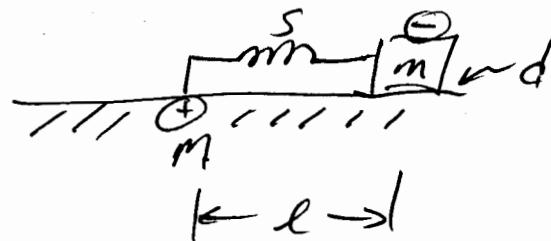
$M \equiv$ mass of nucleus $+Q$

$m \equiv$ " " electrons $-Q$

$d \equiv$ damping/friction coefficient ↔ takes power/energy to move charges

$S \equiv$ spring/tension coefficient ↔ stored power/energy

$\ell \equiv$ displacement distance for centroids of nucleus & electrons

2.9.1 cont.

For a time-harmonic applied electric field, leads to a 2nd order ODE

$$M \frac{d^2 \ell}{dt^2} + d \frac{d\ell}{dt} + s\ell = Q\mathcal{E}(t) = Q E_0 e^{j\omega t}$$

$\underset{\text{"ma"}}{} \quad \underset{\text{"dv"}}{} \quad \underset{\text{"s}\ell\text{"}}{} \qquad \underset{\text{"driving force"}}{}$

$$\frac{d^2 \ell}{dt^2} + 2\alpha \frac{d\ell}{dt} + \omega_0^2 \ell = \frac{Q}{m} E_0 e^{j\omega t}$$

where $\alpha = \frac{d}{2m}$ & $\omega_0 = \sqrt{\frac{s}{m}}$ ↖ very similar to RLC circuits

If $\alpha > \omega_0 \rightarrow$ overdamped

$\alpha = \omega_0 \rightarrow$ critically damped

$\alpha < \omega_0 \rightarrow$ underdamped

complementary (Source-free) Transient ↙   Two solns   ↓ particular soln

$$\ell_c(t) = \begin{cases} \text{underdamped} \\ \text{critically damped} \\ \text{overdamped} \end{cases} \quad + \quad \ell_p(t) = \ell_0 e^{j\omega t}$$

where $\ell_0 = \dfrac{\frac{Q}{m} E_0}{(\omega_0^2 - \omega^2) + j\omega(\frac{d}{m})}$

$$\ell(t) = \ell_c(t) + \ell_p(t)$$

↑

goes to zero as $t \rightarrow \infty$ as long as $d \neq 0$

Assuming an underdamped system ($\alpha < \omega_0$), we can determine the damped resonant frequency (natural frequency) to be

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{\frac{s}{m} - \left(\frac{d}{2m}\right)^2}$$

2.9.1 cont.

If $d=0$ (no damping), $\omega_d = \omega_0 = \sqrt{\frac{s}{m}}$.

At steady-state (i.e., $\ell_c(t) \to 0$), the time-varying polarization is

$$p = p(t) = N_e Q \ell(t) = \frac{N_e \left(\frac{Q^2}{m}\right) \overbrace{E_0 e^{j\omega t}}^{\mathcal{E}(t)}}{(\omega_0^2 - \omega^2) + j\omega(d/m)}$$

$\uparrow$
dipoles
per unit
volume

Back in section 2.2, $\bar{D} = \epsilon_0 \bar{E}_a + \bar{P} = \epsilon_0 \bar{E}_a + \epsilon_0 \chi_e \bar{E}_a$

$^d \epsilon = \epsilon_0 + \frac{|\bar{P}|}{|\bar{E}_a|}$

Now $\frac{p}{\mathcal{E}} = \frac{N_e \left(\frac{Q^2}{m}\right)}{(\omega_0^2 - \omega^2) + j\omega(d/m)} \quad \curvearrowleft$ complex #!

and

complex $\equiv \dot{\epsilon} = \epsilon_0 + \frac{p}{\mathcal{E}} = \epsilon_0 + \frac{N_e \left(\frac{Q^2}{m}\right)}{(\omega_0^2 - \omega^2) + j\omega(d/m)}$
permittivity

$$= \epsilon' - j\epsilon''$$

relative
complex $\equiv \dot{\epsilon}_r = \frac{\dot{\epsilon}}{\epsilon_0} = \epsilon_r' - j\epsilon_r'' = 1 + \frac{\frac{N_e Q^2}{\epsilon_0 m}}{(\omega_0^2 - \omega^2) + j\omega(d/m)}$
permittivity

$\uparrow$
multiply top
& bottom by
complex conj.
of denom.

w/ $\epsilon_r' = 1 + \frac{\frac{N_e Q^2}{\epsilon_0 m}(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\omega d/m)^2}$

$\epsilon_r'' = \frac{N_e Q^2}{\epsilon_0 m}\left[\frac{\omega \, d/m}{(\omega_0^2 - \omega^2)^2 + (\omega d/m)^2}\right]$

2.9.1 cont.

define complex index of refraction $\equiv \dot{n} = \sqrt{\dot{\epsilon_r}}$

Note: If $d = 0$, $\epsilon_r' = 1 + \dfrac{\dfrac{N_e Q^2}{\epsilon_o m}}{w_o^2 - w^2}$ & $\epsilon_r'' = 0$

How does having $\dot{\epsilon} = \epsilon' - j\epsilon''$ impact Maxwell's Equations? Ampere's Law now becomes (phasor form)

$$\overline{\nabla} \times \overline{H} = \overline{J_i} + \overline{J_c} + jw\dot{\epsilon}\overline{E} = \overline{J_t}$$

$$= \overline{J_i} + \sigma_s \overline{E} + jw(\epsilon' - j\epsilon'')\overline{E}$$

$$= \overline{J_i} + (\sigma_s + w\epsilon'')\overline{E} + jw\epsilon'\overline{E}$$

$$= \overline{J_i} + \underbrace{\sigma_e \overline{E}}_{\text{eff. cond. current dens.}} + jw\epsilon'\overline{E} \leftarrow \begin{array}{l}\text{effective} \\ \text{displacement} \\ \text{current density}\end{array}$$

Equivalent conductivity $\equiv \sigma_e = \sigma_s + w\epsilon'' = \sigma_s + \sigma_a$

alternating field conductivity $\equiv \sigma_a = w\epsilon''$

static conductivity $\equiv \sigma_s$

Some materials have modest σ_s and larger σ_a (think microwave oven)

2.9.1 cont.

We can express the total current density

$$\bar{J}_t = \bar{J}_i + \sigma_e \bar{E} + j\omega\epsilon'\bar{E} = \bar{J}_i + j\omega\epsilon'\left(1 - j\frac{\sigma_e}{\omega\epsilon'}\right)\bar{E}$$

$$= \bar{J}_i + j\omega\epsilon'(1 - j\tan\delta_e)\bar{E}$$

Define

effective
loss tangent $\equiv \tan\delta_e = \dfrac{\sigma_e}{\omega\epsilon'} = \dfrac{\sigma_s}{\omega\epsilon'} + \overset{\omega\epsilon''}{\overbrace{\dfrac{\sigma_a}{\omega\epsilon'}}}$

$$= \frac{\sigma_s}{\omega\epsilon'} + \frac{\epsilon''}{\epsilon'} = \tan\delta_s + \tan\delta_a$$

static
loss tangent $\equiv \tan\delta_s = \dfrac{\sigma_s}{\omega\epsilon'}$

alternating
loss tangent $\equiv \tan\delta_a = \dfrac{\epsilon''}{\epsilon'}$

$\Rightarrow$ Manufacturers of PCBs and TLs when
   giving conductivity σ, loss tangent $\tan\delta$,
   and relative permittivity ϵ_r at selected
   frequencies typically mean σ_e, $\tan\delta_e$,
   and ϵ_r'. See Table 2-5.

$\Rightarrow$ Look at the effective conduction + displacement
   current densities on complex plane

$$\bar{J}_{cd} = \bar{J}_{cc} + \bar{J}_{de}$$

$$= j\omega\epsilon'\left(1 - j\frac{\sigma_e}{\omega\epsilon'}\right)\bar{E}$$

$\bar{J}_{cd} = j\omega\epsilon'$

$\bar{J}_{de} = j\omega\epsilon'\bar{E}$

$\bar{J}_{cc} = \sigma_e\bar{E}$

$\Rightarrow \tan\delta_e = \dfrac{\sigma_e}{\omega\epsilon'}$

TABLE 2-5   Dielectric constants and loss tangents of typical dielectric materials

Material	ε'_r	$\tan \delta$
Air	1.0006	
Alcohol (ethyl)	25	0.1
Aluminum oxide	8.8	6×10^{-4}
Bakelite	4.74	22×10^{-3}
Carbon dioxide	1.001	
Germanium	16	
Glass	4–7	1×10^{-3}
Ice	4.2	0.1
Mica	5.4	6×10^{-4}
Nylon	3.5	2×10^{-2}
Paper	3	8×10^{-3}
Plexiglas	3.45	4×10^{-2}
Polystyrene	2.56	5×10^{-5}
Porcelain	6	14×10^{-3}
Pyrex glass	4	6×10^{-4}
Quartz (fused)	3.8	7.5×10^{-4}
Rubber	2.5–3	2×10^{-3}
Silica (fused)	3.8	7.5×10^{-4}
Silicon	11.8	
Snow	3.3	0.5
Sodium chloride	5.9	1×10^{-4}
Soil (dry)	2.8	7×10^{-2}
Styrofoam	1.03	1×10^{-4}
Teflon	2.1	3×10^{-4}
Titanium dioxide	100	15×10^{-4}
Water (distilled)	80	4×10^{-2}
Water (sea)	81	4.64
Wood (dry)	1.5–4	1×10^{-2}

2.9.1 cont.

Ex. Rogers Corporation RO4003C hydrocarbon/ ceramic laminate substrate. From the data sheet

$$\rho_s = 1.7 \times 10^{10} \qquad \Rightarrow \underline{\sigma_s = 5.88 \times 10^{-15} \; S/m}$$

$$\tan \delta = \tan \delta_e = 0.0021 \; @ \; 2.5 \, GHz$$
$$= 0.0027 \; @ \; 10 \, GHz$$

$$\epsilon_{re} = \epsilon_r' = 3.67 \; @ \; 2.5 \, GHz$$
$$= 3.651 \; @ \; 10 \, GHz$$

Find the effective conductivity, imaginary part of $\dot{\epsilon}$, and $\dot{\epsilon}_r$, the complex permittivity and relative permittivity.

@ 2.5 GHz $\qquad \sigma_e = \omega \epsilon' \tan \delta_e = 2\pi(2.5 \times 10^9)3.67 \epsilon_0 (0.0021)$

$$\underline{\sigma_e = 0.001072 \; S/m = 1.072 \; mS/m}$$

$$\epsilon'' = \frac{\sigma_e - \sigma_s}{\omega} = \frac{0.001072 - 5.88 \times 10^{-15}}{2\pi(2.5 \times 10^9)} = \underline{6.824 \times 10^{-14} \; F/m}$$

$$\underline{\epsilon_r'' = \frac{\epsilon''}{\epsilon_0} = 0.007707} \quad \text{(pretty small compared to } \epsilon_r' = 3.67\text{)}$$

@ 10 GHz $\qquad \sigma_e = 2\pi(10 \times 10^9)3.651(8.8542 \times 10^{-12})0.0027$

$$\underline{\sigma_e = 0.005484 \; S/m = 5.484 \; mS/m}$$

$$\epsilon'' = \frac{\sigma_e - \sigma_s}{\omega} = \frac{0.005484 - 5.88 \times 10^{-15}}{2\pi(10 \times 10^9)} = \underline{8.7282 \times 10^{-14} \; F/m}$$

$$\underline{\epsilon_r'' = \frac{\epsilon''}{\epsilon_0} = 0.009858} \quad \text{(small compared to } \epsilon_r' = 3.651\text{)}$$

Consider the extremes

1. <u>Good dielectrics</u> $\left(\tan\delta_e = \frac{\sigma_e}{\omega\varepsilon'} \ll 1\right)$

$$\overline{J}_{cd} \simeq j\omega\varepsilon'\overline{E} \quad \text{(nearly all displacement current density)}$$

2. <u>Good conductors</u> $\left(\tan\delta_e = \frac{\sigma_e}{\omega\varepsilon'} \gg 1\right)$

$$\overline{J}_{cd} \simeq \sigma_e\overline{E} \quad \text{(nearly all conduction current density)}$$

Going back to section 2.2, it was mentioned that electric polarization could occur due to different mechanisms: 1) dipole/orientational, 2) ionic/molecular, & 3) electronic. Depending on the material, 1, 2, or 3 of these mechanisms may apply and typically would occur @ different frequencies/frequency ranges.

Writing the phasor polarization vector as a function of frequency

$$\overline{P}(\omega) = \varepsilon_o \chi_e(\omega)\overline{E}_a(\omega)$$

we'll note that the electric susceptibility is a function of frequency. Further, $\chi_e(\omega)$ must be complex since $\dot{\varepsilon}$ is complex.

$$X_e(\omega) = X_e'(\omega) - j\,X_e''(\omega)$$

$$= \left[X_{ed}'(\omega) + X_{ei}'(\omega) + X_{ee}'(\omega)\right] - j\left[X_{ed}''(\omega) + X_{ei}''(\omega) + X_{ee}''(\omega)\right]$$

dipole   ionic   electronic   dipole   ionic   electronic

In general, $X_e'(\omega) = X_e'(-\omega)$ real part is even

$$X_e''(-\omega) = -X_e''(\omega) \quad \text{imag. part is odd}$$

Figures 2-18 & 2-19 show general samples as to how $\dot{X}_e$ & $\dot{\epsilon}_r$ can vary with frequency

→ Note, both X_e'' & ϵ_r'' peak @/near resonant frequencies while X_e' & ϵ_r' dip (negative slope), called anomalous dispersion.

→ Other than near resonant frequencies $|\dot{\epsilon}_r|$ & ϵ_r' exhibit positive slopes, i.e., normal dispersion.
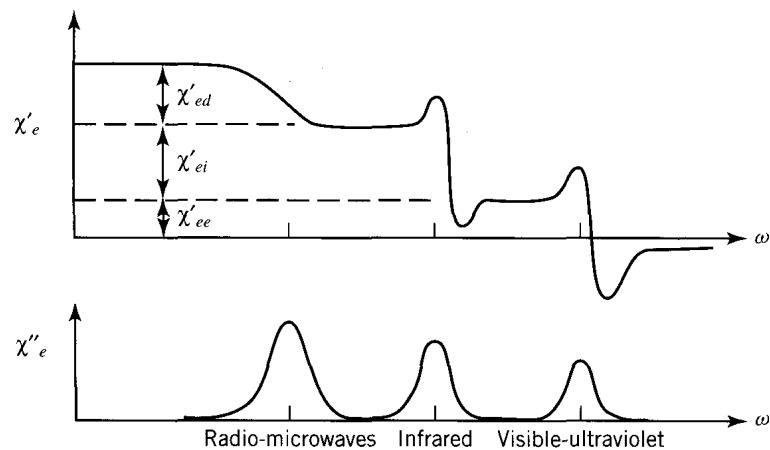
Figure 2-18 Electric susceptibility (real and imaginary) variations as a function of frequency for a typical dielectric.
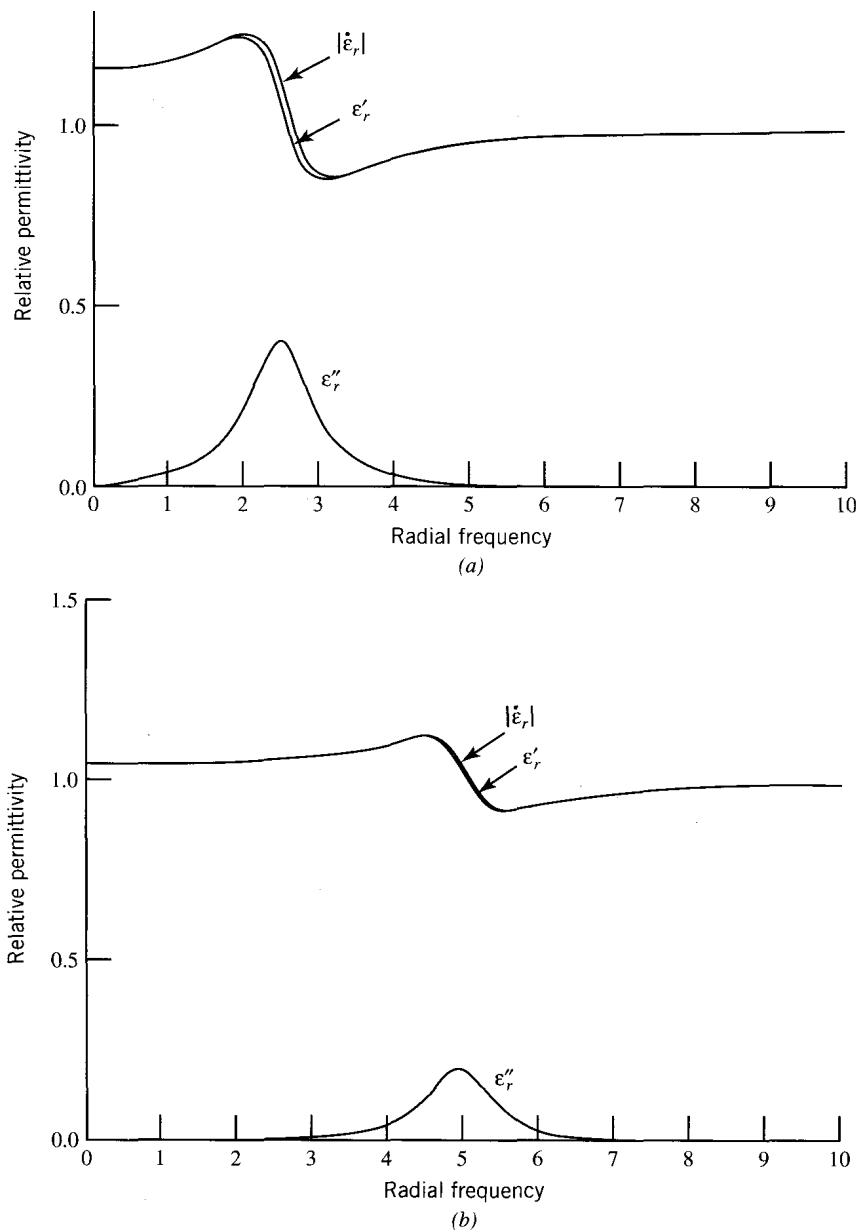

Figure 2-19 Typical frequency variations of real and imaginary parts of relative permittivity of dielectrics. $(a)\, N_e Q^2/\varepsilon_0 m = 1, d/m = 1, \alpha/\omega_0 = 1/5, \omega_0 = 2.5.\,(b)\, N_e Q^2/\varepsilon_0 m = 1, d/m = 1, \alpha/\omega_0 = 1/10, \omega_0 = 5.$

For our mechanical analogy, we assumed a single resonant frequency. However, actual material can have multiple resonances due to different mechanisms. To allow for this possibility, or P resonances, let—

$$\frac{P}{\mathcal{E}} = \sum_{s=1}^{P} \frac{Ne\frac{Q^2}{m}}{(\omega_s^2 - \omega^2) + j\frac{\omega d}{m}}$$

$$\dot{\mathcal{E}} = \epsilon' - j\epsilon'' = \epsilon_0 + \sum_{s=1}^{P} \frac{Ne\frac{Q^2}{m}}{(\omega_s^2 - \omega^2) + j\frac{\omega d}{m}}$$

$$\dot{\epsilon_r} = \epsilon_r' - j\epsilon_r'' = 1 + \sum_{s=1}^{P} \frac{Ne\frac{Q^2}{\epsilon_0 m}}{(\omega_s^2 - \omega^2) + j\frac{\omega d}{m}}$$

$$\epsilon_r' = 1 + \sum_{s=1}^{P} \frac{\frac{Ne Q^2}{\epsilon_0 m}(\omega_s^2 - \omega^2)}{(\omega_s^2 - \omega^2)^2 + \left(\frac{\omega d}{m}\right)^2}$$

$$\epsilon_r'' = \sum_{s=1}^{P} \frac{Ne Q^2}{\epsilon_0 m} \frac{\omega\frac{d}{m}}{(\omega_s^2 - \omega^2)^2 + \left(\frac{\omega d}{m}\right)^2}$$

$\Rightarrow \epsilon_r'$ & ϵ_r'' are related by the Kramers-Kronig equations

$$\epsilon_r'(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{\omega'\epsilon_r''(\omega')}{\omega'^2 - \omega^2}\, d\omega'$$

$$\epsilon_r''(\omega) = \frac{2\omega}{\pi}\int_0^\infty \frac{1 - \epsilon_r'(\omega')}{\omega'^2 - \omega^2}\, d\omega'$$

2.9.1 cont.

For many materials, the Debye equation can be used to find the complex relative permittivity (as well as $\sigma_e(\omega)$)

$$\dot{\varepsilon}_r(\omega) = \varepsilon_{r\infty}' + \frac{\varepsilon_{rs}' - \varepsilon_{r\infty}'}{1 + j\omega\tau_e}$$

where

$$\text{new relaxation time constant} \equiv \tau_e = \tau\,\frac{\varepsilon_{rs}' + 2}{\varepsilon_{r\infty}' + 2}$$

$$\varepsilon_r'(\omega) = \varepsilon_{r\infty}' + \frac{\varepsilon_{rs}' - \varepsilon_{r\infty}'}{1 + (\omega\tau_e)^2}$$

$$\varepsilon_r''(\omega) = \frac{(\varepsilon_{rs}' - \varepsilon_{r\infty}')\,\omega\tau_e}{1 + (\omega\tau_e)^2}$$

$$\Rightarrow \omega_e = \frac{1}{\tau_e}$$

$$f_e = \frac{1}{2\pi\tau_e}$$

$\underbrace{\phantom{f_e = \frac{1}{2\pi\tau_e}}}$
relaxation frequencies

provided that the following are known

$\varepsilon_{rs}' \equiv$ static relative permittivity $(\omega=0)$

$\varepsilon_{r\infty}' \equiv$ relative permittivity as $\omega \to \infty$

τ_e or τ

$\sigma_o \equiv$ static conductivity (S/m)

$\Rightarrow$ This is good for a single resonance. More terms can be added to account for multiple resonances.