Chapter 1 Time-Varying and Time-Harmonic Electromagnetic Fields

1.2 Maxwell's Equations

→ Actually, the work of many scientists – Faraday, Ampere, Gauss, Volta, Lenz, Coulomb, ... – that were combined into a set of equations by James Clerk Maxwell (Scottish physicist + mathematician)

1.2.1 Differential Form of Maxwell's Equations

→ describes relationships between fields and sources at any point in space at any time

→ Except where there are discontinuities in charge or current distributions, the fields are 'well behaved' – single-valued, bounded, + continuous functions of position + time w/ continuous derivatives.

Faraday's Law $\bar{\nabla} \times \bar{\mathcal{E}} = -\bar{\mathcal{M}}_i - \dfrac{\partial \bar{\mathcal{B}}}{\partial t} = -\bar{\mathcal{M}}_i - \bar{\mathcal{M}}_d = -\bar{\mathcal{M}}_t$

Ampere's Law $\bar{\nabla} \times \bar{\mathcal{H}} = \bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \dfrac{\partial \bar{\mathcal{D}}}{\partial t} = \bar{\mathcal{J}}_{ic} + \dfrac{\partial \bar{\mathcal{D}}}{\partial t} = \bar{\mathcal{J}}_{ic} + \bar{\mathcal{J}}_d = \bar{\mathcal{J}}_t$

Gauss' Law $\bar{\nabla} \cdot \bar{\mathcal{D}} = q_{ev}$

$\bar{\nabla} \cdot \bar{\mathcal{B}} = q_{mv}$

1.2.1 cont.

$\bar{\mathcal{E}} \equiv$ electric field intensity vector (V/m)

$\bar{\mathcal{H}} \equiv$ magnetic field intensity vector (A/m)

$\bar{\mathcal{D}} \equiv$ electric flux density vector (C/m^2)

$\bar{\mathcal{B}} \equiv$ magnetic flux density vector (Wb/m^2)

$\bar{\mathcal{J}}_i \equiv$ impressed/source electric current density vector $\left(\frac{A}{m^2}\right)$

$\bar{\mathcal{J}}_c \equiv$ conduction electric current density vector (A/m^2)

$\bar{\mathcal{J}}_d \equiv$ displacement electric current density vector (A/m^2)
(Maxwell's key contribution)

$= \frac{\partial \bar{\mathcal{D}}}{\partial t}$ $\Rightarrow$ In dielectrics, it partly is the motion of bound charges, i.e., true current.
$\Rightarrow$ Early writings speak of 'ether' or 'aether' conducting or allowing movement of bound charges

$\bar{\mathcal{M}}_i \equiv$ impressed/source magnetic current density vector $\left(\frac{V}{m^2}\right)$
$\rightarrow$ new addition (compared to undergraduate courses) part of 'generalized' current to balance Maxwell's Eqns
$\rightarrow$ Not real, but can represent physical problems

$\bar{\mathcal{M}}_d \equiv$ displacement magnetic current density vector (V/m^2)

$= \frac{\partial \bar{\mathcal{B}}}{\partial t}$ $\Rightarrow$ named to balance the $\bar{\mathcal{J}}_d$ term

$q_{ev} \equiv$ electric charge density (C/m^3)

$q_{mv} \equiv$ magnetic charge density (Wb/m^3) $\leftarrow$ Another New addition

1.2.1 cont.

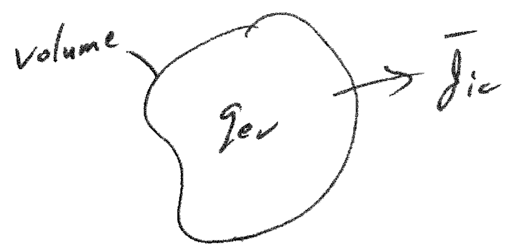
We can combine the impressed & conduction electric current densities into a single term, i.e.,

$$\bar{J}_{ic} = \bar{J}_i + \bar{J}_c \quad (A/m^2)$$

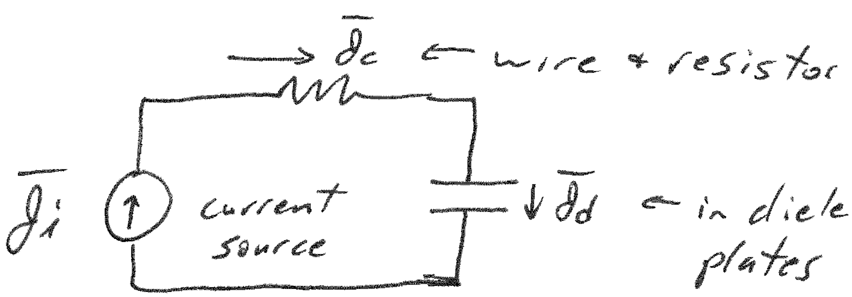
and, in turn, this can be related to the electric charge density by the <u>continuity equation</u>

$$\bar{\nabla} \cdot \bar{J}_{ic} = -\frac{\partial q_{ev}}{\partial t}$$

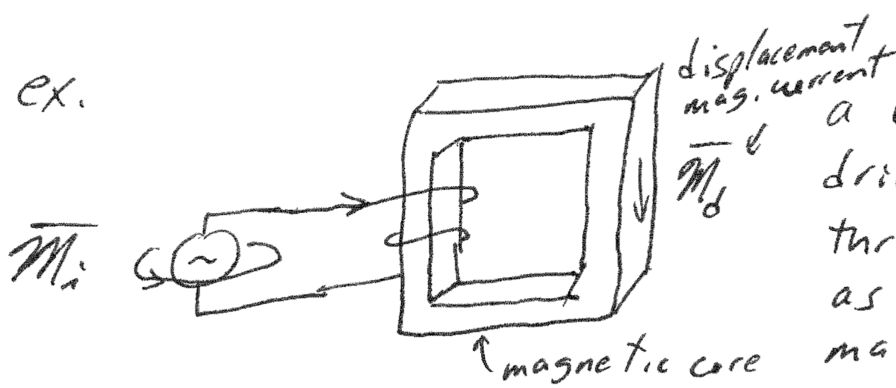
→ relates the net flow of current out of a small volume to the decrease in electric charge w/in the volume.



ex.



ex.



a voltage source driving current through coil <u>acts</u> as an <u>impressed</u> mag. current source

1.2.2 Integral Form of Maxwell's Eqns

→ describes inter-relationships between fields and sources over extended region of space

→ typically can only be analytically solved for highly symmetric problems

→ fields + derivatives of fields do not need continuous distributions

→ Use Stokes's Theorem $\oint_C \bar{A}\cdot d\bar{\ell} = \iint_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s}$ (RHR)

and

Divergence Theorem $\oiint_S \bar{A}\cdot d\bar{s} = \iiint_V \bar{\nabla}\cdot\bar{A}\, dv$

to integrate differential forms of Maxwell's Equations to get

Faraday's Law $\displaystyle \oint_C \bar{\mathcal{E}}\cdot d\bar{\ell} = -\iint_S \bar{\mathcal{M}}_i\cdot d\bar{s} - \frac{\partial}{\partial t}\iint_S \bar{\mathcal{B}}\cdot d\bar{s}$

⇒ w/out $\bar{\mathcal{M}}_i$, this equation tells us the open-circuit emf is related to time rate of decrease of mag. flux

Ampere's Law

$$\oint_C \bar{\mathcal{H}}\cdot d\bar{\ell} = \iint_S \bar{\mathcal{J}}_{ic}\cdot d\bar{s} + \frac{\partial}{\partial t}\iint \bar{\mathcal{D}}\cdot d\bar{s} = \iint_S \bar{\mathcal{J}}_{ic}\cdot d\bar{s} + \iint_S \bar{\mathcal{J}}_d\cdot d\bar{s}$$

⇒ Tells us the line integral of $\bar{\mathcal{H}}$ is equal to current enclosed by closed path C.

1.2.2 cont.

Gauss' Law $\oiint_S \bar{D} \cdot d\bar{s} = \iiint_V q_{ev}\, dv = \mathcal{Q}_e$ ← script Q

⇒ Total electric flux through a closed surface is equal to total charge enclosed.

'Magnetic' Gauss' Law

$$\oiint_S \bar{B} \cdot d\bar{s} = \iiint_V q_{mv}\, dv = \mathcal{Q}_m$$

⇒ Remember no magnetic charges have been found (yet), we are using fictions mag. charges to represent some problems

Continuity Equation

$$\oiint_S \bar{J}_{ic} \cdot d\bar{s} = -\frac{\partial}{\partial t} \iiint_V q_{ev}\, dv = -\frac{\partial \mathcal{Q}_e}{\partial t}$$

1.3 Constitutive Parameters and Relations

⇒ Help us to describe how EM fields interact w/ matter/charged particles

⇒ More extensive discussion/coverage in Chap 2.

In the time-domain

(1) $\bar{\mathcal{D}} = \hat{\varepsilon} * \bar{\mathcal{E}}$ $\quad * \equiv$ convolution

$\hat{\varepsilon} \equiv$ time-varying electric permittivity

⇒ If $\hat{\varepsilon}$ is a constant, ε, then, $\bar{\mathcal{D}} = \varepsilon\bar{\mathcal{E}}$

⇒ Free space $\hat{\varepsilon} = \varepsilon_0 = 8.8541878 \times 10^{-12}$ F/m

(NOT $\frac{10^{-9}}{36\pi}$ F/m)

In the time-domain

(2) $\bar{\mathcal{B}} = \hat{\mu} * \bar{\mathcal{H}}$ where $\hat{\mu} \equiv$ time-varying magnetic permeability

⇒ If $\hat{\mu}$ is a constant, μ, then, $\bar{\mathcal{B}} = \mu\bar{\mathcal{H}}$

⇒ Free space $\hat{\mu} = \mu_0 = 4\pi \times 10^{-7}$ H/m

In the time-domain

(3) $\bar{\mathcal{J}}_c = \hat{\sigma} * \bar{\mathcal{E}}$ where $\hat{\sigma} \equiv$ time-varying elec. conductivity

⇒ If $\hat{\sigma} = \sigma$ (constant) $\bar{\mathcal{J}}_c = \sigma\bar{\mathcal{E}}$

⇒ Free space $\hat{\sigma} = 0$

1.3 cont.

⇒ These three equations are called the constitutive relations and $\hat{\epsilon}$, $\bar{\mu}$, + $\hat{\sigma}$ are called the constitutive parameters

⇒ In the frequency-domain, the convolution turns into a product, e.g.

$$\bar{D} = \epsilon \bar{E}$$
$$\bar{B} = \mu \bar{H}$$
$$\bar{J}_c = \sigma \bar{E}$$

Note: ϵ, μ, + σ can be complex!

⇒ Use the constitutive parameters to characterize/describe electrical/EM properties of materials, e.g., dielectrics, conductors, magnetic materials, ...

⇒ Linear vs nonlinear Do they depend on field strength?

⇒ homogenous vs inhomogenous Do they depend on location?

⇒ isotropic vs. anisotropic Do they depend on direction?

⇒ nondispersive vs. dispersive Do they depend on frequency?

1.4 Circuit-Field Relations

→ Relate Maxwell's equations to circuit quantities + equations

1.4.1 Kirchhoff's Voltage Law (KVL)

Start w/ the integral form of Faraday's Law

$$\oint_c \bar{\mathcal{E}} \cdot d\bar{\ell} = -\iint_S \bar{\mathcal{M}}_i \cdot d\bar{s} - \frac{\partial}{\partial t}\iint_S \bar{\mathcal{B}} \cdot d\bar{s} \quad \text{(Field)}$$

Note $\sum V_{drops} = \oint_c \bar{\mathcal{E}} \cdot d\bar{\ell}$ (V) voltage drops around circuit contour C

Assume $\bar{\mathcal{M}}_i = 0$ (No impressed magnetic current densities)

Note: $\iint_S \bar{\mathcal{B}} \cdot d\bar{s} = \psi_m \equiv$ magnetic flux (Wb) (through surface enclosed by circuit)

Then $$-\frac{\partial}{\partial t}\iint_S \bar{\mathcal{B}} \cdot d\bar{s} = -\frac{\partial \psi_m}{\partial t} = -\frac{\partial (L_S i)}{\partial t} = -L_S \frac{\partial i}{\partial t}$$

where $L_S \equiv$ stray inductance of the circuit (constant)

and $i \equiv$ current flowing around circuit

This yields:

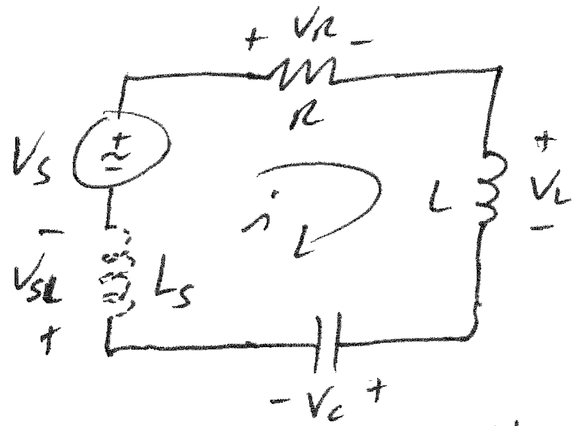
$$\sum V_{drops} = -\frac{\partial \psi_m}{\partial t} = \overbrace{-L_S \frac{\partial i}{\partial t}}^{V_{SL}} \quad \text{(Circuit)}$$

At DC, we get $\underline{\sum V_{drops} = 0}$

1.4.1 cont.

* Often, $L_S \simeq 0$ or $\frac{\partial i}{\partial t} \simeq 0$ and we use the DC form of KVL

ex.



Series RLC circuit

⇓

$$\sum V_{drops} = -V_S + V_R + V_L + V_C = -V_{SL} = -L_S \frac{\partial i}{\partial t}$$

1.4.2 Kirchhoff's Current Law (KCL)

Start w/ the integral Continuity equation

$$\oint\!\!\oint_S \bar{J}_{ic} \cdot d\bar{s} = -\frac{\partial}{\partial t} \iiint_V q_{ev} \, dv = -\frac{d Q_e}{dt} \quad \text{(Field)}$$

Note $\sum i_{out} = \oint\!\!\oint_S \bar{J}_{ic} \cdot d\bar{s}$ (A) current passing (leaving) through closed surface S

Define $Q_e = C_S V$ where C_S is the stray capacitance of the circuit

and V is the associated voltage

1.4.2 cont.

This yields:

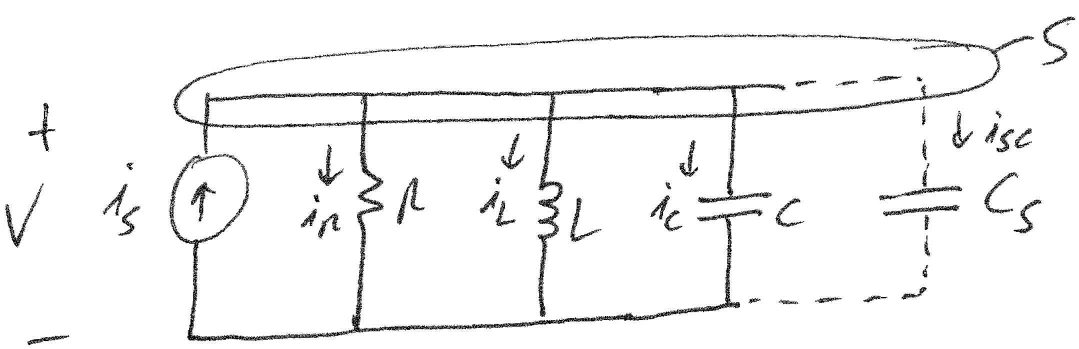
$$\sum i_{out} = \frac{-\partial Q_e}{\partial t} = -\frac{\partial (C_S V)}{\partial t} = -\overbrace{C_S \frac{\partial V}{\partial t}}^{i_{SC}} \quad \text{(Circuit)}$$

Includes displacement current NOT through wires.

At DC, we get $\underline{\underline{\sum i_{out} = 0}}$

* Often, $C_S \simeq 0$ or $\frac{\partial V}{\partial t} \simeq 0$ and we use the DC form of KCL

ex. Parallel RLC circuit



$$\Downarrow$$

$$\sum i_{out} = -i_S + i_R + i_L + i_C = -i_{SC} = -C_S \frac{\partial V}{\partial t}$$

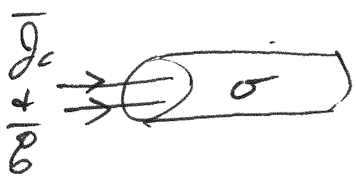
1.4.3 Element Laws

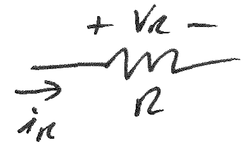
Besides KVL + KCL, we have relations between voltage + current for circuit elements

Ohm's Law / Resistor

Field Relation for conduction current density $\Longleftrightarrow$ Circuit Relation

$$\bar{\mathcal{J}}_c = \sigma \bar{\mathcal{E}} \qquad\qquad i_R = \frac{1}{R} v_R = G v_R$$





Inductor

Define $\Psi_m = L i$

Field Relation $\Longleftrightarrow$ Circuit Relation

$$\bar{\mathcal{B}} = \mu \bar{\mathcal{H}} \qquad\qquad \Psi_m = L i_L$$

Then $\bar{\mathcal{M}}_d = \dfrac{\partial \bar{\mathcal{B}}}{\partial t} = \mu \dfrac{\partial \bar{\mathcal{H}}}{\partial t}$

and since $\dfrac{\partial}{\partial t} \iint_S \bar{\mathcal{B}} \cdot d\bar{s} = \dfrac{\partial \Psi_m}{\partial t} = L \dfrac{\partial i_L}{\partial t} = v_L$

Field Relation Circuit Relation

$$\bar{\mathcal{M}}_d = \mu \frac{\partial \bar{\mathcal{H}}}{\partial t} \Longleftrightarrow v_L = L \frac{\partial i_L}{\partial t}$$

1.4.3 cont

Capacitor Similar to the inductor

Field Relation — Circuit Relation

$$\bar{\mathcal{D}} = \epsilon \bar{\mathscr{E}} \Longleftrightarrow \mathcal{Q}_e = C V_e$$

$$\bar{\mathcal{J}}_d = \epsilon \frac{\partial \bar{\mathscr{E}}}{\partial t} \Longleftrightarrow i_c = C \frac{\partial V_c}{\partial t}$$

TABLE 1-2 Relations between electromagnetic field and circuit theories

Field theory	Circuit theory
1. $\mathscr{E}$ (electric field intensity)	1. v (voltage)
2. $\mathscr{H}$ (magnetic field intensity)	2. i (current)
3. $\mathscr{D}$ (electric flux density)	3. q_{ev} (electric charge density)
4. $\mathscr{B}$ (magnetic flux density)	4. q_{mv} (magnetic charge density)
5. $\mathscr{J}$ (electric current density)	5. i_e (electric current)
6. $\mathscr{M}$ (magnetic current density)	6. i_m (magnetic current)
7. $\mathscr{J}_d = \varepsilon \dfrac{\partial \mathscr{E}}{\partial t}$ (electric displacement current density)	7. $i = C \dfrac{dv}{dt}$ (current through a capacitor)
8. $\mathscr{M}_d = \mu \dfrac{\partial \mathscr{H}}{\partial t}$ (magnetic displacement current density)	8. $v = L \dfrac{di}{dt}$ (voltage across an inductor)
9. *Constitutive relations*	9. *Element laws*
(a) $\mathscr{J}_c = \sigma \mathscr{E}$ (electric conduction current density)	(a) $i = Gv = \dfrac{1}{R} v$ (Ohm's law)
(b) $\mathscr{D} = \varepsilon \mathscr{E}$ (dielectric material)	(b) $\mathcal{Q}_e = Cv$ (charge in a capacitor)
(c) $\mathscr{B} = \mu \mathscr{H}$ (magnetic material)	(c) $\psi = Li$ (flux of an inductor)
10. $\oint_C \mathscr{E} \cdot d\boldsymbol{\ell} = -\dfrac{\partial}{\partial t} \iint_S \mathscr{B} \cdot d\mathbf{s}$ (Maxwell–Faraday equation)	10. $\sum v = -L_s \dfrac{\partial i}{\partial t} \simeq 0$ (Kirchhoff's voltage law)
11. $\oiint_S \mathscr{J}_{ic} \cdot d\mathbf{s} = -\dfrac{\partial}{\partial t} \iiint_V q_{ev}\, dv = -\dfrac{\partial \mathcal{Q}_e}{\partial t}$ (continuity equation)	11. $\sum i = -\dfrac{\partial \mathcal{Q}_e}{\partial t} = -C_s \dfrac{\partial v}{\partial t} \simeq 0$ (Kirchhoff's current law)
12. *Power and energy densities*	12. *Power and energy*
(a) $\oiint_S (\mathscr{E} \times \mathscr{H}) \cdot d\mathbf{s}$ (instantaneous power)	(a) $\mathcal{P} = vi$ (power–voltage–current relation)
(b) $\iiint_V \sigma \mathscr{E}^2 dv$ (dissipated power)	(b) $\mathcal{P}_d = Gv^2 = \dfrac{1}{R} v^2$ (power dissipated in a resistor)
(c) $\dfrac{1}{2} \iiint_V \varepsilon \mathscr{E}^2 dv$ (electric stored energy)	(c) $\dfrac{1}{2} C v^2$ (energy stored in a capacitor)
(d) $\dfrac{1}{2} \iiint_V \mu \mathscr{H}^2 dv$ (magnetic stored energy)	(d) $\dfrac{1}{2} L i^2$ (energy stored in an inductor)

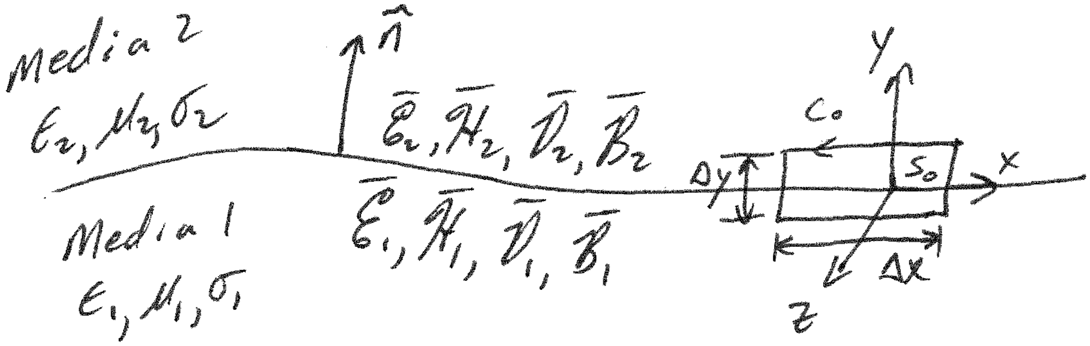
1.5 Boundary Conditions

⇒ What happens at the interface or boundary between materials/media w/ different electrical properties (σ, μ, ϵ)

⇒ Use integral form of Maxwell's Equations to derive boundary conditions

1.5.1 Finite Conductivity Media

Tangential



As shown, we have a rectangular contour C_0 bounding a surface area S_0 straddling the boundary. Further, we are <u>locally</u> going to assume the Cartesian coordinates shown.

* Assume No charges or sources at the boundary. (The no charges will be true assuming finite conductivities.)

1.5.1 cont.

First, let's apply Faraday's Law

$$\oint_{C_0} \bar{\mathcal{E}} \cdot d\bar{l} = -\iint_{S_0} \underbrace{\bar{\mathcal{M}}_i}_{\to 0} \cdot \underbrace{d\bar{s}}_{\to 0} - \frac{\partial}{\partial t}\iint_{S_0} \bar{\mathcal{B}} \cdot d\bar{s}$$

Letting $\Delta y \to 0$ which implies $S_0 \to 0$

$$\underset{\text{(bottom)}}{\bar{\mathcal{E}}_1 \cdot \hat{a}_x \Delta x} - \underset{\text{(top)}}{\bar{\mathcal{E}}_2 \cdot \hat{a}_x \Delta x} = 0$$

Since the x-components of $\bar{\mathcal{E}}_1$ + $\bar{\mathcal{E}}_2$ are tangential (parallel) to the boundary,

we get $\mathcal{E}_{1x}\, \Delta x - \mathcal{E}_{2x}\, \Delta x = 0$

$$\hookrightarrow \underline{\mathcal{E}_{1t} = \mathcal{E}_{2t}} \qquad \sigma_1 + \sigma_2 \text{ finite}$$

or $\underline{\hat{n} \times (\bar{\mathcal{E}}_2 - \bar{\mathcal{E}}_1) = 0}$

In words, the tangential electric field is continuous across the boundary between two finite conductivity media when there are no impressed magnetic current densities $\bar{\mathcal{M}}_i$ at the boundary.

$\Rightarrow$ Tangential electric flux discontinuous $\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$

1.5.1 cont.

Next, let's apply Ampere's Law w/ the assumption that there are no impressed sources at the boundary

$$\oint_{C_0} \bar{\mathcal{H}} \cdot d\bar{\ell} = \iint_{S_0} \overset{\to 0}{\bar{\mathcal{J}}_i} \cdot \underset{\to 0}{d\bar{s}} + \iint_{S_0} \bar{\mathcal{J}}_c \cdot d\bar{s} + \frac{\partial}{\partial t} \iint_{S_0} \bar{\mathcal{D}} \cdot d\bar{s}$$

Again, let $\Delta y \to 0 \Rightarrow S_0 \to 0$

$$\bar{\mathcal{H}}_1 \cdot \hat{a}_x \Delta x - \bar{\mathcal{H}}_2 \cdot \hat{a}_x \Delta x = 0 + 0$$

$$\mathcal{H}_{1x} \Delta x - \mathcal{H}_{2x} \Delta x = 0$$

Since the x-components of $\bar{\mathcal{H}}_1$ + $\bar{\mathcal{H}}_2$ are tangential to the boundary, we get

$$\underline{\mathcal{H}_{1t} = \mathcal{H}_{2t}}$$

or

$$\underline{\hat{n} \times (\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_1) = 0}$$

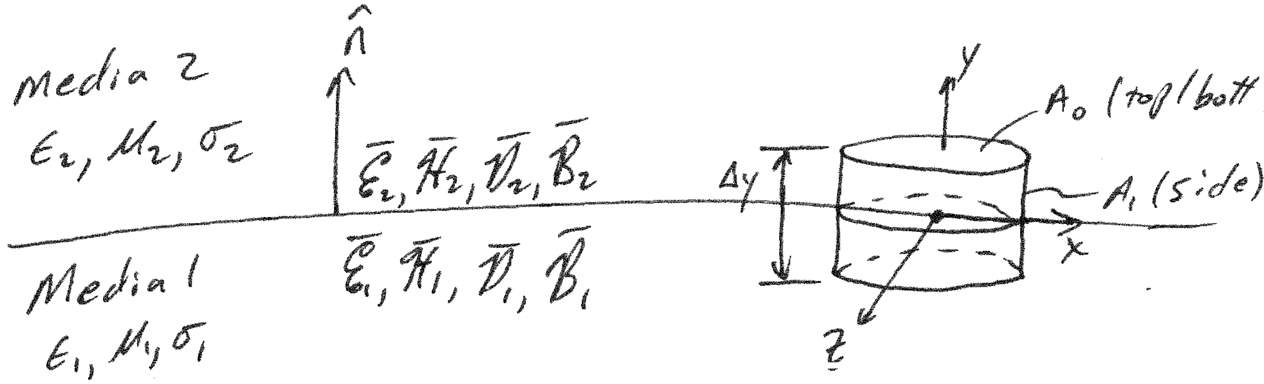
σ_1 + σ_2 finite

In words, the tangential magnetic field is continuous across the boundary between two finite conductivity media where there are <u>no</u> impressed electric current densities $\bar{\mathcal{J}}_i$ at the boundary.

$\Rightarrow$ Tangential mag. flux density discontinuous $\frac{\mathcal{B}_{1t}}{\mu_1} = \frac{\mathcal{B}_{2t}}{\mu_2}$

1.5.1 cont.

Normal



Applying Gauss' Law w/ the assumption there are no charges at/on the boundary (reasonable w/ finite σ_1 & σ_2)

$$\oiint_S \bar{D}\cdot d\bar{s} = \iiint_V \cancelto{0}{q_v}\, dv$$

$$\iint_{A_0} \bar{D}_2 \cdot d\bar{s}_y + \iint_{\substack{A_1 \\ Top}} \bar{D}_2 \cdot d\bar{s}_{side} + \iint_{\substack{A_1 \\ Bott}} \bar{D}_1 \cdot d\bar{s}_{side} + \iint_{A_0} \bar{D}_1 \cdot -d\bar{s}_y = 0$$

letting $\Delta y \to 0 \Rightarrow A_1 \to 0$

$$\bar{D}_2 \cdot \hat{a}_y A_0 + 0 + 0 - \bar{D}_1 \cdot \hat{a}_y A_0 = 0$$

Noting that the y-direction is NORMAL to boundary the boundary, we get:

$$\underline{D_{1n} = D_{2n}}$$

or

$$\underline{\hat{n}\cdot(\bar{D}_2 - \bar{D}_1) = 0}$$

σ_1 & σ_2 finite

1.5.1 cont.

In words, the normal components of the electric flux density vector are continuous across the boundary of finite conductivity media w/out sources (charges)

If put in terms of the electric field

$$\mathcal{D}_{1n} = \underline{\epsilon_1 \mathcal{E}_{1n} = \epsilon_2 \mathcal{E}_{2n}} = \mathcal{D}_{2n} \quad \begin{pmatrix}\text{neglect} \\ \text{convolution}\end{pmatrix}$$

OR

$$\underline{\hat{n} \cdot (\epsilon_2 \bar{\mathcal{E}}_{2n} - \epsilon_1 \bar{\mathcal{E}}_{1n}) = 0} \qquad \sigma_1 \text{ \& } \sigma_2 \text{ finite}$$

$\Rightarrow$ Normal components of electric field are discontinuous.

Next, use our last Maxwell Eq'n w/ the assumption that there are no magnetic charges on boundary

$$\oint\!\!\!\oint_S \bar{\mathcal{B}} \cdot d\bar{s} = \iiint_V \overset{0}{q_{mv}}\, dv$$

to get

$$\underline{\mathcal{B}_{1n} = \mathcal{B}_{2n}}$$

or

$$\underline{\hat{n} \cdot (\bar{\mathcal{B}}_2 - \bar{\mathcal{B}}_1) = 0}$$

1.5.1 cont.

In words, the normal component of the magnetic flux density vector is continuous across a boundary w/ no sources (charges)

If put in terms of the magnetic field

$$\mathcal{B}_{1n} = \underline{\mu_1 \mathcal{H}_{1n} = \mu_2 \mathcal{H}_{2n}} = \mathcal{B}_{2n} \qquad \text{(neglect convolution)}$$

or

$$\hat{n} \cdot (\mu_2 \bar{\mathcal{H}}_2 - \mu_1 \bar{\mathcal{H}}_1) = 0$$

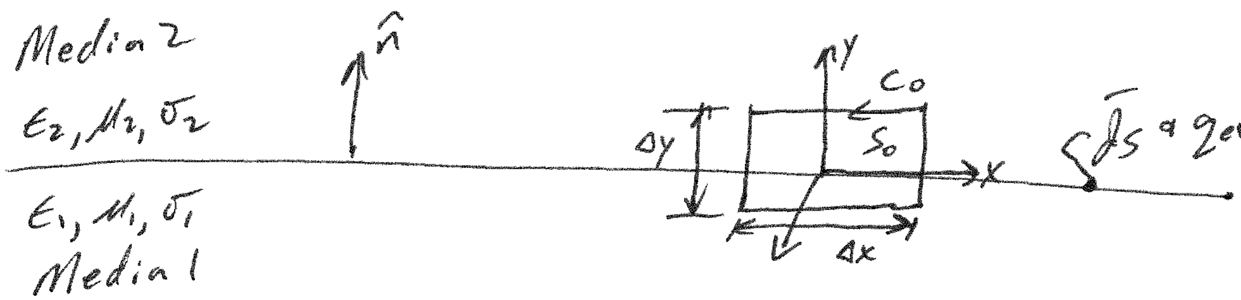
$\Rightarrow$ Normal components of magnetic field are discontinuous.

1.5.2 Infinite Conductivity Media

What occurs if there are electric sources and/or charges at the boundary, or if either of the media is a perfect electric conductor (PEC)? $\Rightarrow$ Must include / allow for surface current density $\bar{J}_s$ (A/m)

$\Rightarrow$ Must allow for surface electric charge density q_{es} (C/m²)

Tangential



The application of Faraday's Law is unchanged by allowing $\bar{J}_s$ &/or q_{ev} on boundary.

$$\mathcal{E}_{1t} = \mathcal{E}_{2t} \Rightarrow \frac{\mathcal{D}_{1t}}{\epsilon_1} = \frac{\mathcal{D}_{2t}}{\epsilon_2}$$

or $\hat{n} \times (\bar{\mathcal{E}}_2 - \bar{\mathcal{E}}_1) = 0$

However, the application of Ampere's Law does change

$$\oint_{C_0} \bar{\mathcal{H}} \cdot d\bar{\ell} = \iint_{S_0} \bar{J}_i \cdot d\bar{s} + \iint_{S} \bar{J}_c \cdot d\bar{s} + \frac{\partial}{\partial t} \iint_{S_0} \bar{\mathcal{D}} \cdot d\bar{s}$$

Again, let $\Delta y \to 0 \Rightarrow S_0 \to 0$

1.5.2 cont.

$$(\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2)\cdot \hat{a}_x \Delta x = \lim_{\Delta y \to 0} [\bar{\mathcal{J}}_i \cdot \hat{a}_z \Delta x \Delta y] + 0 + 0$$

$$= \bar{\mathcal{J}}_s \cdot \hat{a}_z \Delta x$$

only surface current density

dividing by Δx & re-arranging

$$(\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2)\cdot \hat{a}_x - \bar{\mathcal{J}}_s \cdot \hat{a}_z = 0$$

Note: $\hat{a}_x = \hat{a}_y \times \hat{a}_z$

$$(\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2)\cdot (\hat{a}_y \times \hat{a}_z) - \bar{\mathcal{J}}_s \cdot \hat{a}_z = 0$$

Use vector identity $\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{C}\cdot(\bar{A}\times\bar{B})$ on LHS, to get

$$\hat{a}_z \cdot [(\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2)\times \hat{a}_y] - \hat{a}_z \cdot \bar{\mathcal{J}}_s = 0$$

True IFF $(\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2)\times \hat{a}_y - \bar{\mathcal{J}}_s = 0$

$$(\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2)\times \hat{a}_y = \bar{\mathcal{J}}_s$$

Use $\bar{A}\times\bar{B} = -\bar{B}\times\bar{A}$

$$\hat{a}_y \times (\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_1) = \bar{\mathcal{J}}_s$$

Note: $\hat{a}_y = \hat{n}$

$$\underline{\underline{\hat{n} \times (\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_1) = \bar{\mathcal{J}}_s}}$$

$\Rightarrow$ Tangential magnetic field is discontinuous by an amount equal to the electric surface current density

1.5.2 cont.

What occurs to the tangential fields, if media 1 is a PEC (i.e., $\sigma_1 \to \infty$)?

$$\Rightarrow \bar{\mathcal{E}}_1 = 0 \Rightarrow \bar{\mathcal{D}}_1 = 0$$

This in turn causes

$$\mathcal{E}_{2t} = \underbrace{\mathcal{E}_{1t}}_{0} = 0 \Rightarrow \frac{\mathcal{D}_{2t}}{\epsilon_2} = 0 \Rightarrow \mathcal{D}_{2t} = 0$$

or

$$\underline{\hat{n} \times \bar{\mathcal{E}}_2 = 0}$$

No tangential $\mathcal{E}$ next to PEC!

In turn, from the differential form of Faraday's Law

$$\bar{\nabla} \times \underbrace{\bar{\mathcal{E}}_1}_{0} = 0 = -\cancelto{0}{\mathcal{M}_{i1}} - \frac{\partial \bar{\mathcal{B}}_1}{\partial t} \Rightarrow \bar{\mathcal{B}}_1 = 0$$

$$(\text{assumes } \mu_1 \text{ finite}) \Rightarrow \bar{\mathcal{H}}_1 = 0$$

Now, for $\sigma_1 \to \infty$,

$$\hat{n} \times (\bar{\mathcal{H}}_2 - \underbrace{\bar{\mathcal{H}}_1}_{0}) = \bar{\mathcal{J}}_s$$

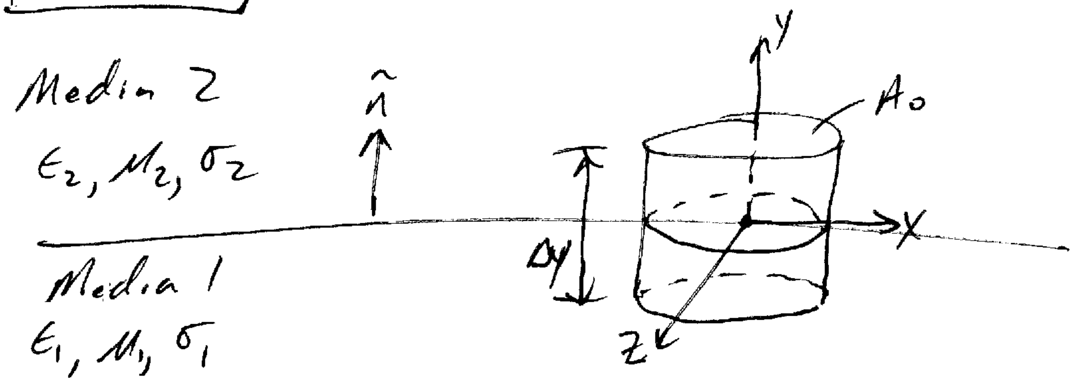
becomes

$$\underline{\hat{n} \times \bar{\mathcal{H}}_2 = \bar{\mathcal{J}}_s \Rightarrow |\bar{\mathcal{H}}_{2t}| = \mathcal{H}_{2t} = \mathcal{J}_s = |\bar{\mathcal{J}}_s|}$$

$\Rightarrow$ Tangential $\mathcal{H}$ is discontinuous next to a PEC by an amount equal to induced electric surface current density.

1.5.2 cont.

Normal



The application of Gauss' Law is changed by the possibility of a surface charge density q_{es}

$$\oiint_S \bar{D} \cdot d\bar{s} = \iiint_V q_{ev}\, dv$$

Now, as $\Delta y \to 0$, we get

$$\left(\bar{D}_2 - \bar{D}_1\right) \cdot \hat{a}_y A_0 = q_{es} A_0$$

$$\Rightarrow \quad \hat{n} = \hat{a}_y \Rightarrow \underline{\hat{n} \cdot \left(\bar{D}_2 - \bar{D}_1\right) = q_{es}}$$

or $\underline{D_{2n} - D_{1n} = q_{es}}$

or $\underline{\hat{n} \cdot \left(\epsilon_2 \bar{\mathcal{E}}_2 - \epsilon_1 \bar{\mathcal{E}}_1\right) = q_{es}}$

Both electric flux + electric field vectors are discontinuous across boundary w/ q_{es}.

1.5.2 cont.

The application of our last Maxwell eq'n is unchanged by q_{es}

$$\oiint_s \bar{\mathcal{B}} \cdot d\bar{s} = \iiint_v q_{mv}\, dv$$

$$\Downarrow$$

$$\underline{\mathcal{B}_{1n} = \mathcal{B}_{2n}} \quad \text{or} \quad \underline{\hat{n} \cdot (\bar{\mathcal{B}}_2 - \bar{\mathcal{B}}_1) = 0}$$

$$\underline{\mu_1 \mathcal{H}_{1n} = \mu_2 \mathcal{H}_{2n}}$$

What happens to normal field components if medial is a PEC?

Per earlier, $\bar{\mathcal{E}}_1 = \bar{\mathcal{D}}_1 = \bar{\mathcal{B}}_1 = \bar{\mathcal{H}}_1 = 0$ (μ_1 finite).

Therefore,
$$\underline{\hat{n} \cdot \bar{\mathcal{D}}_2 = q_{es} = \mathcal{D}_{2n}}$$

or
$$\underline{\hat{n} \cdot \bar{\mathcal{E}}_2 = \frac{q_{es}}{\epsilon_2} = \mathcal{E}_{2n}}$$

and
$$\underline{\mathcal{B}_{2n} = \mathcal{H}_{2n} = 0}$$

$\Rightarrow$ The normal components of the electric field + flux are discontinuous next to a PEC.

1.5.3 Sources Along Boundaries

If we allow for $\bar{\mathcal{M}}_s$ (V/m) or $\bar{\mathcal{J}}_s$ (A/m), the magnetic + electric surface current densities, or for q_{ms} $\left(\frac{Wb}{m^2}\right)$ or q_{es} (C/m²), the magnetic + electric surface charge densities, the boundary conditions are:

$$\left.\begin{aligned} -\hat{n} \times (\bar{\mathcal{E}}_2 - \bar{\mathcal{E}}_1) &= \bar{\mathcal{M}}_s \\ \hat{n} \times (\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_1) &= \bar{\mathcal{J}}_s \end{aligned}\right\} \text{Tangential}$$

$$\left.\begin{aligned} \hat{n} \cdot (\bar{\mathcal{D}}_2 - \bar{\mathcal{D}}_1) &= q_{es} \\ \hat{n} \cdot (\bar{\mathcal{B}}_2 - \bar{\mathcal{B}}_1) &= q_{ms} \end{aligned}\right\} \text{Normal}$$

when media 1 and media 2 are NOT perfect conductors.

TABLE 1-3 Boundary conditions on instantaneous electromagnetic fields

	General	Finite conductivity media, no sources or charges $\sigma_1, \sigma_2 \neq \infty$ $\mathcal{J}_s = 0; q_{es} = 0$ $\mathcal{M}_s = 0; q_{ms} = 0$	Medium 1 of infinite electric conductivity $(\mathcal{E}_1 = \mathcal{H}_1 = 0)$ $\sigma_1 = \infty; \sigma_2 \neq \infty$ $\mathcal{M}_s = 0; q_{ms} = 0$	Medium 1 of infinite magnetic conductivity $(\mathcal{E}_1 = \mathcal{H}_1 = 0)$ $\mathcal{J}_s = 0; q_{es} = 0$
Tangential electric field intensity	$-\hat{\mathbf{n}} \times (\mathcal{E}_2 - \mathcal{E}_1) = \mathcal{M}_s$	$\hat{\mathbf{n}} \times (\mathcal{E}_2 - \mathcal{E}_1) = 0$	$\hat{\mathbf{n}} \times \mathcal{E}_2 = 0$	$-\hat{\mathbf{n}} \times \mathcal{E}_2 = \mathcal{M}_s$
Tangential magnetic field intensity	$\hat{\mathbf{n}} \times (\mathcal{H}_2 - \mathcal{H}_1) = \mathcal{J}_s$	$\hat{\mathbf{n}} \times (\mathcal{H}_2 - \mathcal{H}_1) = 0$	$\hat{\mathbf{n}} \times \mathcal{H}_2 = \mathcal{J}_s$	$\hat{\mathbf{n}} \times \mathcal{H}_2 = 0$
Normal electric flux density	$\hat{\mathbf{n}} \cdot (\mathcal{D}_2 - \mathcal{D}_1) = q_{es}$	$\hat{\mathbf{n}} \cdot (\mathcal{D}_2 - \mathcal{D}_1) = 0$	$\hat{\mathbf{n}} \cdot \mathcal{D}_2 = q_{es}$	$\hat{\mathbf{n}} \cdot \mathcal{D}_2 = 0$
Normal magnetic flux density	$\hat{\mathbf{n}} \cdot (\mathcal{B}_2 - \mathcal{B}_1) = q_{ms}$	$\hat{\mathbf{n}} \cdot (\mathcal{B}_2 - \mathcal{B}_1) = 0$	$\hat{\mathbf{n}} \cdot \mathcal{B}_2 = 0$	$\hat{\mathbf{n}} \cdot \mathcal{B}_2 = q_{ms}$

1.5.3 cont.

Note: Perfect Magnetic Conductors (PMC) are also added.

PMCs do NOT physically exist, but can be used to simplify some EM problems

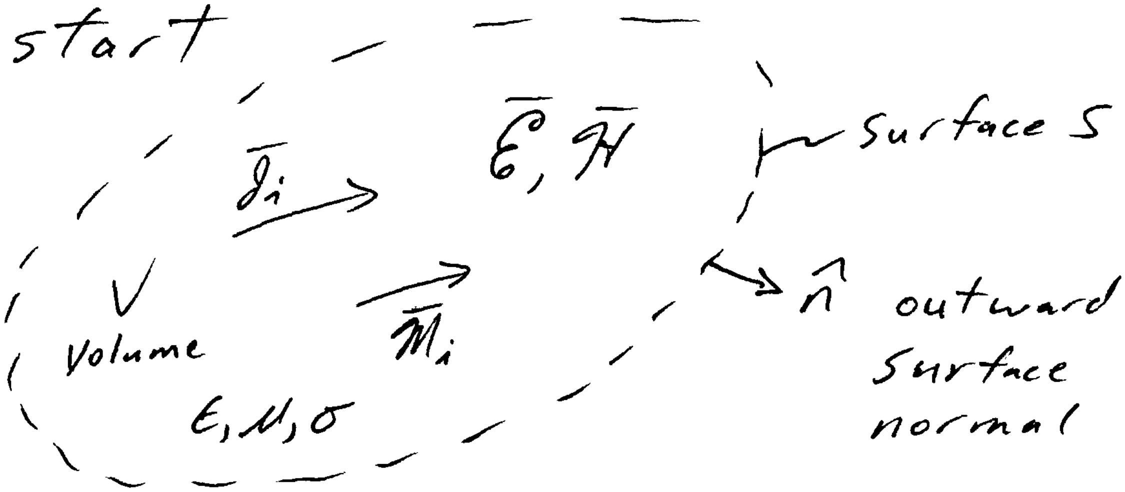
PMCs: $\bar{\mathcal{E}}_1 = \bar{\mathcal{H}}_1 = 0$, $\underline{\underline{\bar{\mathcal{J}}_s = 0}}$, $q_{es} = 0$
(media 1)

but <u>can</u> have $\bar{\mathcal{M}}_s$ + q_{ms}

1.6 Power and Energy

$\Rightarrow$ Pay homage to John Henry Poynting (British phycist) and his 1884 paper "On the Transfer of Energy in the Electromagnetic Field"

To start



Faraday's & Ampere's Laws

$$\bar{\nabla} \times \bar{\mathcal{E}} = -\bar{\mathcal{M}}_i - \frac{\partial \bar{\mathcal{B}}}{\partial t} = -\bar{\mathcal{M}}_i - \bar{\mathcal{M}}_d$$

$$\bar{\nabla} \times \bar{\mathcal{H}} = \bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \frac{\partial \bar{\mathcal{D}}}{\partial t} = \bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d$$

Take dot product of $\bar{\mathcal{H}}$ w/ Faraday's and $\bar{\mathcal{E}}$ w/ Ampere's Law

$$\bar{\mathcal{H}} \cdot (\bar{\nabla} \times \bar{\mathcal{E}}) = -\bar{\mathcal{H}} \cdot (\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_d)$$

$$\bar{\mathcal{E}} \cdot (\bar{\nabla} \times \bar{\mathcal{H}}) = \bar{\mathcal{E}} \cdot (\bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d)$$

Then, subtract the second eq'n from first

$$\bar{\mathcal{H}} \cdot (\bar{\nabla} \times \bar{\mathcal{E}}) - \bar{\mathcal{E}} \cdot (\bar{\nabla} \times \bar{\mathcal{H}}) = -\bar{\mathcal{H}} \cdot (\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_d) - \bar{\mathcal{E}} \cdot (\bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d)$$

1.6 cont.

use vector identity $\bar{\nabla}\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\bar{\nabla}\times\bar{A}) - \bar{A}\cdot(\bar{\nabla}\times\bar{B})$ to get

$$\bar{\nabla}\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}}) = -\bar{\mathcal{H}}\cdot(\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_d) - \bar{\mathcal{E}}\cdot(\bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d)$$

$$\boxed{\bar{\nabla}\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}}) + \bar{\mathcal{H}}\cdot(\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_d) + \bar{\mathcal{E}}\cdot(\bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d) = 0}$$

$\Rightarrow$ differential form of <u>Poynting Theorem</u>

To get integral form, integrate over vol. V

$$\underbrace{\iiint_V \bar{\nabla}\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}})\,dv}_{\text{use divergence theorem}} + \iiint_V \left[\bar{\mathcal{H}}\cdot(\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_d) + \bar{\mathcal{E}}\cdot(\bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d)\right] dV = 0$$

$$\boxed{\oiint_S (\bar{\mathcal{E}}\times\bar{\mathcal{H}})\cdot d\bar{s} + \iiint_V \left[\bar{\mathcal{H}}\cdot(\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_d) + \bar{\mathcal{E}}\cdot(\bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c + \bar{\mathcal{J}}_d)\right] dV = 0}$$

$\Rightarrow$ integral form of Poynting Theorem

$\Rightarrow$ Both deal w/ conservation of energy

Definitions + interpretations –

Poynting vector $\equiv \bar{\mathcal{S}} = \bar{\mathcal{E}}\times\bar{\mathcal{H}}$ (W/m^2) Power density

$P_e = \oiint_S (\bar{\mathcal{E}}\times\bar{\mathcal{H}})\cdot d\bar{s} = \oiint_S \bar{\mathcal{S}}\cdot d\bar{s}$ is the <u>power</u> leaving volume V through surface S (in the EM field).

1.6 cont.

Supplied power density $\equiv p_s = -(\bar{\mathcal{H}} \cdot \bar{\mathcal{M}}_i + \bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_i)$ (W/m³)

dissipated power density $\equiv p_d = \bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_c = \bar{\mathcal{E}} \cdot \sigma \bar{\mathcal{E}} = \sigma \mathcal{E}^2$ (W/m³)

The term $\bar{\mathcal{H}} \cdot \bar{\mathcal{M}}_d$ has to do w/ the time rate of change in the magnetic energy density w_m

$$\bar{\mathcal{H}} \cdot \bar{\mathcal{M}}_d = \bar{\mathcal{H}} \cdot \frac{\partial \bar{\mathcal{B}}}{\partial t} = \mu \bar{\mathcal{H}} \cdot \frac{\partial \bar{\mathcal{H}}}{\partial t} = \tfrac{1}{2} \mu \frac{\partial \mathcal{H}^2}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\mu \mathcal{H}^2\right) \quad (\text{W/m}^3)$$

where $w_m = \tfrac{1}{2} \mu \mathcal{H}^2$ (J/m³)

The term $\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_d$ has to do w/ the time rate of change in the electric energy density w_e

$$\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_d = \bar{\mathcal{E}} \cdot \frac{\partial \bar{\mathcal{D}}}{\partial t} = \epsilon \bar{\mathcal{E}} \cdot \frac{\partial \bar{\mathcal{E}}}{\partial t} = \tfrac{1}{2} \epsilon \frac{\partial \mathcal{E}^2}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\epsilon \mathcal{E}^2\right) \quad (\text{W/m}^3)$$

where $w_e = \tfrac{1}{2} \epsilon \mathcal{E}^2$ (J/m³)

Now, looking at the volume integral of these terms

Supplied power $\equiv P_s = -\iiint_V (\bar{\mathcal{H}} \cdot \bar{\mathcal{M}}_i + \bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_i)\, dv = \iiint_V p_s\, dv$ (W)

1.6 cont.

dissipated power $\equiv \mathcal{P}_d = \iiint_V (\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_c)\, dv = \iiint_V \sigma \mathcal{E}^2 dV = \iiint_V p_d\, dV$ (W)

Time rate of magnetic energy $\mathcal{W}_m$ (J) stored w/in V $\equiv \iiint_V (\bar{\mathcal{H}} \cdot \bar{\mathcal{M}}_d)\, dV = \frac{\partial}{\partial t} \iiint \frac{1}{2} \mu \mathcal{H}^2 dV$

$= \frac{\partial}{\partial t} \iiint w_m\, dV = \frac{\partial \mathcal{W}_m}{\partial t}$ (W)

Time rate of electric energy $\mathcal{W}_e$ (J) stored w/in V $\equiv \iiint_V (\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_d)\, dv = \frac{\partial}{\partial t} \iiint \frac{1}{2} \epsilon \mathcal{E}^2 dV$

$= \frac{\partial}{\partial t} \iiint w_e\, dV = \frac{\partial \mathcal{W}_e}{\partial t}$ (W)

w/ these definitions, we can re-write the integral form of the Poynting Theorem as

$$\mathcal{P}_e - \mathcal{P}_s + \mathcal{P}_d + \frac{\partial}{\partial t}(\mathcal{W}_e + \mathcal{W}_m) = 0$$

or

$$\underbrace{\mathcal{P}_s}_{\text{supplied}} = \underbrace{\mathcal{P}_e}_{\text{leaving}} + \underbrace{\mathcal{P}_d}_{\text{dissipated}} + \underbrace{\frac{\partial}{\partial t}(\mathcal{W}_e + \mathcal{W}_m)}_{\text{change in stored energy}}$$

which is why it is a <u>conservation of power law</u> (equation)!

1.7 Time-Harmonic Electromagnetic Fields

⇒ Many (most?) practical problems involve signals + EM waves that are time-harmonic, i.e., can be represented in terms of cos( ) or sin( ) functions (remember Fourier Series).

⇒ Here, the time-dependence can be represented by $e^{j\omega t}$ and we can express +/or analyze the EM fields in terms of their phasors.

⇒ Euler's Identity $e^{\pm jA} = \cos A \pm j\sin A$

so, $\mathcal{R}e\{V_o e^{j\omega t}\} = V_o \cos(\omega t)$

We'll define for time-harmonic EM waves

$$\bar{\mathcal{E}}(x,y,z;t) = \mathcal{R}e\left[\bar{E}(x,y,z)e^{j\omega t}\right] \quad (V/m)$$

$$\bar{\mathcal{H}}(x,y,z;t) = \mathcal{R}e\left[\bar{H}(x,y,z)e^{j\omega t}\right] \quad (A/m)$$

$$\bar{\mathcal{D}}(x,y,z;t) = \mathcal{R}e\left[\bar{D}(x,y,z)e^{j\omega t}\right] \quad (C/m^2)$$

$$\bar{\mathcal{B}}(x,y,z;t) = \mathcal{R}e\left[\bar{B}(x,y,z)e^{j\omega t}\right] \quad (Wb/m^2)$$

$$\bar{\mathcal{J}}(x,y,z;t) = \mathcal{R}e\left[\bar{J}(x,y,z)e^{j\omega t}\right] \quad (A/m^2)$$

$$\mathcal{q}(x,y,z;t) = \mathcal{R}e\left[q(x,y,z)e^{j\omega t}\right] \quad (C)$$

↑ Instantaneous (time-domain) ↑ complex spatial forms (i.e., phasors) ↖ peak values (NOT RMS)

1.7.1 Maxwell's Equations in Differential and Integral Forms

⇒ Replace time-domain fields w/ phasors

⇒ $\frac{\partial \bar{\mathcal{N}}}{\partial t} \Rightarrow j\omega \bar{N}$ since $\frac{\partial (\bar{N} e^{j\omega t})}{\partial t} = \bar{N} j\omega e^{j\omega t}$

and we divide out the $e^{j\omega t}$ terms

⇒ Easier to do EM problem solutions w/ phasors and then convert back to time-domain (if necessary)

⇒ Also, $\hat{\sigma} * \bar{\mathcal{E}} \Rightarrow \sigma \bar{E}$! Constitutive eq'ns easier

TABLE 1-4 Instantaneous and time-harmonic forms of Maxwell's equations and continuity equation in differential and integral forms

Instantaneous	Time harmonic
Differential form	
$\nabla \times \boldsymbol{\mathscr{E}} = -\boldsymbol{\mathscr{M}}_i - \frac{\partial \boldsymbol{\mathscr{B}}}{\partial t}$	$\nabla \times \mathbf{E} = -\mathbf{M}_i - j\omega \mathbf{B}$
$\nabla \times \boldsymbol{\mathscr{H}} = \boldsymbol{\mathscr{J}}_i + \boldsymbol{\mathscr{J}}_c + \frac{\partial \boldsymbol{\mathscr{D}}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J}_i + \mathbf{J}_c + j\omega \mathbf{D}$
$\nabla \cdot \boldsymbol{\mathscr{D}} = \mathscr{q}_{ev}$	$\nabla \cdot \mathbf{D} = q_{ev}$
$\nabla \cdot \boldsymbol{\mathscr{B}} = \mathscr{q}_{mv}$	$\nabla \cdot \mathbf{B} = q_{mv}$
$\nabla \cdot \boldsymbol{\mathscr{J}}_{ic} = -\frac{\partial \mathscr{q}_{ev}}{\partial t}$	$\nabla \cdot \mathbf{J}_{ic} = -j\omega q_{ev}$
Integral form	
$\oint_C \boldsymbol{\mathscr{E}} \cdot d\boldsymbol{\ell} = -\iint_S \boldsymbol{\mathscr{M}}_i \cdot d\mathbf{s} - \frac{\partial}{\partial t} \iint_S \boldsymbol{\mathscr{B}} \cdot d\mathbf{s}$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\iint_S \mathbf{M}_i \cdot d\mathbf{s} - j\omega \iint_S \mathbf{B} \cdot d\mathbf{s}$
$\oint_C \boldsymbol{\mathscr{H}} \cdot d\boldsymbol{\ell} = \iint_S \boldsymbol{\mathscr{J}}_i \cdot d\mathbf{s} + \iint_S \boldsymbol{\mathscr{J}}_c \cdot d\mathbf{s} + \frac{\partial}{\partial t} \iint_S \boldsymbol{\mathscr{D}} \cdot d\mathbf{s}$	$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \iint_S \mathbf{J}_i \cdot d\mathbf{s} + \iint_S \mathbf{J}_c \cdot d\mathbf{s} + j\omega \iint_S \mathbf{D} \cdot d\mathbf{s}$
$\oiint_S \boldsymbol{\mathscr{D}} \cdot d\mathbf{s} = \mathscr{Q}_e$	$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q_e$
$\oiint_S \boldsymbol{\mathscr{B}} \cdot d\mathbf{s} = \mathscr{Q}_m$	$\oiint_S \mathbf{B} \cdot d\mathbf{s} = Q_m$
$\oiint_S \boldsymbol{\mathscr{J}}_{ic} \cdot d\mathbf{s} = -\frac{\partial \mathscr{Q}_e}{\partial t}$	$\oiint_S \mathbf{J}_{ic} \cdot d\mathbf{s} = -j\omega Q_e$

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

1.7.2 Boundary Conditions

⇒ In a similar fashion, we can update our derived boundary conditions for the time-harmonic case.

TABLE 1-5 Boundary conditions on time-harmonic electromagnetic fields

	General	Finite conductivity media, no sources or charges $\sigma_1, \sigma_2 \neq \infty$ $\mathbf{J}_s = \mathbf{M}_s = 0$ $q_{es} = q_{ms} = 0$	Medium 1 of infinite electric conductivity $(\mathbf{E}_1 = \mathbf{H}_1 = 0)$ $\sigma_1 = \infty; \sigma_2 \neq \infty$ $\mathbf{M}_s = 0; q_{ms} = 0$	Medium 1 of infinite magnetic conductivity $(\mathbf{E}_1 = \mathbf{H}_1 = 0)$ $\mathbf{J}_s = 0; q_{es} = 0$
Tangential electric field intensity	$-\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{M}_s$	$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$	$\hat{\mathbf{n}} \times \mathbf{E}_2 = 0$	$-\hat{\mathbf{n}} \times \mathbf{E}_2 = \mathbf{M}_s$
Tangential magnetic field intensity	$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$	$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0$	$\hat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{J}_s$	$\hat{\mathbf{n}} \times \mathbf{H}_2 = 0$
Normal electric flux density	$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = q_{es}$	$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 0$	$\hat{\mathbf{n}} \cdot \mathbf{D}_2 = q_{es}$	$\hat{\mathbf{n}} \cdot \mathbf{D}_2 = 0$
Normal magnetic flux density	$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = q_{ms}$	$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$	$\hat{\mathbf{n}} \cdot \mathbf{B}_2 = 0$	$\hat{\mathbf{n}} \cdot \mathbf{B}_2 = q_{ms}$

Advanced Engineering Electromagnetics (Second Edition), Balanis, Wiley, 2012, ISBN-10: 0470589485, ISBN-13: 978-0470589489.

What if media 1 is a very good conductor, but NOT a PEC?

Now, the surface current is approximately related to the tangential magnetic field

$$\underline{\bar{J}_s = \hat{n} \times \bar{H}_2}$$

⇒ reality is that current density drops exponentially from surface into media 1 (skin effect)

1.7.2 cont.

Looking ahead to chap 4, define surface impedance of media 1 as

$$Z_S = R_S + jX_S = (1+j)\sqrt{\frac{\omega\mu_1}{2\sigma_1}}$$

Now, the tangential electric field in media 2 is NOT zero (will be small)

$$\underline{\bar{E}_{t2} = Z_S\,\bar{J}_S = Z_S\,\hat{n}\times\bar{H}_2}$$

Now, we must consider what happens to the normal components. As it turns out, they <u>are</u> dependent on the <u>tangential</u> field components in this case.

Use Faraday's & Ampere's equations w/ tangential $\bar{E}$ (Faraday) and $\bar{H}$ (Ampere) on LHS

$$\bar{\nabla}\times\bar{E} = -\cancelto{0}{\bar{M}_i} - j\omega\bar{B} = -j\omega\mu\bar{H}$$

$$\bar{\nabla}\times\bar{H} = \cancelto{0}{\bar{J}_i} + \bar{J}_c + j\omega\bar{D}$$

1.7.3 Power and Energy

Let/note $\bar{\mathcal{E}} = Re\left[\bar{E} e^{j\omega t}\right] = \frac{1}{2}\left[\bar{E} e^{j\omega t} + (\bar{E} e^{j\omega t})^*\right]$

$$\bar{\mathcal{H}} = Re\left[\bar{H} e^{j\omega t}\right] = \frac{1}{2}\left[\bar{H} e^{j\omega t} + (\bar{H} e^{j\omega t})^*\right]$$

Then, the instantaneous Poynting vector is

$$\bar{\mathcal{S}} = \bar{\mathcal{E}} \times \bar{\mathcal{H}} = \frac{1}{2}\left[\bar{E} e^{j\omega t} + \bar{E}^* e^{-j\omega t}\right] \times \frac{1}{2}\left[\bar{H} e^{j\omega t} + \bar{H}^* e^{-j\omega t}\right]$$

$$= \frac{1}{2}\Big[\underbrace{Re(\bar{E} \times \bar{H}^*)}_{\text{constant wrt } t} + \underbrace{Re(\bar{E} \times \bar{H} e^{j2\omega t})}_{\text{varies @ freq } 2\omega}\Big] \quad (W/m^2)$$

$$\bar{\mathcal{S}}_{ave} = \frac{1}{T}\int_{t_0}^{t_0+T} \bar{\mathcal{S}}\, dt = \underline{\bar{S}_{ave} = \frac{1}{2} Re(\bar{E} \times \bar{H}^*)} \equiv \text{Time-ave Poynting vector}$$

Note: The imaginary part of $\bar{E} \times \bar{H}^*$ is related to reactive power.

To derive the Poynting Theorem for the time-harmonic form, dot $\bar{H}^*$ w/ Faraday's Law and $\bar{E}$ w/ Ampere's Law complex conjugate

$$\bar{H}^* \cdot (\bar{\nabla} \times \bar{E}) = -\bar{H}^* \cdot \bar{M}_i - j\omega\mu \underbrace{\bar{H} \cdot \bar{H}^*}_{|\bar{H}|^2}$$

$$\bar{E} \cdot (\bar{\nabla} \times \bar{H}^*) = \bar{E} \cdot \bar{J}_i^* + \sigma \underbrace{\bar{E} \cdot \bar{E}^*}_{|E|^2} - j\omega\epsilon \underbrace{\bar{E} \cdot \bar{E}^*}_{|E|^2}$$

1.7.3 cont.

Next subtract the second eq'n from the first and apply $\bar{\nabla}\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\bar{\nabla}\times\bar{A}) - \bar{A}\cdot(\bar{\nabla}\times\bar{B})$

$$\bar{\nabla}\cdot(\bar{H}^*\times\bar{E}) = -\bar{\nabla}\cdot(\bar{E}\times\bar{H}^*) = \bar{H}^*\cdot\bar{M}_i + \bar{E}\cdot\bar{J}_i^* + \sigma|\bar{E}|^2 + j\omega\left[\mu|\bar{H}|^2 - \epsilon|\bar{E}|^2\right]$$

Divide by 2 to get (W/m^3)

$$-\bar{\nabla}\cdot(\tfrac{1}{2}\bar{E}\times\bar{H}^*) = \tfrac{1}{2}\bar{H}^*\cdot\bar{M}_i + \tfrac{1}{2}\bar{E}\cdot\bar{J}_i^* + \tfrac{1}{2}\sigma|\bar{E}|^2 + j2\omega\left[\tfrac{\mu}{4}|\bar{H}|^2 - \tfrac{\epsilon}{4}|\bar{E}|^2\right]$$

Poynting Theorem in differential form for time-harmonic case.

To get integral form, again integrate over volume V & use divergence theorem and some algebra

$$\boxed{\begin{aligned} -\tfrac{1}{2}\iiint_V(\bar{H}^*\cdot\bar{M}_i + \bar{E}\cdot\bar{J}_i^*)\,dv &= \oiint_S(\tfrac{1}{2}\bar{E}\times\bar{H}^*)\cdot d\bar{S} \\ &\quad + \tfrac{1}{2}\iiint_V\sigma|\bar{E}|^2\,dv \quad (W) \\ &\quad + j2\omega\iiint_V\left[\tfrac{1}{4}\mu|\bar{H}|^2 - \tfrac{1}{4}\epsilon|\bar{E}|^2\right]dv \end{aligned}}$$

Here, we define -

supplied complex power (W) $\equiv P_S = -\tfrac{1}{2}\iiint_V(\bar{H}^*\cdot\bar{M}_i + \bar{E}\cdot\bar{J}_i^*)\,dv$

1.7.3 cont.

Exiting complex power (W) $\equiv P_e = \oiint_S \left(\frac{1}{2}\bar{E} \times \bar{H}^*\right) \cdot d\bar{s}$

Dissipated real power (W) $\equiv P_d = \frac{1}{2} \iiint_V \sigma |\bar{E}|^2 dv$

Time-ave magnetic energy (J) $\equiv \bar{W}_m = \iiint_V \frac{1}{4} \mu |\bar{H}|^2 dv$

Time-ave electric energy (J) $\equiv \bar{W}_e = \iiint_V \frac{1}{4} \epsilon |\bar{E}|^2 dv$

which leads to

$$\underbrace{P_s}_{\text{complex}} = P_e + \underbrace{P_d}_{\text{real}} + \underbrace{j2\omega(\bar{W}_m - \bar{W}_E)}_{\text{reactive/imaginary}}$$

Note: If ϵ or μ are complex, the real portions of the last terms are combined w/ P_d as they'll be losses.

1.7.3 cont.

For the time-harmonic case, the relations between field quantities and circuit theory are given in Table 1-6.

TABLE 1-6 Relations between time-harmonic electromagnetic field and steady-state a.c. circuit theories

Field theory	Circuit theory
1. $\mathbf{E}$ (electric field intensity)	1. v (voltage)
2. $\mathbf{H}$ (magnetic field intensity)	2. i (current)
3. $\mathbf{D}$ (electric flux density)	3. q_{ev} (electric charge density)
4. $\mathbf{B}$ (magnetic flux density)	4. q_{mv} (magnetic charge density)
5. $\mathbf{J}$ (electric current density)	5. i_e (electric current)
6. $\mathbf{M}$ (magnetic current density)	6. i_m (magnetic current)
7. $\mathbf{J}_d = j\omega\varepsilon\mathbf{E}$ (electric displacement current density)	7. $i = j\omega C v$ (current through a capacitor)
8. $\mathbf{M}_d = j\omega\mu\mathbf{H}$ (magnetic displacement current density)	8. $v = j\omega L i$ (voltage across an inductor)
9. *Constitutive relations*	9. *Element laws*
(a) $\mathbf{J}_c = \sigma\mathbf{E}$ (electric conduction current density)	(a) $i = Gv = \frac{1}{R}v$ (Ohm's law)
(b) $\mathbf{D} = \varepsilon\mathbf{E}$ (dielectric material)	(b) $Q_e = Cv$ (charge in a capacitor)
(c) $\mathbf{B} = \mu\mathbf{H}$ (magnetic material)	(c) $\psi = Li$ (flux of an inductor)
10. $\oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = -j\omega\iint_S \mathbf{B}\cdot d\mathbf{s}$ (Maxwell–Faraday equation)	10. $\sum v = -j\omega L_s i \simeq 0$ (Kirchhoff's voltage law)
11. $\oiint_S \mathbf{J}_{ic}\cdot d\mathbf{s} = -j\omega\iiint_V q_{ev}\,dv = -\frac{\partial Q_e}{\partial t}$ (continuity equation)	11. $\sum i = -j\omega Q_e = -j\omega C_s v \simeq 0$ (Kirchoff's current law)
12. *Power and energy densities*	12. *Power and energy* (v and i represent peak values)
(a) $\frac{1}{2}\oiint_S (\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{s}$ (complex power)	(a) $P = \frac{1}{2}vi$ (power-voltage-current relation)
(b) $\frac{1}{2}\iiint_V \sigma\lvert\mathbf{E}\rvert^2 dv$ (dissipated real power)	(b) $P_d = \frac{1}{2}Gv^2 = \frac{1}{2}\frac{v^2}{R}$ (power dissipated in a resistor)
(c) $\frac{1}{4}\iiint_V \varepsilon\lvert\mathbf{E}\rvert^2 dv$ (time-average electric stored energy)	(c) $\frac{1}{4}Cv^2$ (energy stored in a capacitor)
(d) $\frac{1}{4}\iiint_V \mu\lvert\mathbf{H}\rvert^2 dv$ (time-average magnetic stored energy)	(d) $\frac{1}{4}Li^2$ (energy stored in an inductor)

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