Chapter 6 Arrays: Linear, …

Introduction

Why? → High directivity / gains

→ control radiation pattern

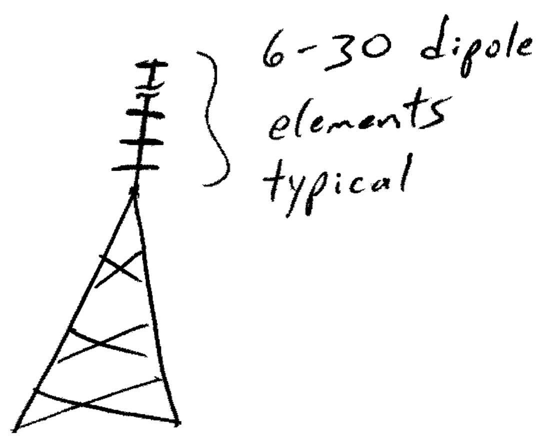
→ can change characteristics by changing elements (e.g., scan, track multiple targets, introduce nulls, ...)

→ More flexible than a dish, horn, or other high gain antenna

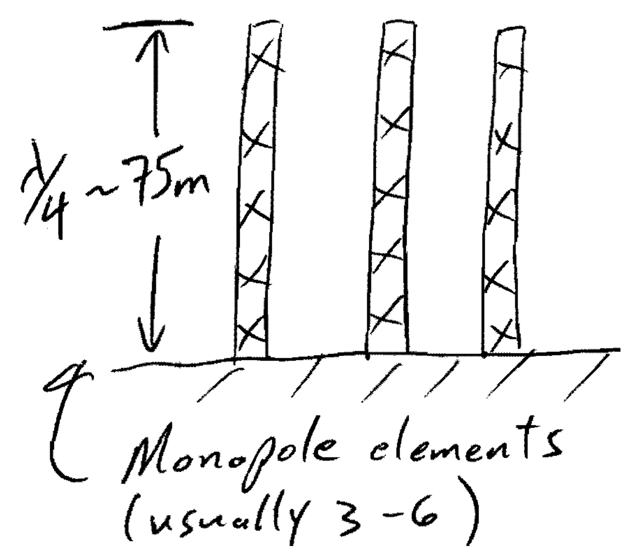
Forms: individual elements can be whips / monopoles, dipoles, horns, dishes, ...

Examples of linear arrays:

Television &/or FM Radio



AM Radio ($\lambda \simeq 300$ m)



How?

1) physical configuration/layout (linear, circular, ... arrays)

2) relative spacing between elements

3) Magnitude of excitation for each element

4) Relative phase for/between elements

5) Radiation pattern of individual elements

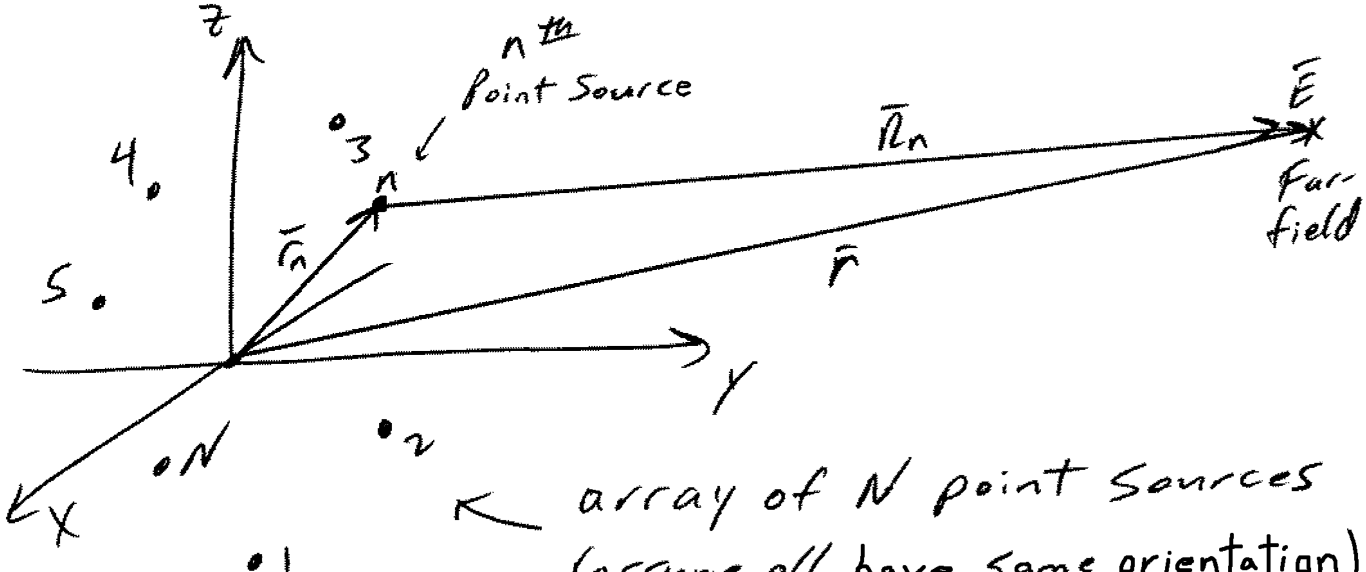
* Ideally, the individual fields add up vectorally in the far-field independent of one another. In reality, the array elements will interact to some extent. (mutual coupling)

(Detour from text)

Point Sources

phasor

$$\bar{E}_{FF}(x,y,z) = \hat{a}\,\frac{j\omega\mu I}{4\pi r}\,e^{-jkr}$$ (ignore polarization for now)



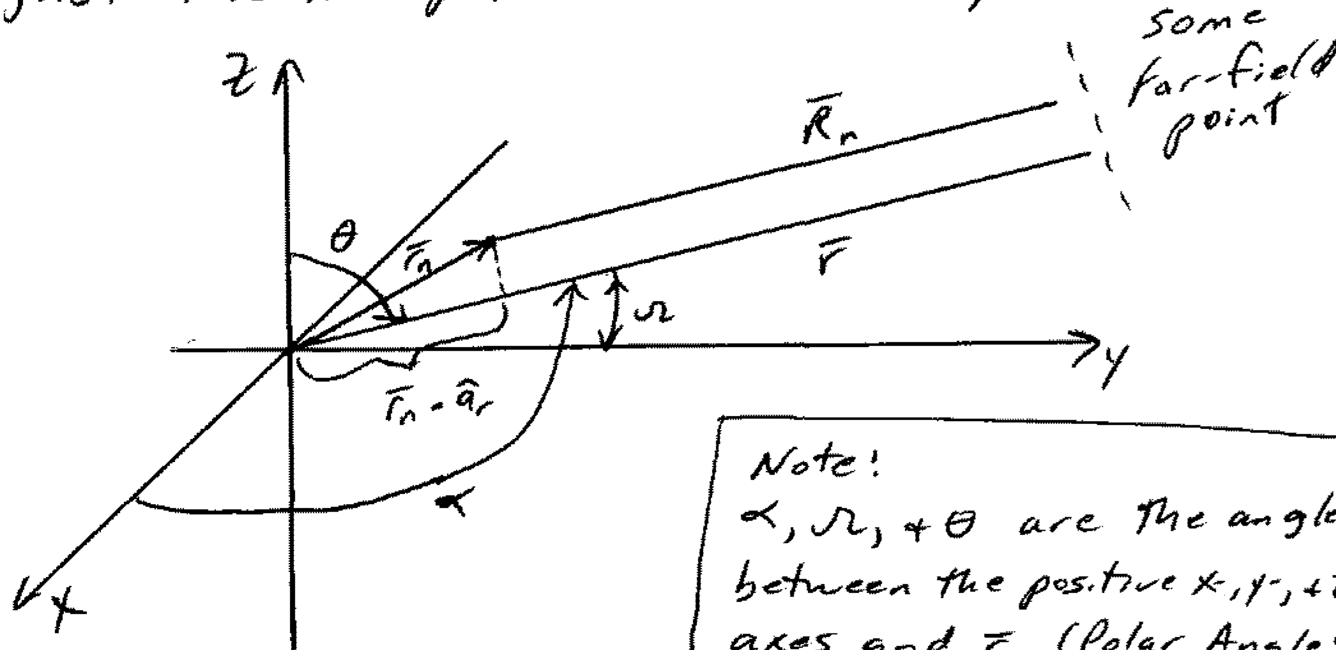
array of N point sources (assume all have same orientation)

Assuming no interaction between the point sources, the far-field $\bar{E}$ is simply the superposition of the far-fields for each point source

$$\bar{E} = \hat{a}\,\frac{j\omega\mu}{4\pi}\sum_{n=1}^{N} I_n\,\frac{e^{-jkR_n}}{R_n}$$

Note: Each point source in array can have a unique location (x_n, y_n, z_n), amplitude $|I_n|$ & phase $\measuredangle I_n$. Therefore, the far-field $\bar{E}$ is a function of $\underline{5N}$ variables

In the far-field, we can make some approximations (similar to chap 4 & 5) to make our analysis easier. Looking at just the n^{th} point in our array



Note!
α, Ω, & θ are the angles between the positive x, y, & z-axes and $\bar{r}$ (Polar Angles)

If we are in the far-field, we can assume that R_n & $\bar{r}$ are parallel.

Then,

$$R_n = r - (\bar{r}_n \cdot \hat{a}_r)$$

projection of $\bar{r}_n$ onto $\hat{a}_r$

$$= r - \left[\left(\hat{a}_x x_n + \hat{a}_y y_n + \hat{a}_z z_n\right) \cdot \left(\frac{\hat{a}_x x + \hat{a}_y y + \hat{a}_z z}{r}\right)\right]$$

$$= r - \left(x_n \frac{x}{r} + y_n \frac{y}{r} + z_n \frac{z}{r}\right)$$

$$\boxed{R_n = r - (x_n \cos\alpha + y_n \cos\Omega + z_n \cos\theta)}$$

$\cos\alpha$, $\cos\Omega$, & $\cos\theta$ are known as the <u>direction cosines</u> of $\bar{r}$

From vector algebra, we can express the direction cosines in terms of spherical coordinates:

$$\cos\alpha = \frac{x}{r} = \sin\theta\cos\phi$$

$$\cos\Omega = y/r = \sin\theta\sin\phi$$

$$\cos\theta = z/r = \cos\theta$$

So, for <u>phase</u>, we will use this approx. for R_n. For amplitude (much less sensitive), we will use $R_n \simeq r$

Going back to the total field $\bar{E}$, we get

$$\bar{E} = \hat{a}\frac{j\omega\mu}{4\pi}\sum_{n=1}^{N} I_n \frac{e^{-jk(r - x_n\cos\alpha - y_n\cos\Omega - z_n\cos\theta)}}{r}$$

$$= \underbrace{\left[\frac{\hat{a}\, j\omega\mu e^{-jkr}}{4\pi r}\right]}_{\text{Element Pattern of the point sources}} \underbrace{\left[\sum_{n=1}^{N} I_n e^{jk(x_n\cos\alpha + y_n\cos\Omega + z_n\cos\theta)}\right]}_{\text{Array Factor}}$$

In general, we can write the antenna pattern for <u>any</u> antenna in the far-field as

$$\bar{E}(r,\theta,\phi) = \bar{E}_{FF}(\theta,\phi)\, I \frac{e^{-jkr}}{r}$$

Assuming all the antennas are identically oriented (i.e., θ + ϕ have same meaning for each element) in an N-element array w/ phasor excitations I_n, the overall field will be

$$\boxed{\bar{E} = \underbrace{\left[\bar{E}_{FF}(\theta,\phi)\frac{e^{-jkr}}{r}\right]}_{\text{Element Pattern}}\underbrace{\left[\sum_{n=1}^{N} I_n e^{jk(x_n\cos\alpha + y_n\cos\Omega + z_n\cos\theta)}\right]}_{\text{Array Factor}}}$$

Element Pattern
→ element type/pattern
→ orientation
→ polarization

Array Factor
→ excitation info
→ location info

Note how the array factor did not change.

What if we have a linear array along the z-axis? Then, $x_n = y_n = 0$ which greatly simplifies our expression for $\bar{E}$.

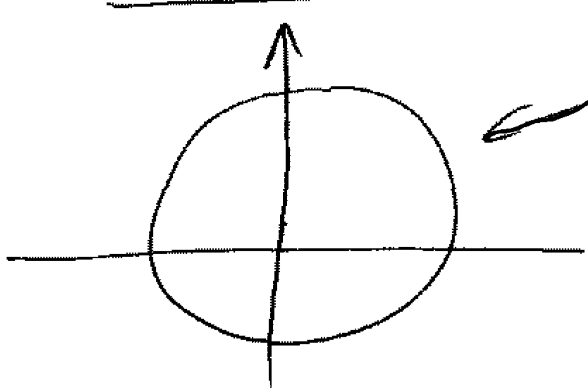
Let's look at some examples for point source arrays located along the z-axis

$$E = \left[\frac{j\omega\mu}{4\pi} \frac{e^{-jkr}}{r} \right] \left[\sum_{n=1}^{N} I_n e^{jkz_n \cos\theta} \right]$$

$$E_{norm} = \frac{E}{\text{element pattern}} = \sum_{n=1}^{N} I_n e^{jkz_n \cos\theta}$$

Note: No ϕ dependence!

Element Pattern



just a sphere, no θ or ϕ dependence

Linear Array of Point Sources Example (Fall 2003)

$k := 0..359$ $\qquad\qquad$ $theta_k := k \cdot \frac{\pi}{180}$

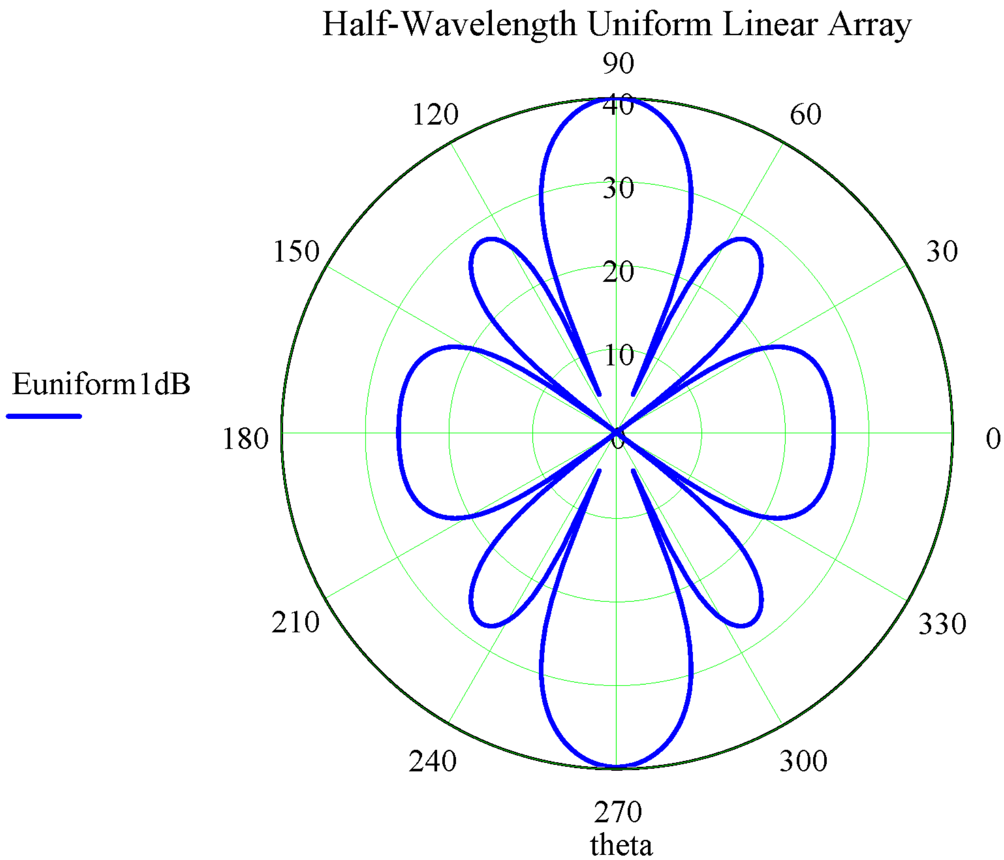
$N := 5$ $\qquad$ $n := 0..(N-1)$ $\qquad$ 5 point source elements in linear array

$kz1_n := 2 \cdot \pi \cdot n \cdot 0.5$ $\qquad$ Elements spaced every half-wavelength starting at $z = 0$

Uniform current distribution (same magnitude and phase)

$Iuniform_n := 1$ $\qquad$ $$Euniform1_k := \sum_{n=0}^{N-1} Iuniform_n \cdot e^{j \cdot kz1_n \cdot \cos(theta_k)}$$

$$Euniform1dB_k := if\left(\frac{|Euniform1_k|}{|Euniform1_{90}|} < 0.01, 0, 40 + 20 \cdot \log\left(\frac{|Euniform1_k|}{|Euniform1_{90}|}\right)\right)$$



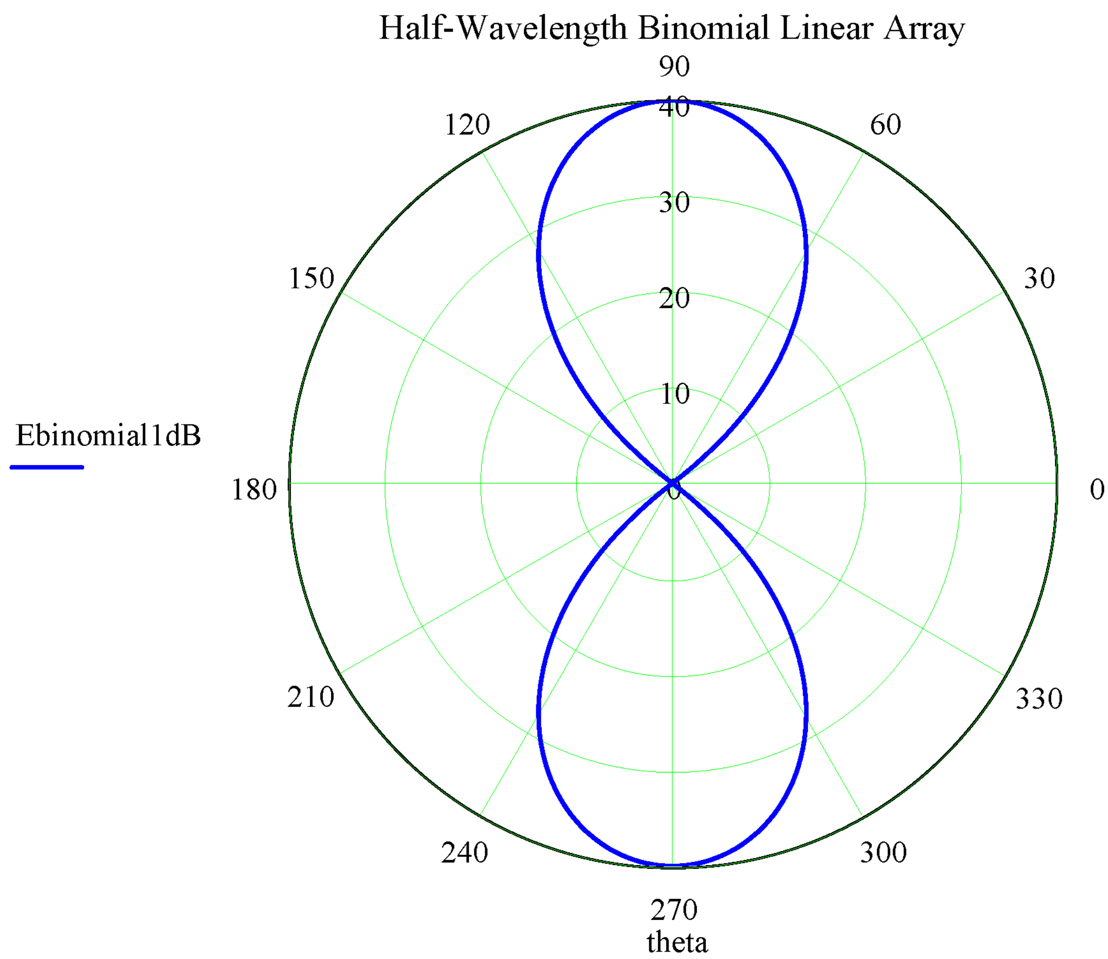
Note: 3 dB beamwidth = 20.3 degrees, Maximum sidelobe level = -12 dB

<u>Binomial current distribution (different magnitudes but same phase)</u>

$\text{Ibinomial}_0 := 1 \qquad \text{Ibinomial}_1 := 4 \qquad \text{Ibinomial}_2 := 6 \qquad \text{Ibinomial}_3 := 4 \qquad \text{Ibinomial}_4 := 1$

$$\text{Ebinomial1}_k := \sum_{n=0}^{N-1} \text{Ibinomial}_n \cdot e^{j \cdot kz1_n \cdot \cos(\text{theta}_k)}$$

$$\text{Ebinomial1dB}_k := \text{if}\left(\frac{\left|\text{Ebinomial1}_k\right|}{\left|\text{Ebinomial1}_{90}\right|} < 0.01, 0, 40 + 20 \cdot \log\left(\frac{\left|\text{Ebinomial1}_k\right|}{\left|\text{Ebinomial1}_{90}\right|}\right)\right)$$



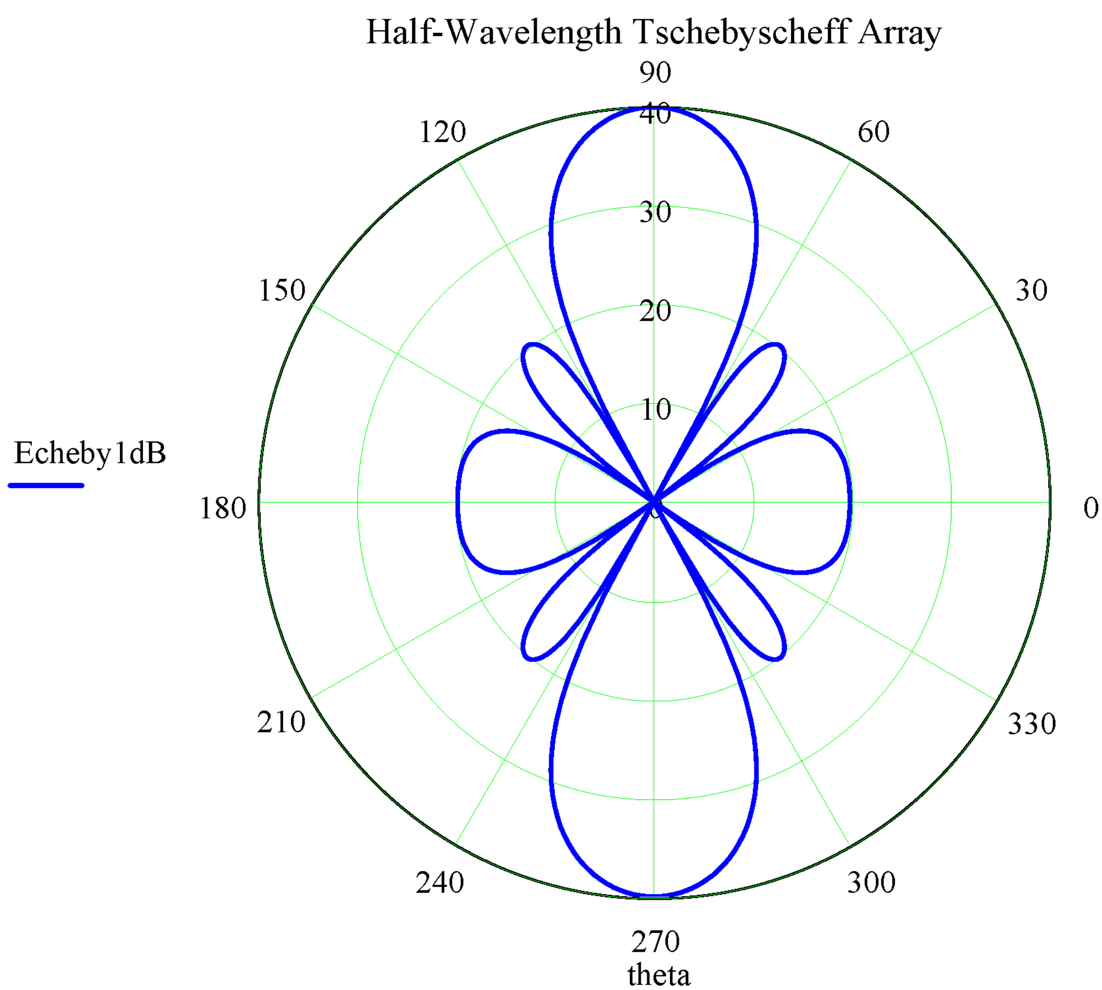
Note: 3 dB beamwidth = 30 degrees, Maximum sidelobe level = -infinity

Tschebyscheff current distribution (different magnitudes but same phase)

$Icheby_0 := 1$ $Icheby_1 := 1.6$ $Icheby_2 := 1.9$ $Icheby_3 := 1.6$ $Icheby_4 := 1$

$$Echeby1_k := \sum_{n=0}^{N-1} Icheby_n \cdot e^{j \cdot kz1_n \cdot \cos(theta_k)}$$

$$Echeby1dB_k := \text{if}\left(\frac{|Echeby1_k|}{|Echeby1_{90}|} < 0.01, 0, 40 + 20 \cdot \log\left(\frac{|Echeby1_k|}{|Echeby1_{90}|}\right)\right)$$



Note: 3 dB beamwidth = 23 degrees, Maximum sidelobe level = -19.7 dB

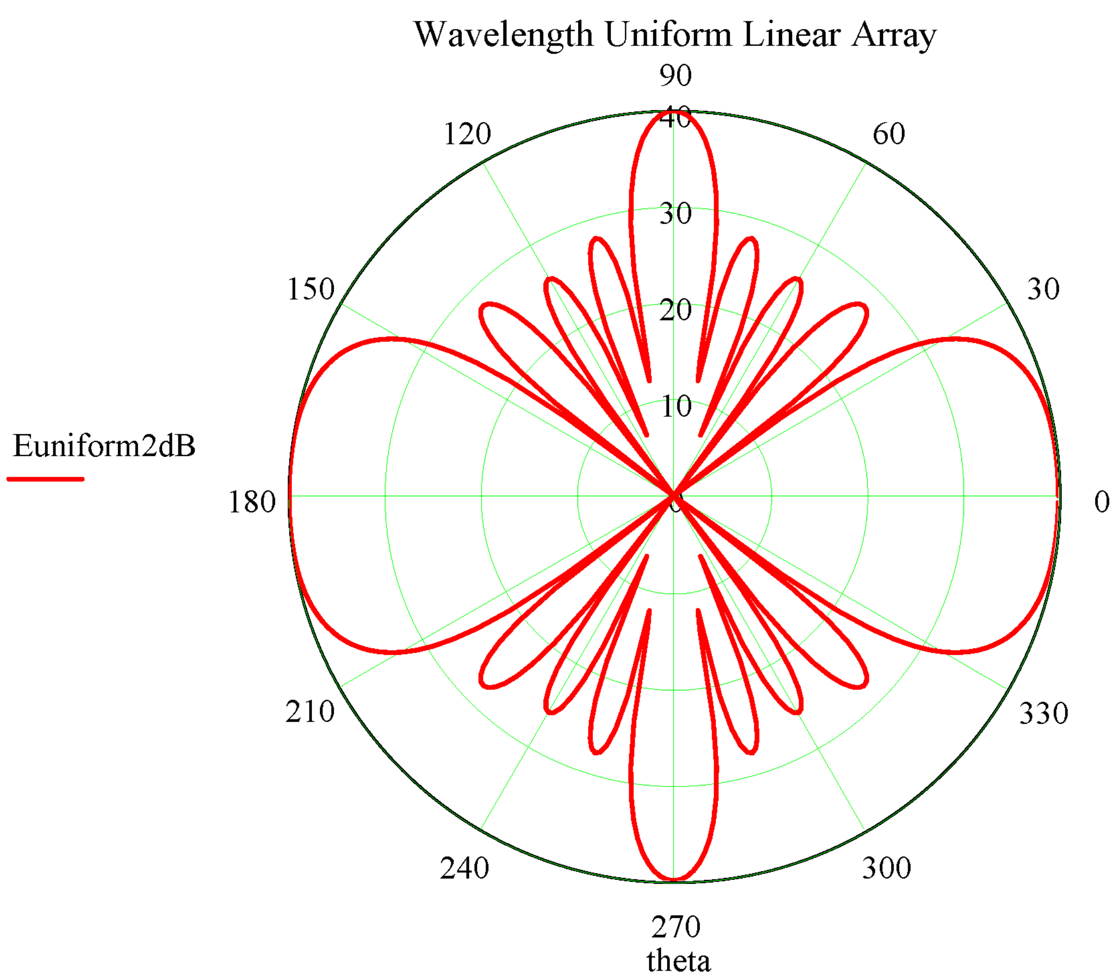
Elements spaced every wavelength starting at *z* = 0

$$kz2_n := 2 \cdot \pi \cdot n \cdot 1.0$$

Uniform current distribution (same magnitude and phase)

$$Euniform2_k := \sum_{n=0}^{N-1} Iuniform_n \cdot e^{j \cdot kz2_n \cdot \cos(theta_k)}$$

$$Euniform2dB_k := if\left(\frac{|Euniform2_k|}{|Euniform2_0|} < 0.01, 0, 40 + 20 \cdot \log\left(\frac{|Euniform2_k|}{|Euniform2_0|}\right)\right)$$

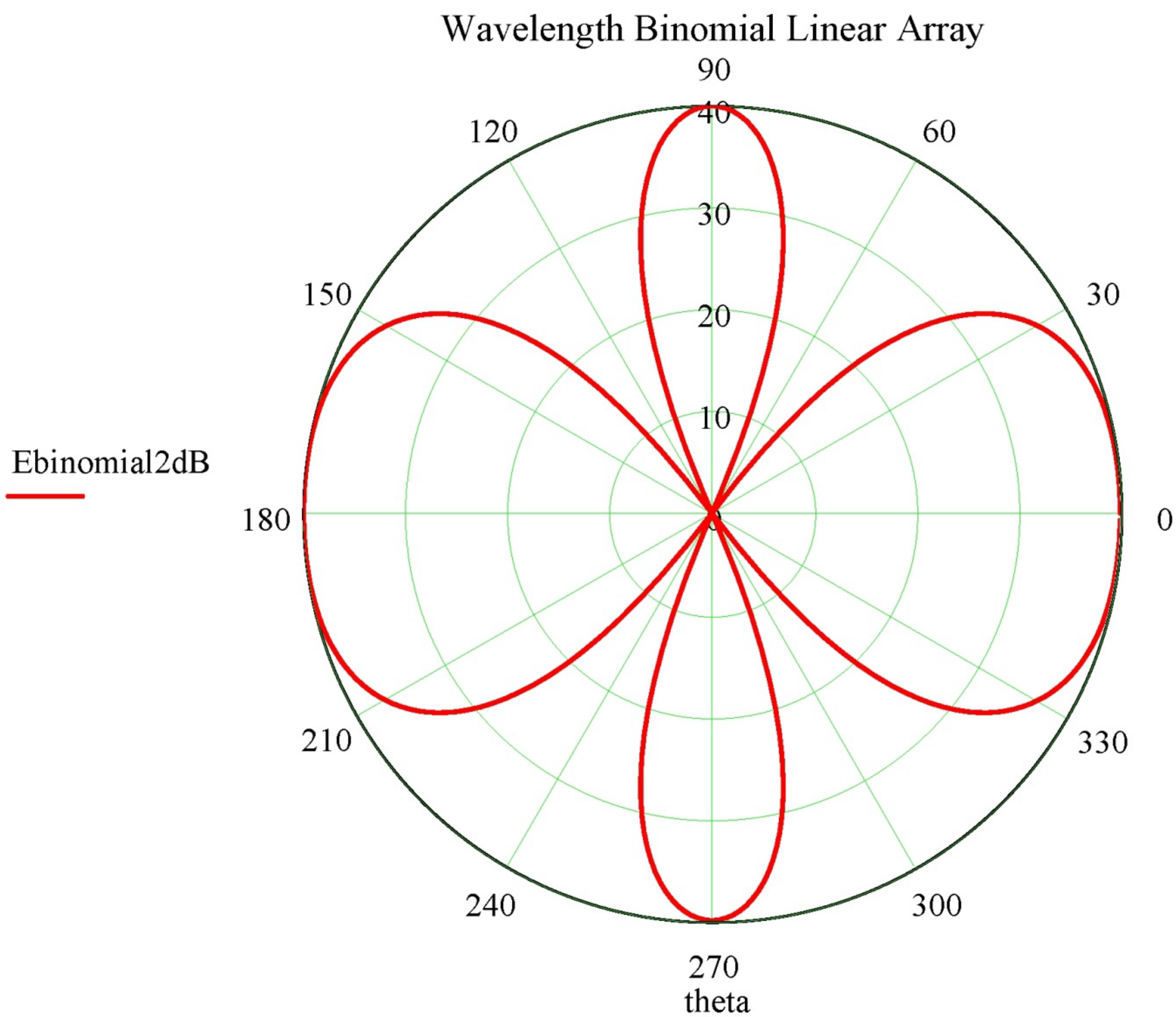


Note: 3 dB beamwidth = 49 degrees, Maximum sidelobe level = 0 dB ("Grating" Lobes)

Binomial current distribution (different magnitudes but same phase)

$$\text{Ebinomial2}_k := \sum_{n=0}^{N-1} \text{Ibinomial}_n \cdot e^{j \cdot kz2_n \cdot \cos(\text{theta}_k)}$$

$$\text{Ebinomial2dB}_k := \text{if}\left(\frac{|\text{Ebinomial2}_k|}{|\text{Ebinomial2}_0|} < 0.01, 0, 40 + 20 \cdot \log\left(\frac{|\text{Ebinomial2}_k|}{|\text{Ebinomial2}_0|}\right)\right)$$

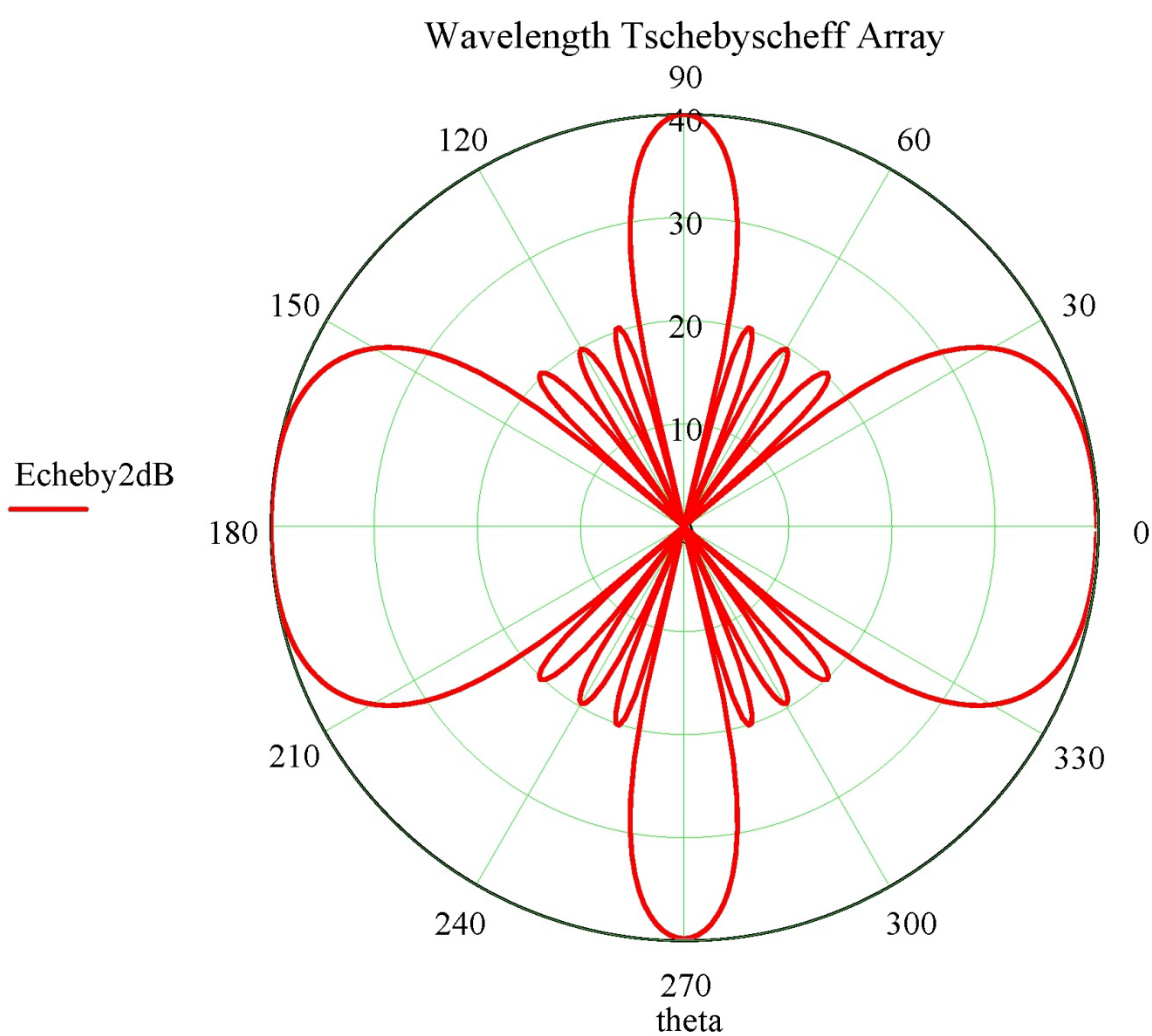


Note: 3 dB beamwidth = 59 degrees, Maximum sidelobe level = 0 dB ("Grating" Lobes)

Tschebyscheff current distribution (different magnitudes but same phase)

$$Echeby2_k := \sum_{n=0}^{N-1} Icheby_n \cdot e^{j \cdot kz2_n \cdot \cos(theta_k)}$$

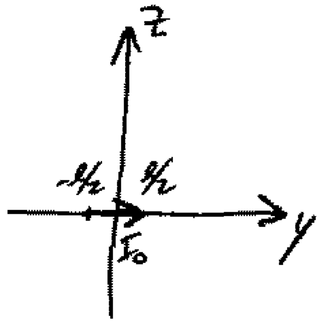
$$Echeby2dB_k := if\left(\frac{|Echeby2_k|}{|Echeby2_0|} < 0.01, 0, 40 + 20 \cdot \log\left(\frac{|Echeby2_k|}{|Echeby2_0|}\right)\right)$$



Note: 3 dB beamwidth = 52 degrees, Maximum sidelobe level = 0 dB ("Grating" Lobes)

6.2 Two-Element Array

Infinitesimal dipole (pointing in y-direction)



$$\bar{I}_e = \hat{a}_y I_o \qquad \bar{r}' = y'\hat{a}_y$$

$$\bar{r} = r\hat{a}_r \qquad |\bar{r}-\bar{r}'| \simeq r$$

$$\bar{A} = \hat{a}_y \frac{\mu_o I_o l}{4\pi r} e^{-jkr}$$

$$\bar{A} = \frac{\mu_o I_o l}{4\pi r} e^{-jkr}\left[\sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi\right]$$

In the far-field

$$\bar{E} = -j\omega A_\theta \hat{a}_\theta - j\omega A_\phi \hat{a}_\phi \qquad (3\text{-}58a)$$

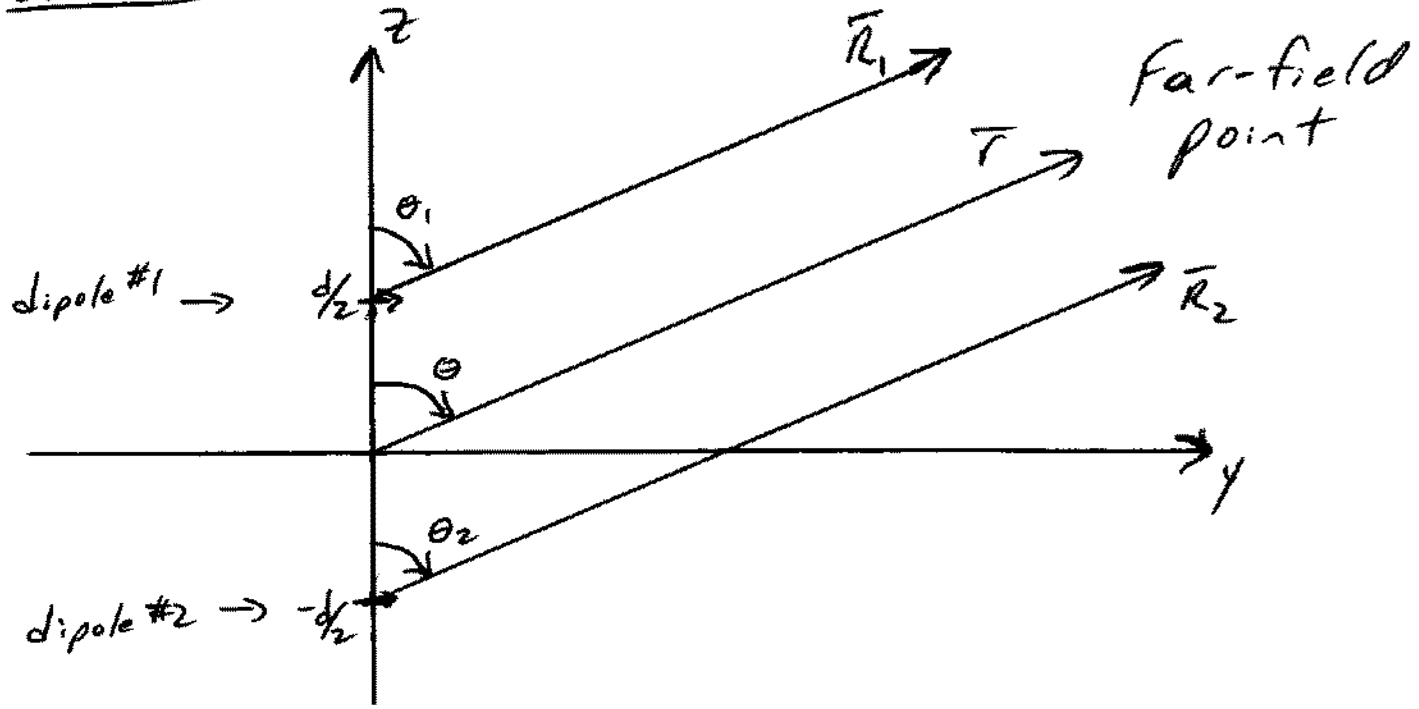
$$= \frac{-j\omega\mu_o I_o l}{4\pi r} e^{-jkr}\left[\cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi\right]$$

use $\omega\mu_o = k\eta$

$$\bar{E} = -j\eta\frac{k I_o l}{4\pi r} e^{-jkr}\left[\cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi\right]$$

Now, let's build a 2-element array of infinitesimal dipoles in the y-z plane placed along the z-axis at $z = -d/2$ and $d/2$. We'll let $\bar{I}_1 = \hat{a}_y I_o e^{+j\beta/2}$ and $\bar{I}_2 = \hat{a}_y I_o e^{-j\beta/2}$ (phase difference of β between the dipoles exciting currents, same magnitude)

6.2 cont.



→ Since elements are along z-axis, $x_n\cos\alpha = y_n\cos\gamma = 0$

$$\bar{E} = \left[\left(\frac{-j\eta k \ell}{4\pi r}\right) e^{-jkr}\left[\cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi\right]\right]\left[\begin{array}{l} I_0 e^{j\beta/2} e^{jk\frac{d}{2}\cos\theta} \\ + I_0 e^{-j\beta/2} e^{-jk\frac{d}{2}\cos\theta} \end{array}\right]$$

Element Factor   Array factor

→ Considering only the plane $\phi = \pi/2$ (positive half of y-z plane)

$$\bar{E} = \left[\hat{a}_\theta \frac{-j\eta k I_0 \ell}{4\pi r} e^{-jkr}\cos\theta\right]\left[e^{j\frac{1}{2}(kd\cos\theta+\beta)} + e^{-j\frac{1}{2}(kd\cos\theta+\beta)}\right]$$

Field of single element located @ origin   Array factor (AF)

6.2 cont.

Use the identity that $\cos A = \dfrac{e^{+jA} + e^{-jA}}{2}$

to get:

$$\bar{E} = \left[\hat{a}_\theta \frac{-j\eta k I_o \ell}{4\pi r} e^{-jkr} \cos\theta\right]\left[2\cos\left(\tfrac{1}{2}(kd\cos\theta + \beta)\right)\right]$$

→ By controlling the spacing d and phase β between the elements, the array factor can be manipulated to give desired results.

One item of interest when designing arrays is where the nulls ($|\bar{E}| = 0$) occur.

$$\frac{|\bar{E}|}{|\bar{E}|_{max}} = |\cos\theta|\left|\cos\left[\tfrac{1}{2}(kd\cos\theta + \beta)\right]\right| = E_n$$

normalized electric field magnitude

To find nulls (located at θ_n)

$$E_n = 0 = |\cos\theta_n|\left|\cos\left[\tfrac{1}{2}(kd\cos\theta_n + \beta)\right]\right|$$

From the $\cos\theta_n$ term of the element factor

$$\cos\theta_n = 0 \Rightarrow \theta_n = 90^\circ$$

6.2 cont.

From the array factor

$$\cos\left[\tfrac{1}{2}\left(kd\cos\theta_n+\beta\right)\right]=0$$

when $\tfrac{1}{2}\left(kd\cos\theta_n+\beta\right)=\pm\pi/2,\ \pm3\pi/2,\ \ldots$

$$=\pm\left(\frac{2n+1}{2}\right)\pi \quad n=0,1,2,\ldots$$

$$kd\cos\theta_n+\beta=\pm(2n+1)\pi$$

$$\cos\theta_n=\frac{-\beta\pm(2n+1)\pi}{kd}=\frac{-\beta\pm(2n+1)\pi}{\left(2\pi d/\lambda\right)}$$

$$\underline{\underline{\theta_n=\cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta\pm(2n+1)\pi\right)\right]}}$$

Be Careful, depending on the spacing d and/or phasing β, the argument can be >1 or <-1 which implies that there are no nulls

$d\geq\lambda/2$ in order to get a real null when $n=0$

e.g. $\beta=0$, $d=\lambda/4$

$$\theta_n=\cos^{-1}\left[\frac{\lambda}{2\pi d}\left(\pm(2n+1)\right)\pi\right]$$

$$=\cos^{-1}\left[2\left(\pm(2n+1)\right)\right]\Rightarrow \underline{\underline{\text{No sol'n}}}$$

only null $\underline{\underline{\theta_n=90^\circ}}$ from element factor

6.2 cont.

ex. $\beta = \pi/2$, $d = \lambda/4$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\pi/2 \pm (2n+1)\pi\right)\right]$$

$$= \cos^{-1}\left[2\left(-1/2 \pm (2n+1)\right)\right]$$

$$= \cos^{-1}\left[-1 \pm (4n+2)\right]$$

not possible

$n=0$ $\quad \theta_n = \cos^{-1}\left[-1 \pm 2\right] = \cos^{-1}(1)$ or $\cancel{\cos^{-1}(-3)}$

$\theta_n = 0$

$n=1$ $\quad \theta_n = \cancel{\cos^{-1}\left[-1 \pm 6\right]}$ Not possible

So $\theta_n = \underbrace{0°}_{\text{Array}}$ and $\underbrace{90°}_{\text{element factor}}$

ex. $\beta = -\pi/2$, $d = \lambda/4$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(+\pi/2 \pm (2n+1)\pi\right)\right]$$

$$= \cos^{-1}\left[2\left(1/2 \pm (2n+1)\right)\right]$$

$$= \cos^{-1}\left[1 \pm (4n+2)\right]$$

not possible

$n=0$ $\quad \theta_n = \cos^{-1}\left[1 \pm 2\right] = \cos^{-1}(-1)$ or $\cancel{\cos^{-1}(3)}$

$\theta_n = 180°$

So $\theta_n = \underbrace{90°}_{\text{Element}}$ and $\underbrace{180°}_{\text{Array}}$

6.2 cont

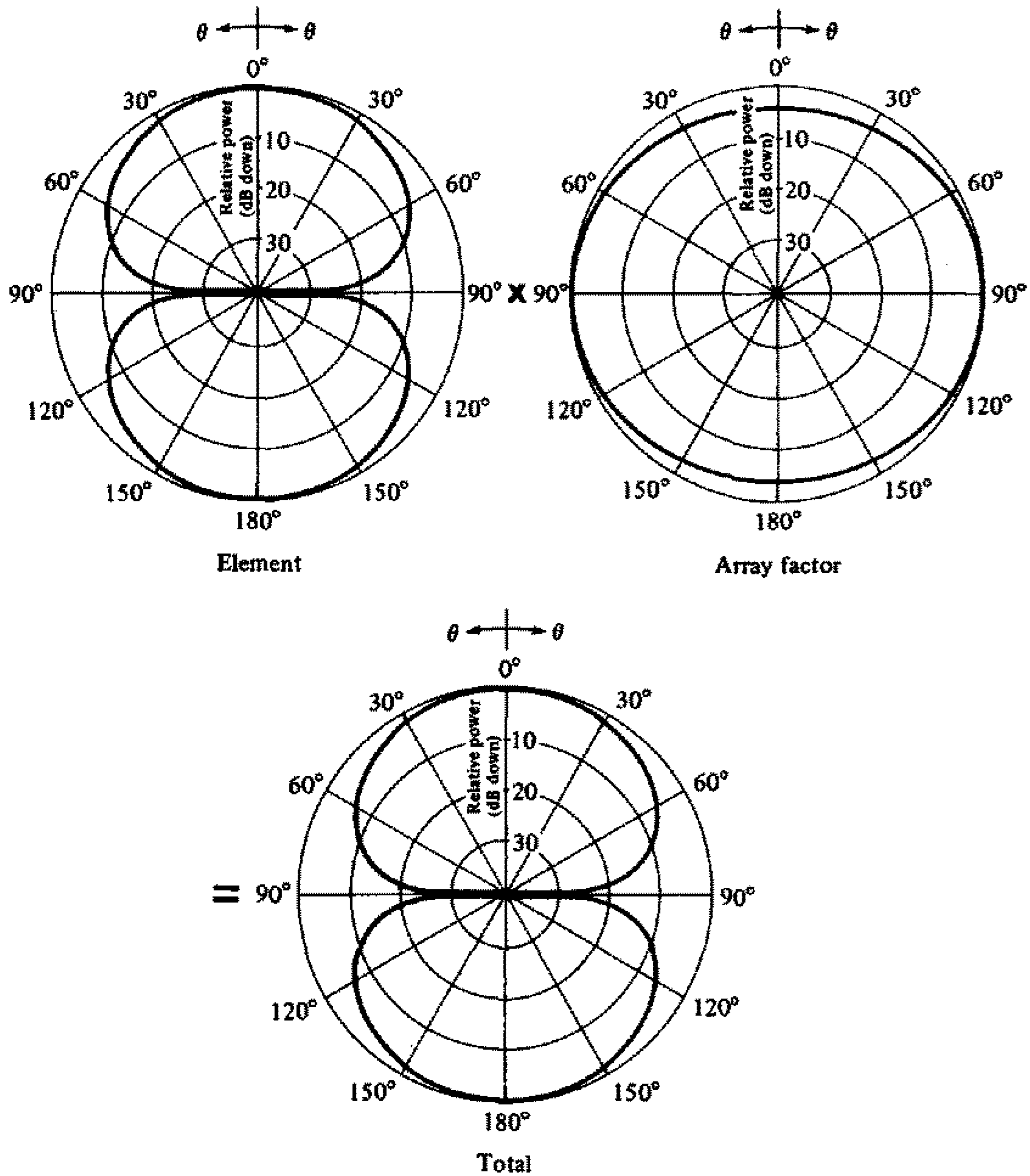


Figure 6.3 Element, array factor, and total field patterns of a two-element array of infinitesimal horizontal dipoles with identical phase excitation ($\beta = 0°$, $d = \lambda/4$).

[Antenna Theory (Fourth Edn) by Balanis]

$\beta = 0$ & $d = \lambda/4$

6.2 cont.

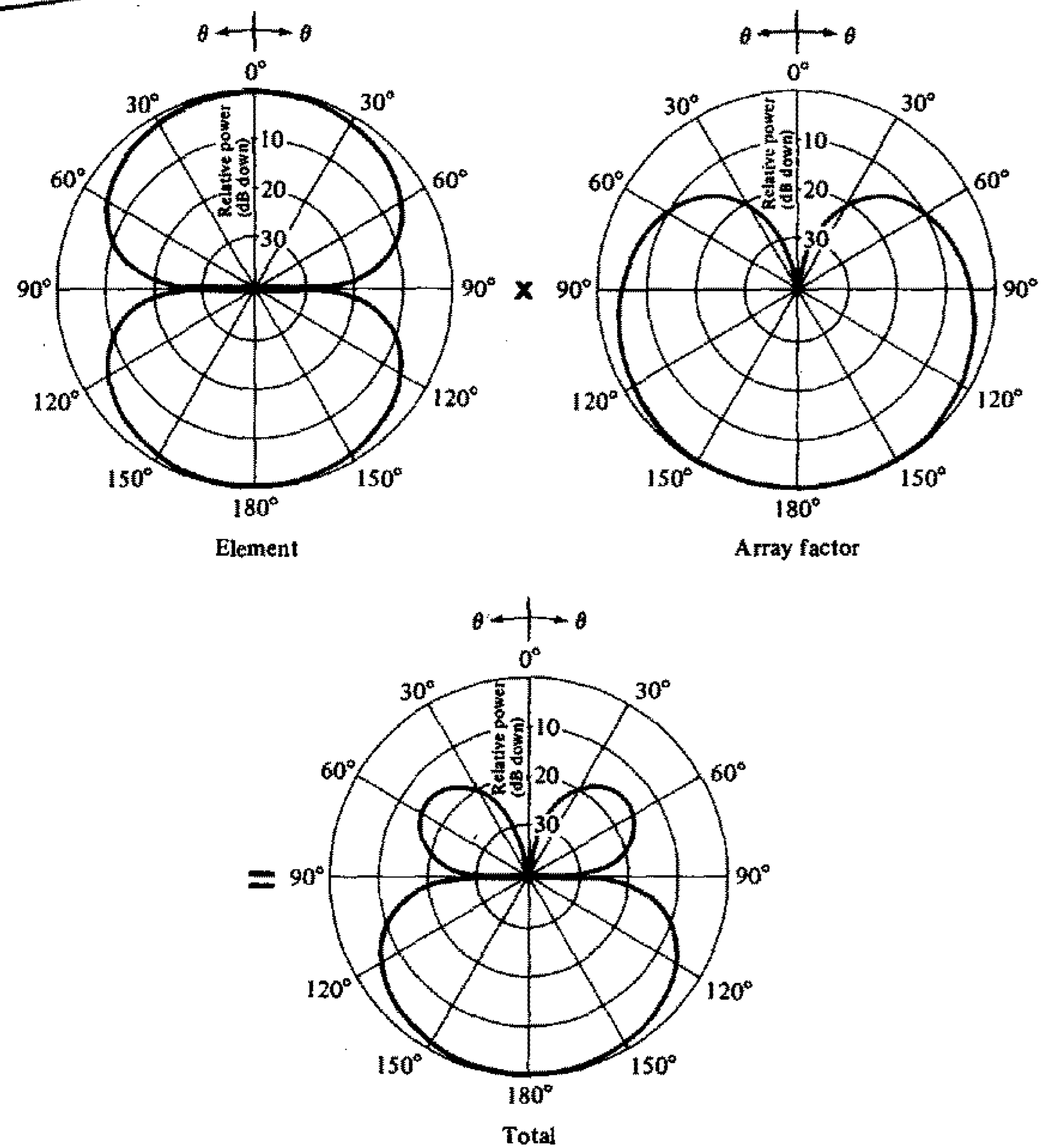


Figure 6.4 Pattern multiplication of element, array factor, and total array patterns of a two-element array of infinitesimal horizontal dipoles with (a) $\beta = +90°$, $d = \lambda/4$. [Antenna Theory (Fourth Edn) by Balanis]

$\beta = +90°$ & $d = \lambda/4$

6,2 cont.

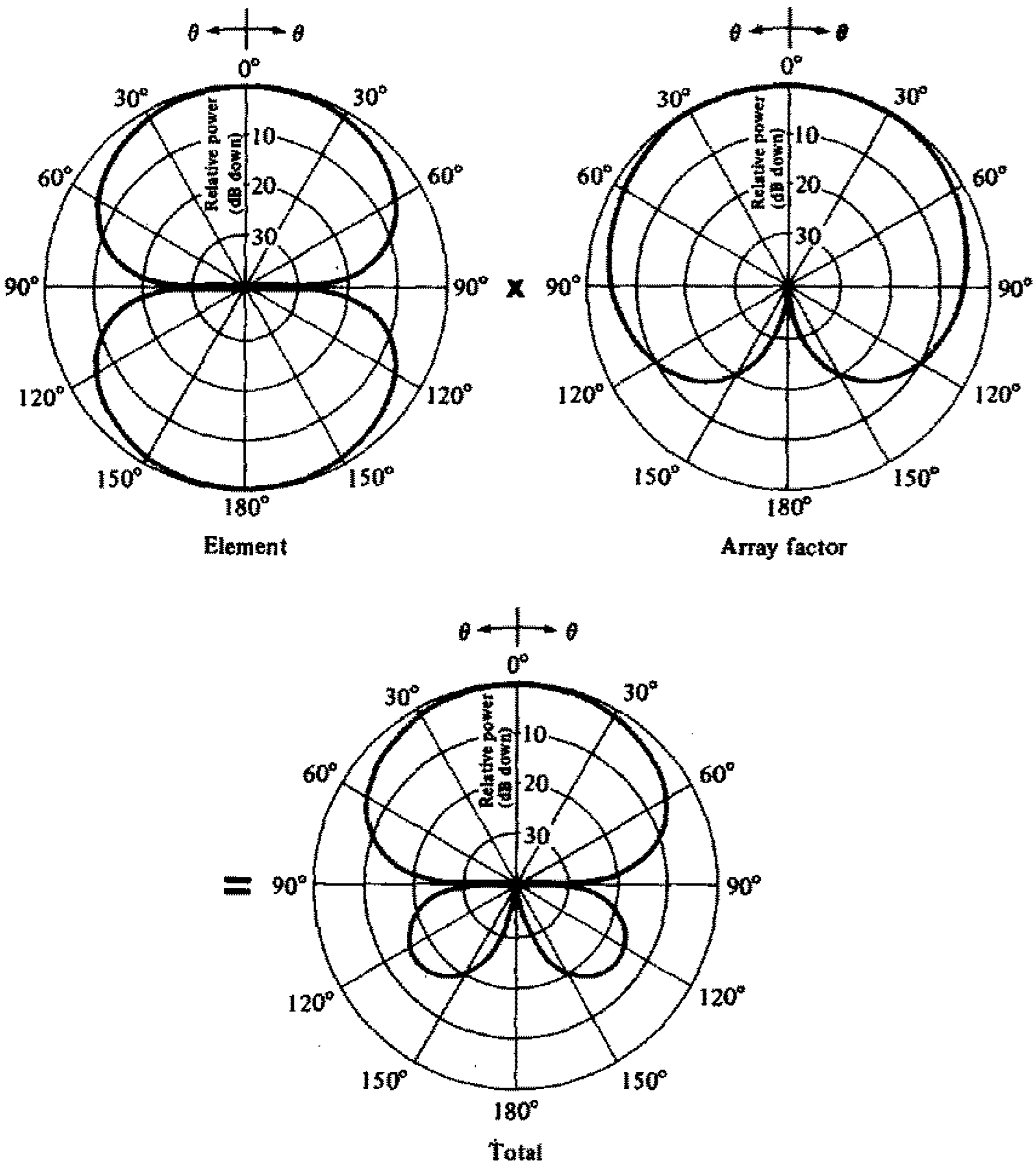


Figure 6.4 (b) Continued ($\beta = -90°$, $d = \lambda/4$).

[Antenna Theory (Fourth Edn) by Balanis]

$\beta = -90°$ & $d = \lambda/4$

6.3 N-Element Linear Array: Uniform Amplitude and Spacing

Characteristics:

→ located along z-axis

→ uniform spacing $d \Rightarrow z_n = (n-1)d$

→ uniform excitation amplitude $|I_n| = I_o$

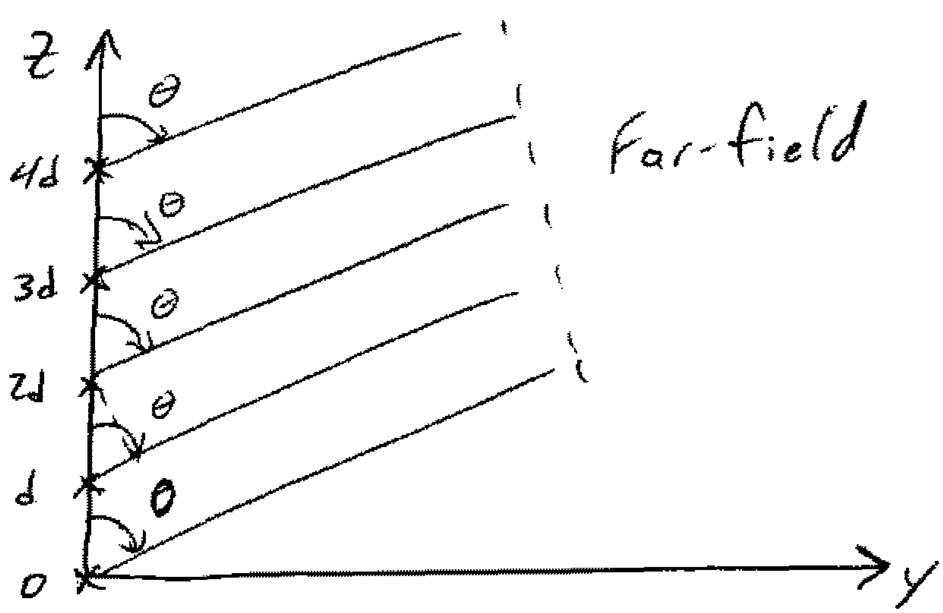
→ linearly progressive phase $\measuredangle I_n = (n-1)\beta$

$$AF = \sum_{n=1}^{N} I_o e^{+j(n-1)\beta} e^{jk(n-1)d\cos\theta}$$

$$= I_o \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)}$$

$$= I_o \sum_{n=1}^{N} e^{j(n-1)\psi}$$

where we define $\psi = kd\cos\theta + \beta$



6.3 cont.

→ We can adjust / select N, d, & β to achieve desired results in AF.

$$\left.\begin{aligned} |AF|_{max} &= N I_0 \\ |AF|_{min} &= 0 \end{aligned}\right\} \text{Two extremes}$$

To get the angle where $|AF|_n = N I_0$, or to place the mainbeam of the array factor at a particular angle θ_{mB}, set

$$\psi = kd\cos\theta_{mB} + \beta = 0$$

$$\boxed{\beta = -kd\cos\theta_{mB}}$$

$$\boxed{\theta_{mB} = \cos^{-1}\left(\frac{-\beta}{kd}\right)}$$

ex. For $d = \lambda/4$ and $\beta = 0$, where is θ_{mB}?

$$\theta_{mB} = \cos^{-1}(0) = 90^\circ \text{ (Broadside)}$$

ex. If $d = \lambda/2$, what is β for $\theta_{mB} = 60^\circ$?

$$\beta = -\frac{2\pi}{\lambda}(\lambda/2)\underbrace{\cos 60^\circ}_{1/2} = -\pi/2 = -90^\circ$$

6.3 cont.

Before proceeding, there is a more compact expression for this array factor

$$(AF)\, e^{j\psi} = I_0\, e^{j\psi} \sum_{n=1}^{N} e^{j(n-1)\psi}$$

$$= I_0 \sum_{n=1}^{N} e^{jn\psi}$$

$$= I_0 \left(e^{j\psi} + e^{j2\psi} + \dots e^{jN\psi} \right)$$

Next, subtract the original AF from both sides

$$(AF)\, e^{j\psi} - AF = I_0 \left(e^{j\psi} + e^{j2\psi} + \dots e^{jN\psi} \right) - I_0 \left(1 + e^{j\psi} + e^{j2\psi} + \dots e^{j(N-1)\psi} \right)$$

$$AF\left(e^{j\psi} - 1 \right) = I_0 \left(e^{jN\psi} - 1 \right)$$

$$\boxed{AF = I_0 \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = I_0\, e^{j\left(\frac{N-1}{2}\right)\psi} \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)} \right]}$$

6.3 cont.

Also, if we move our array center from $z = \frac{N-1}{2} d$ to $z = 0$ (Now 1st element is at $\left(z = -\left(\frac{N-1}{2}\right) d\right)$ or the array goes from $z = -\left(\frac{N-1}{2}\right) d$ to $z = \left(\frac{N-1}{2}\right) d$ instead of $z = 0$ to $z = (N-1) d$

$$\boxed{AF = I_0 \left[\frac{\sin\left(\frac{N}{2} \psi\right)}{\sin\left(\psi/2\right)} \right]}$$

as $\psi \to 0$, $AF = N I_0$ (let $\sin(x) = x$ as $x \to 0$)

To normalize:

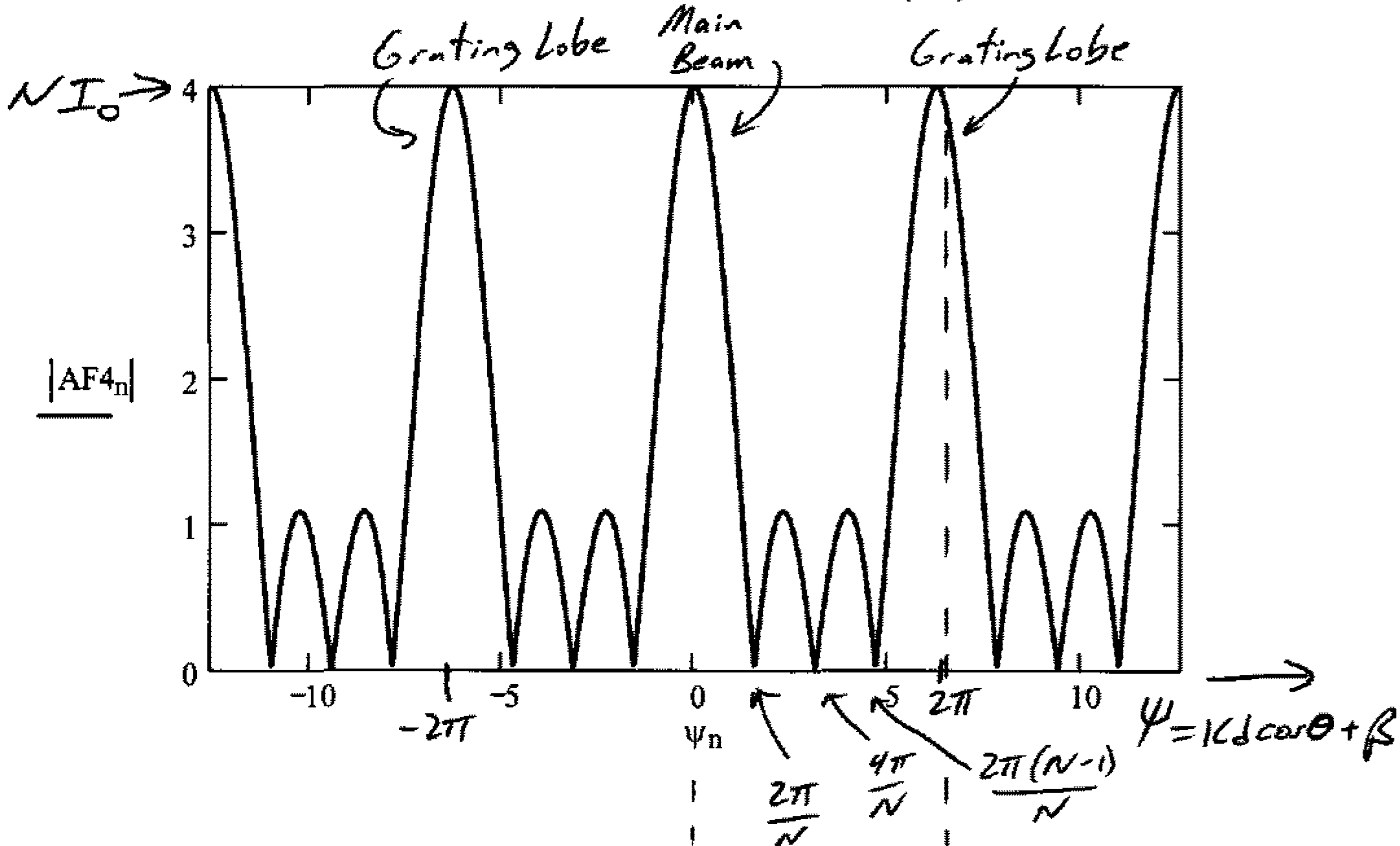
$$(AF)_n = \frac{AF}{N I_0} = 1/N \left[\frac{\sin\left(\frac{N}{2} \psi\right)}{\sin\left(\psi/2\right)} \right]$$

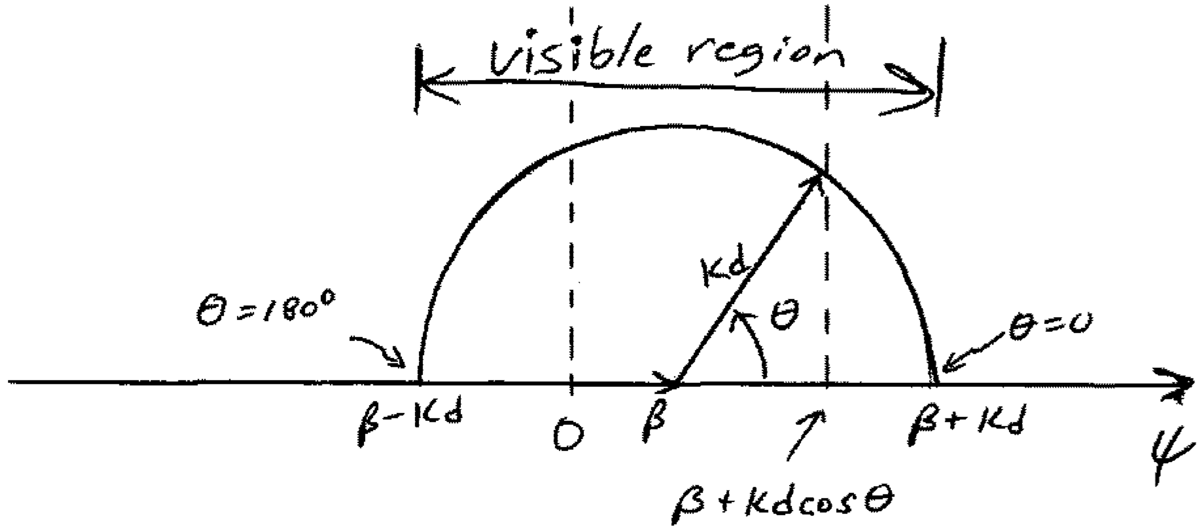
Uniform Array Factor (AF) example

$n := 0..900$ $\qquad \psi_n := (n-450)\cdot\frac{\pi}{111} + 0.001$ $\qquad I0 := 1$

avoid ÷ by 0

N=4 example $\qquad N := 4 \qquad AF4_n := \dfrac{\sin\left(\frac{N}{2}\cdot\psi_n\right)}{\sin\left(\frac{\psi_n}{2}\right)}$





6.3 cont.

General Notes on Uniform Arrays

1) β controls location of Main Beam
2) Increasing N decreases beamwidth
3) Increasing d decreases beamwidth. However, eventually this will result in grating lobes.
4) Maximum value of uniform AF occurs when $\psi = 0$
5) First nulls of uniform AF @ $\psi = \pm 2\pi/N$

   Second nulls of uniform AF @ $\psi = \pm \frac{4\pi}{N}$

   ⋮
6) Last nulls of uniform AF @ $\psi = \pm\left(2\pi - 2\pi/N\right)$

$$= \pm 2\pi\left(1 - 1/N\right)$$

$$= \pm 2\pi\left(\frac{N-1}{N}\right)$$

7) Uniform AF repeats @ $\pm 2\pi$ intervals in ψ

6.3 cont.

Notes cont.

8) Maximum sidelobe level ~ 0.25 (-12dB)

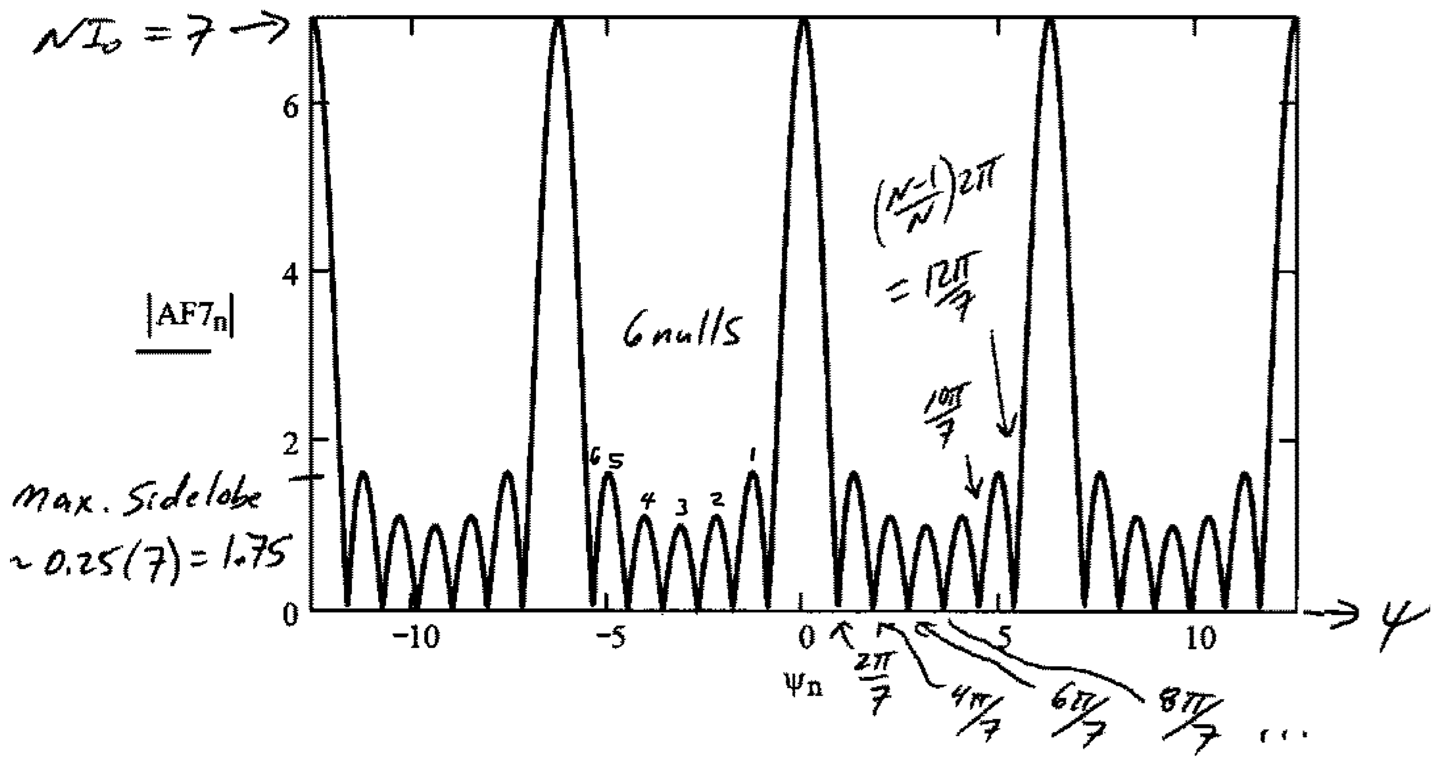
9) N-1 nulls between MB peak ($\psi=0$) and first grating lobes ($\psi = \pm 2\pi$)

10) N-2 sidelobes between MB peak and first grating lobe peak(s)

11) In terms of ψ, the MB is twice as wide as each sidelobe.

N=7 example $N := 7$

$n := 0..900$ $\psi_n := (n - 450)\cdot\frac{\pi}{111} + 0.001$ $I0 := 1$ $AF7_n := \frac{\sin\left(\frac{N}{2}\cdot\psi_n\right)}{\sin\left(\frac{\psi_n}{2}\right)}$



→ Main Beam narrower than $N = 4$ case

→ # nulls $= 6 = N-1 = 7-1$

→ # sidelobes $= 5 = N-2 = 7-2$

6.3 cont.

Let's find the angle(s) θ_n where the AF has nulls. From $\sin\left(\frac{N}{2}\psi_n\right) = 0$; $\frac{N}{2}\psi_n = \pm n\pi$

$$\psi_n = \pm n\frac{2\pi}{N} = kd\cos\theta_n + \beta \qquad n = 1, 2, \ldots$$

$$\hookrightarrow \boxed{\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm n\frac{2\pi}{N}\right)\right] \quad \begin{matrix} n = 1, 2, 3, \ldots \\ n \neq N, 2N, \ldots \end{matrix}}$$

At what angle(s) θ_m does the AF have maxima? From the $\sin(\frac{1}{2}\psi)$ term of the AF:

$$\psi_m = \pm 2m\pi \quad \text{OR} \qquad m = 0 \text{ Main Beam}$$

$$\tfrac{1}{2}\psi_m = \tfrac{1}{2}(kd\cos\theta_m + \beta) = \pm m\pi \qquad m = 1, 2, 3, \ldots \text{ Grating Lobes}$$

$$\hookrightarrow \boxed{\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm 2m\pi\right)\right] \quad m = 0, 1, \ldots}$$

What about half-power or 3 dB angles θ_h of AF?

$$\frac{N}{2}\psi_h = \frac{N}{2}(kd\cos\theta_h + \beta) = \pm 1.391$$

$$\hookrightarrow \boxed{\theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]}$$

for Main Beam

6.3 cont.

$$\boxed{HPBW = 2\left|\theta_{mB} - \theta_h\right| \quad \left(\text{assume symmetrical pattern}\right)}$$

half-power beamwidth of AF

ex. For $N = 4$, $d = \lambda/3$, and $\beta = \pi/2$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi(\lambda/3)}\left(-\pi/2 \pm n\frac{2\pi}{4}\right)\right] \qquad n = 1, 2, 3, \ldots \;\; \neq 4, 8, \ldots$$

$$= \cos^{-1}\left[3/2\left(-1/2 \pm n\frac{1}{2}\right)\right]$$

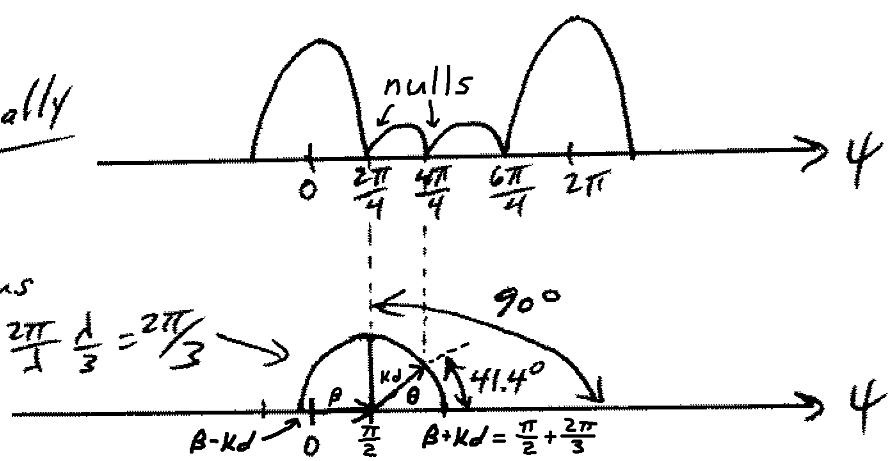
$n=1$ $\quad \theta_n = \cos^{-1}\left[3/2\left(-1/2 \pm 1/2\right)\right] = \underbrace{\cos^{-1}(0)}_{\to 90^\circ}$ or $\cancel{\cos^{-1}(-3/2)}$ Not possible

$n=2$ $\quad \theta_n = \cos^{-1}\left[3/2\left(-1/2 \pm 1\right)\right] = \underbrace{\cos^{-1}(3/4)}_{\to 41.4^\circ}$ $\cancel{\cos^{-1}(-9/4)}$ Not possible

$n=3$ $\quad \theta_n = \cos^{-1}\left[3/2\left(\cancel{-1/2 \pm 3/2}\right)\right]$ not possible

$\underline{\theta_n = 41.4^\circ \text{ and } 90^\circ}$

Graphically



6.3 cont.

ex. cont.

$$\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi(\lambda/3)}\left(-\pi/2 \pm 2m\pi\right)\right] \quad m = 0, 1, 2, \ldots$$

$$= \cos^{-1}\left[3/2\left(-1/2 \pm 2m\right)\right]$$

Main Beam (m=0) $\theta_m = \cos^{-1}(-3/4) = 138.59^\circ$ ← agrees w/ graphical method

m=1 $\theta_m = \cos^{-1}\left[3/2\left(\cancel{-1/2 \pm 2}\right)\right]$ Not possible

$$\theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi(\lambda/3)}\left(-\pi/2 \pm \frac{2.782}{4}\right)\right]$$

$$= \cos^{-1}(-0.41792311) \text{ or } \cos^{-1}(\cancel{-1.082})$$

$\theta_h = 114.7035^\circ$

$$HPBW = 2\,|138.59 - 114.7035^\circ|$$

$HPBW = 47.774^\circ$

6.3 cont. Summary

For **any** Array w/ uniform amplitude + spacing + linearly progressive phase

Main Beam: $\psi_{mB} = kd\cos\theta_{mB} + \beta = 0$

$$\beta = -kd\cos\theta_{mB}$$

$$\theta_{mB} = \cos^{-1}\left(-\beta/kd\right)$$

Nulls: $\psi_n = kd\cos\theta_n + \beta = \pm\frac{n2\pi}{N}$ $\quad n = 1, 2, \ldots$ $\quad n \neq N, 2N, \ldots$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{n2\pi}{N}\right)\right]$$

Maxima: $\frac{1}{2}\psi_m = \frac{1}{2}(kd\cos\theta_m + \beta) = \pm m\pi$

$$\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm 2m\pi\right)\right]$$

$m = 0$ Main Beam
$m = 1, 2, \ldots$ Grating Lobes

half-power points of Main Beam: $\frac{N\psi_h}{2} = \frac{N}{2}(kd\cos\theta_h + \beta) = \pm 1.391$

$$\theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]$$

Half-power Beam Width: $HPBW = 2\left|\theta_{mB} - \theta_h\right|$ ← assumes pattern symmetrical

* Always check to ensure argument of $\cos^{-1}[\;]$ is between −1 and +1

6.3.1 Broadside Array

→ Radiation @ $\theta = 90°$ (normal to axis of array) often desirable (e.g. TV + FM arrays)

→ Best if <u>both</u> element factor and array factor have maxima @ 90° (e.g. $\lambda/2$ dipole, $\lambda/4$ monopole, small loops)

Main beam occurs when $\psi = 0$

so $$\psi = 0 = kd\,\underbrace{\cos 90°}_{0} + \beta \Rightarrow \boxed{\beta = 0}$$

⇒ Want NO progressive phase shift

↳ $\psi = kd\cos\theta$

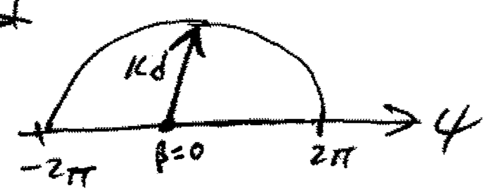
To avoid grating lobes:

$|\psi| < 2\pi$ $\qquad \psi < 2\pi = \frac{2\pi}{\lambda} d\,(1)$

need a good enough margin to well down the "shoulder" of the grating lobe(s).

$$\boxed{d < \lambda}$$

OR from graphical method



$$\underbrace{\cancel{\beta}}_{0} + kd < 2\pi$$

$$\frac{2\pi}{\lambda} d < 2\pi \Rightarrow \underline{d < \lambda}$$

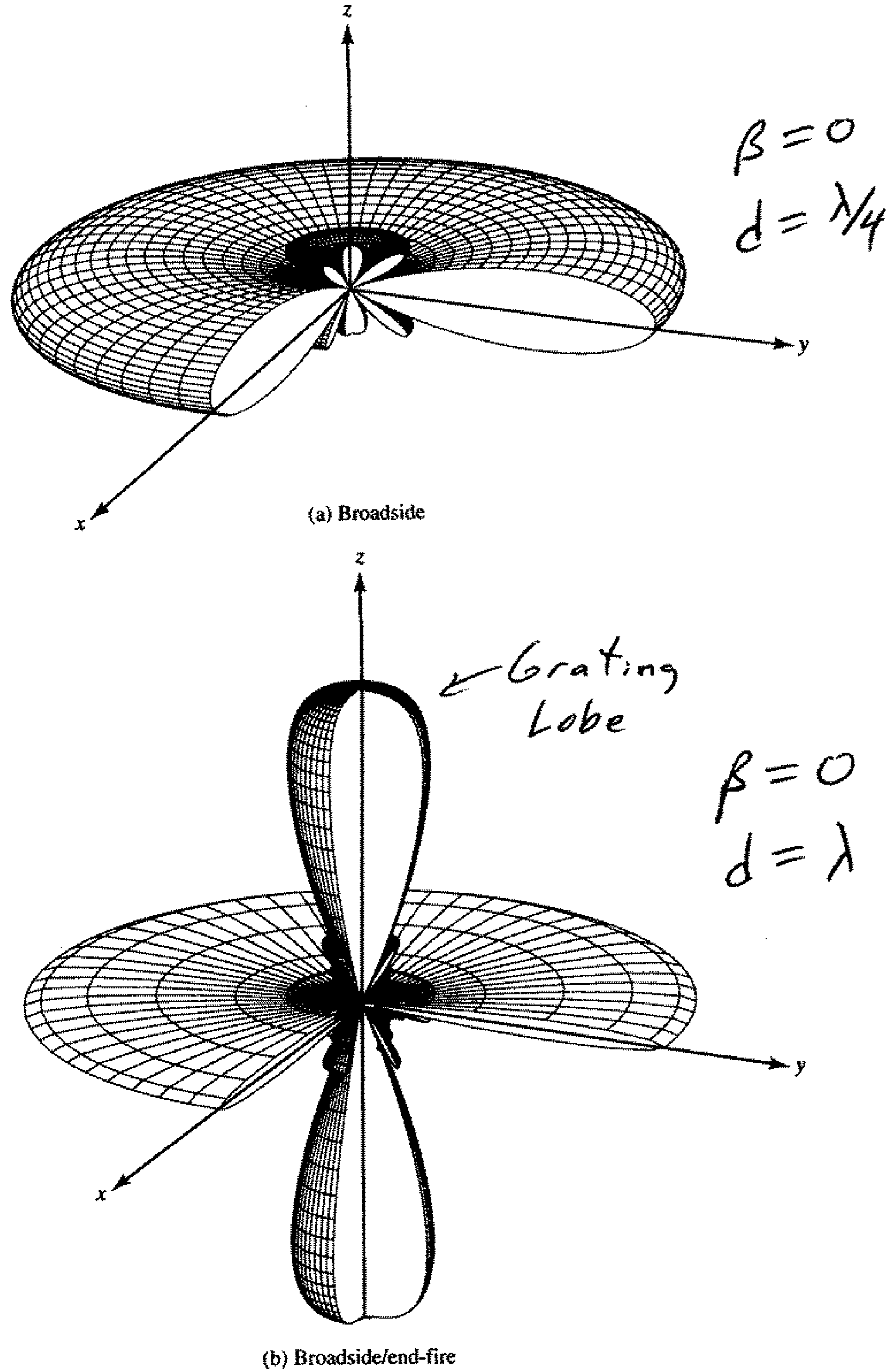


Figure 6.6 **Three-dimensional amplitude patterns for broadside, and broadside/end-fire arrays.** [Antenna Theory (Second Edition) by Balanis]

Table 6.1 NULLS, MAXIMA, HALF-POWER POINTS, AND MINOR LOBE MAXIMA FOR UNIFORM AMPLITUDE BROADSIDE ARRAYS $(\theta_{mB} = 90^\circ, \beta = 0)$

NULLS	$\theta_n = \cos^{-1}\left(\pm \frac{n}{N}\frac{\lambda}{d}\right)$ $n = 1, 2, 3, \ldots$ $n \neq N, 2N, 3N, \ldots$
MAXIMA	$\theta_m = \cos^{-1}\left(\pm \frac{m\lambda}{d}\right)$ $m = 0, 1, 2, \ldots$
HALF-POWER POINTS	$\theta_h \simeq \cos^{-1}\left(\pm \frac{1.391\lambda}{\pi N d}\right)$ $\pi d/\lambda \ll 1$
MINOR LOBE MAXIMA	$\theta_s \simeq \cos^{-1}\left[\pm \frac{\lambda}{2d}\left(\frac{2s+1}{N}\right)\right]$ $s = 1, 2, 3, \ldots$ $\pi d/\lambda \ll 1$

Table 6.2 BEAMWIDTHS FOR UNIFORM AMPLITUDE BROADSIDE ARRAYS

FIRST NULL BEAMWIDTH (FNBW)	$\Theta_n = 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{Nd}\right)\right]$
HALF-POWER BEAMWIDTH (HPBW)	$\Theta_h \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right)\right]$ $\pi d/\lambda \ll 1$
FIRST SIDE LOBE BEAMWIDTH (FSLBW)	$\Theta_s \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{3\lambda}{2dN}\right)\right]$ $\pi d/\lambda \ll 1$

[Antenna Theory (Second Edition) by Balanis]

6.3.2 Ordinary End-Fire Array

→ Radiate Main Beam @ $\theta = 0°$ or $180°$

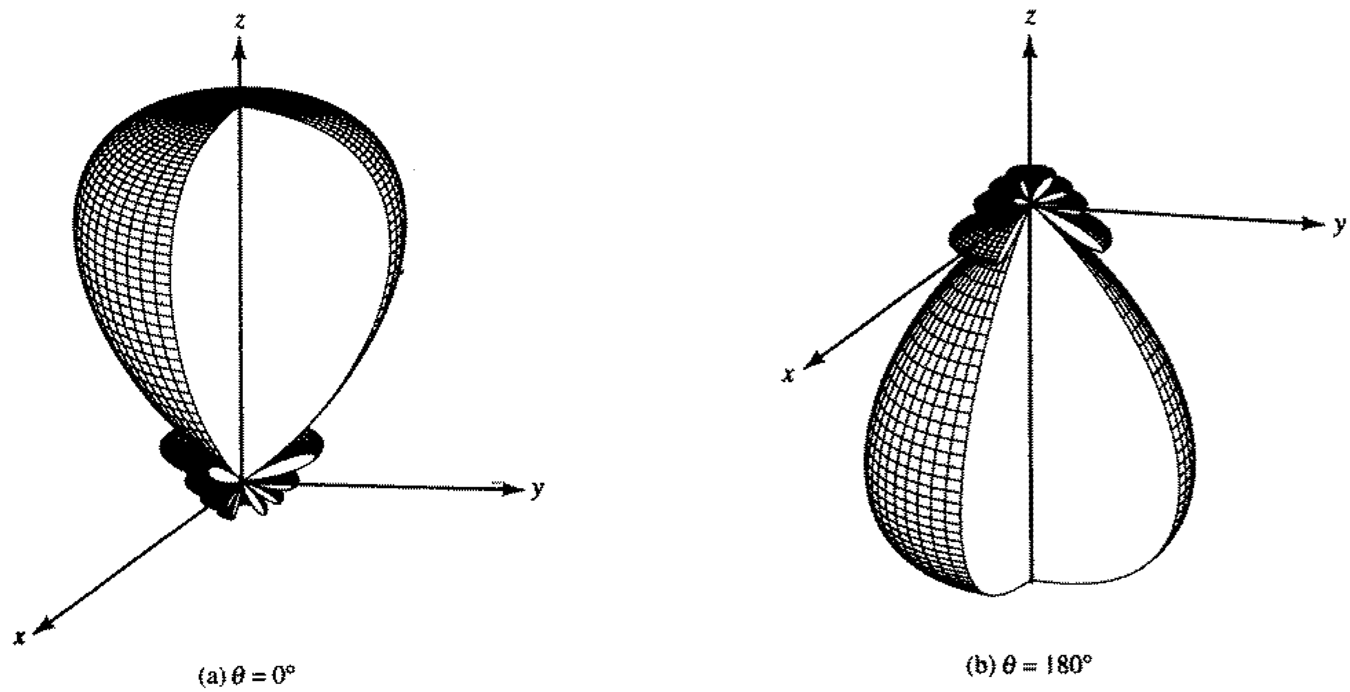


Figure 6.8 Three-dimensional amplitude patterns for end-fire arrays toward $\theta = 0°$ and 180°. [Antenna Theory (Second Edition) by Balanis]

To get Main Beam at $\theta = 0°$

$$\psi = kd \underbrace{\cos 0°}_{\to 1} + \beta = 0 \implies \boxed{\begin{array}{c}\beta = -kd \\ \theta_{mB} = 0°\end{array}}$$

To get Main Beam at $\theta = 180°$

$$\psi = kd \underbrace{\cos 180°}_{\to -1} + \beta = 0 \implies \boxed{\begin{array}{c}\beta = +kd \\ \theta_{mB} = 180°\end{array}}$$

Again to avoid grating lobes $\boxed{d < \lambda}$

→ actually need some margin to get well down "shoulder" of grating lobe(s)

Valid for $\theta_{mB} = 0^\circ$ $(\beta = -kd)$

Table 6.3 NULLS, MAXIMA, HALF-POWER POINTS, AND MINOR LOBE MAXIMA FOR UNIFORM AMPLITUDE ORDINARY END-FIRE ARRAYS

NULLS	$\theta_n = \cos^{-1}\left(1 - \dfrac{n\lambda}{Nd}\right)$ $n = 1, 2, 3, \ldots$ $n \neq N, 2N, 3N, \ldots$
MAXIMA	$\theta_m = \cos^{-1}\left(1 - \dfrac{m\lambda}{d}\right)$ $m = 0, 1, 2, \ldots$
HALF-POWER POINTS	$\theta_h \simeq \cos^{-1}\left(1 - \dfrac{1.391\lambda}{\pi d N}\right)$ $\pi d/\lambda \ll 1$
MINOR LOBE MAXIMA	$\theta_s \simeq \cos^{-1}\left[1 - \dfrac{(2s+1)\lambda}{2Nd}\right]$ $s = 1, 2, 3, \ldots$ $\pi d/\lambda \ll 1$

Table 6.4 BEAMWIDTHS FOR UNIFORM AMPLITUDE ORDINARY END-FIRE ARRAYS

FIRST NULL BEAMWIDTH (FNBW)	$\Theta_n = 2\cos^{-1}\left(1 - \dfrac{\lambda}{Nd}\right)$
HALF-POWER BEAMWIDTH (HPBW)	$\Theta_h \simeq 2\cos^{-1}\left(1 - \dfrac{1.391\lambda}{\pi d N}\right)$ $\pi d/\lambda \ll 1$
FIRST SIDE LOBE BEAMWIDTH (FSLBW)	$\Theta_s \simeq 2\cos^{-1}\left(1 - \dfrac{3\lambda}{2Nd}\right)$ $\pi d/\lambda \ll 1$

[Antenna Theory (Second Edition) by Balanis]

6.3.2 cont.

Nulls, Maxima, Half-power points, for Uniform Amplitude Ordinary End-Fire Array

($\theta_{mB} = \pi = 180^\circ$ & $\beta = +kd$)

Nulls $\quad \theta_n = \cos^{-1}\left(-1 + \frac{\lambda n}{dN}\right) \quad$ $n = 1, 2, \ldots$; $n \neq N, 2N, 3N, \ldots$

Maxima $\quad \theta_m = \cos^{-1}\left(-1 + \frac{m\lambda}{d}\right) \quad$ $m = 0$ mB; $m = 1, 2, \ldots$ Grating Lobes

Half-power Points $\quad \theta_h \simeq \cos^{-1}\left(-1 + \frac{1.391\lambda}{\pi d N}\right) \quad \frac{\pi d}{\lambda} << 1$

Minor Lobe Maxima $\quad \theta_s \simeq \cos^{-1}\left[-1 + \frac{(2s+1)\lambda}{2Nd}\right] \quad$ $s = 1, 2, \ldots$; $\frac{\pi d}{\lambda} << 1$

First Null Beamwidth (FNBW) $\quad \Theta_n = 2\left|\pi - \cos^{-1}\left(-1 + \frac{\lambda}{dN}\right)\right|$

Half-power Beamwidth (HPBW) $\quad \Theta_h \simeq 2\left|\pi - \cos^{-1}\left(-1 + \frac{1.391\lambda}{\pi d N}\right)\right| \quad \frac{\pi d}{\lambda} << 1$

First Sidelobe Beamwidth (FSLBW) $\quad \Theta_s \simeq 2\left|\pi - \cos^{-1}\left(-1 + \frac{3\lambda}{2Nd}\right)\right| \quad \frac{\pi d}{\lambda} << 1$

6.3.3 Phased or Scanning Array

→ Able to move Main Beam from 0°
to 180° and anywhere in between

To get a particular angle (w/ a uniform array):

$$\psi = kd\cos\theta_{mB} + \beta = 0 \Rightarrow \boxed{\beta = -kd\cos\theta_{mB}}$$

→ So, we need to be able to control the progressive phase shift β to scan the main beam → ferrite or diode phase shifters

→ The half-power beam width is

$$\boxed{HPBW = \cos^{-1}\left[\cos\theta_{mB} - \frac{2.782}{Nkd}\right] - \cos^{-1}\left[\cos\theta_{mB} + \frac{2.782}{Nkd}\right]}$$

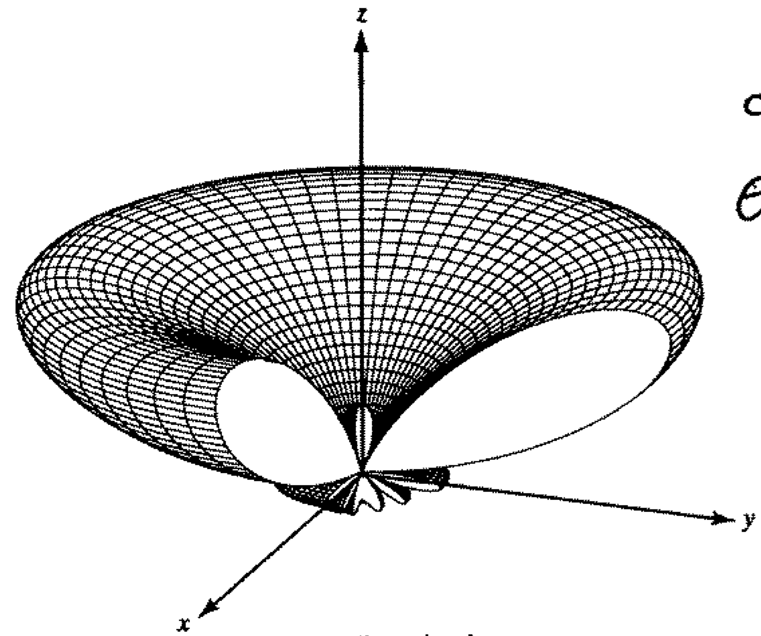
or

$$\boxed{HPBW = \cos^{-1}\left[\cos\theta_m - 0.443\frac{\lambda}{(L+d)}\right] - \cos^{-1}\left[\cos\theta_m + 0.443\frac{\lambda}{(L+d)}\right]}$$

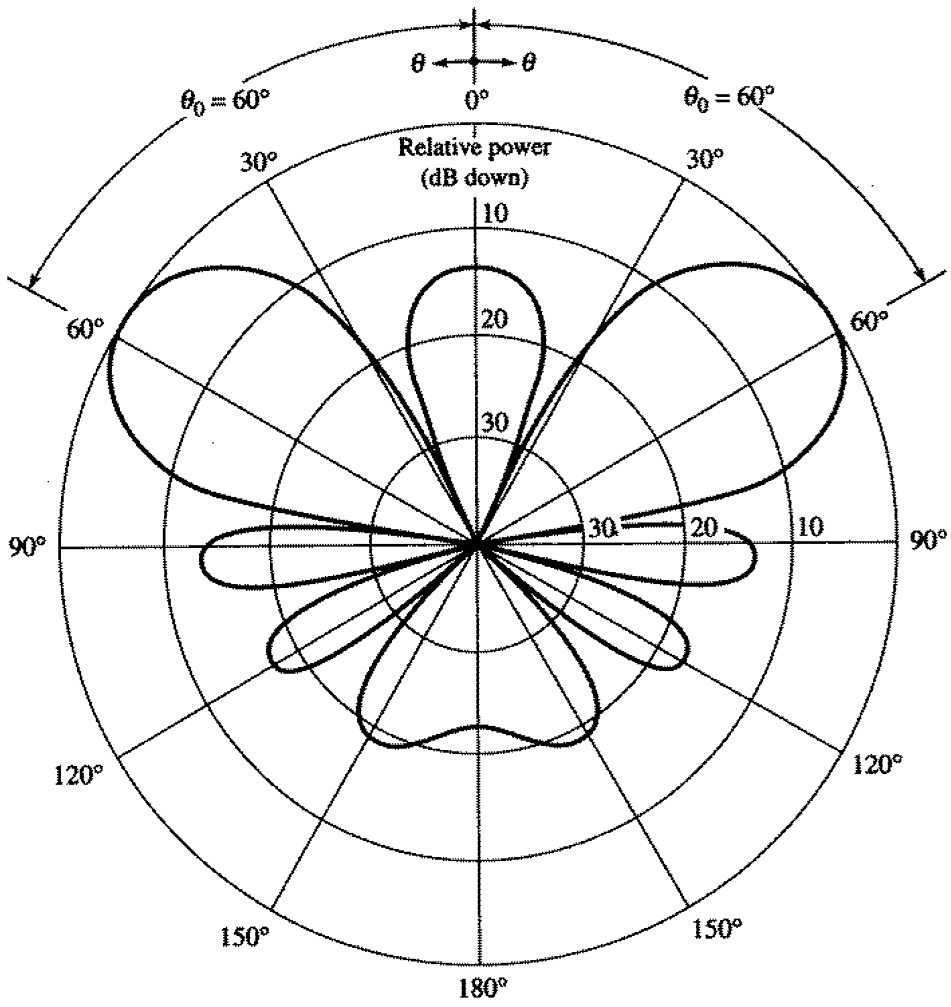
where $L = (N-1)d$ = length of array

$d = \lambda/4$

$\theta_{mB} = 60°$



(a) Three-dimensional



(b) Two-dimensional

Figure 6.10 Three- and two-dimensional array factor patterns of a 10-element uniform amplitude scanning array ($N = 10$, $\beta = -kd \cos \theta_0$, $\theta_0 = 60°$, $d = \lambda/4$.)

[Antenna Theory (Second Edition) by Balanis]

6.3.4 Hansen-Woodyard End-Fire Array

→ In 1938, Hansen + Woodyard found a way to improve the directivity of an end-fire array w/out degrading other characteristics for closely-spaced elements in a very long array (actually sidelobes can get bigger)

$$\boxed{\beta = -\left(kd + \frac{2.92}{N}\right) \simeq -\left(kd + \pi/N\right) \quad \theta_{MB} = 0^\circ}$$

and

$$\boxed{\beta = +\left(kd + \frac{2.92}{N}\right) \simeq \left(kd + \pi/N\right) \quad \theta_{MB} = 180^\circ}$$

→ Won't necessarily get the maximum directivity for finite-length arrays

In addition, @ $\theta_{MB} = 0^\circ$, require:

$$\boxed{|\psi| = |kd \underbrace{\cos 0^\circ}_{1} + \beta| = \pi/N \quad \text{and} \quad |\psi| = |kd \underbrace{\cos 180^\circ}_{-1} + \beta| \simeq \pi}$$

and @ $\theta_{MB} = 180^\circ$:

$$\boxed{|\psi| = |kd \underbrace{\cos 180^\circ}_{-1} + \beta| = \pi/N \quad \text{and} \quad |\psi| = |kd \cos 0^\circ + \beta| \simeq \pi}$$

6.3.4 cont.

These additional requirements lead to

$$\boxed{d = \left(\frac{N-1}{N}\right)\lambda/4 \quad \text{for } \theta_{MB} = 0^\circ \text{ or } 180^\circ}$$

Note: As $N \to \infty$, $d \to \lambda/4$!

Table 6.5 NULLS, MAXIMA, HALF-POWER POINTS, AND MINOR LOBE MAXIMA FOR UNIFORM AMPLITUDE HANSEN-WOODYARD END-FIRE ARRAYS

NULLS	$\theta_n = \cos^{-1}\left[1 + (1 - 2n)\frac{\lambda}{2dN}\right]$ $n = 1, 2, 3, \ldots$ $n \neq N, 2N, 3N, \ldots$
SECONDARY MAXIMA	$\theta_m = \cos^{-1}\left\{1 + [1 - (2m + 1)]\frac{\lambda}{2Nd}\right\}$ $m = 1, 2, 3, \ldots$ $\pi d/\lambda \ll 1$
HALF-POWER POINTS	$\theta_h = \cos^{-1}\left(1 - 0.1398\frac{\lambda}{Nd}\right)$ $\pi d/\lambda \ll 1$ N large
MINOR LOBE MAXIMA	$\theta_s = \cos^{-1}\left(1 - \frac{s\lambda}{Nd}\right)$ $s = 1, 2, 3, \ldots$ $\pi d/\lambda \ll 1$

Valid for $\theta_{MB} = 0$

used $\beta \simeq -(kd + \pi/N)$ approximation

Table 6.6 BEAMWIDTHS FOR UNIFORM AMPLITUDE HANSEN-WOODYARD END-FIRE ARRAYS

FIRST NULL BEAMWIDTH (FNBW)	$\Theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{2dN}\right)$	
HALF-POWER BEAMWIDTH (HPBW)	$\Theta_h = 2\cos^{-1}\left(1 - 0.1398\frac{\lambda}{Nd}\right)$	$\pi d/\lambda \ll 1$ N large
FIRST SIDE LOBE BEAMWIDTH (FSLBW)	$\Theta_s = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$	$\pi d/\lambda \ll 1$

[Antenna Theory (Second Edition) by Balanis]

6.3.4 cont.

Nulls, Maxima, half-power ... for Uniform Amplitude Hansen-Woodyard End-fire Arrays

$\left[\theta_{mB} = \pi \text{ or } 180° \text{ using } \beta \simeq +(kd + \pi/N)\right]$

Nulls: $\theta_n = \cos^{-1}\left[-1 - \frac{\lambda}{2dN}(1 - 2n)\right]$ $\quad n = 1, 2, \ldots$ $\quad n \neq N, 2N, 3N \ldots$

Secondary Maxima (Grating Lobes): $\theta_m = \cos^{-1}\left[-1 - \frac{\lambda}{2dN}\left(1 + (2m+1)\right)\right]$ $\quad m = 1, 2, \ldots$ $\quad \pi d/\lambda \ll 1$

Half-Power points: $\theta_h = \cos^{-1}\left[-1 + 0.1398\frac{\lambda}{Nd}\right]$ $\quad \pi d/\lambda \ll 1$ $\quad N$ large

Minor Lobe Maxima: $\theta_s = \cos^{-1}\left(-1 + \frac{s\lambda}{Nd}\right)$ $\quad s = 1, 2, \ldots$ $\quad \pi d/\lambda \ll 1$

First Null Beamwidth (FNBW) $\Theta_n = 2\left|\pi - \cos^{-1}\left(-1 + \frac{\lambda}{2dN}\right)\right|$

Half-power Beamwidth (HPBW) $\Theta_h = 2\left|\pi - \cos^{-1}\left(-1 + 0.1398\frac{\lambda}{Nd}\right)\right|$ $\quad \frac{\pi d}{\lambda} \ll 1$ $\quad N$ large

First Sidelobe Beamwidth (FSLBW) $\Theta_s = 2\left|\pi - \cos^{-1}\left(-1 + \frac{\lambda}{Nd}\right)\right|$ $\quad \frac{\pi d}{\lambda} \ll 1$

6.3.4 cont.

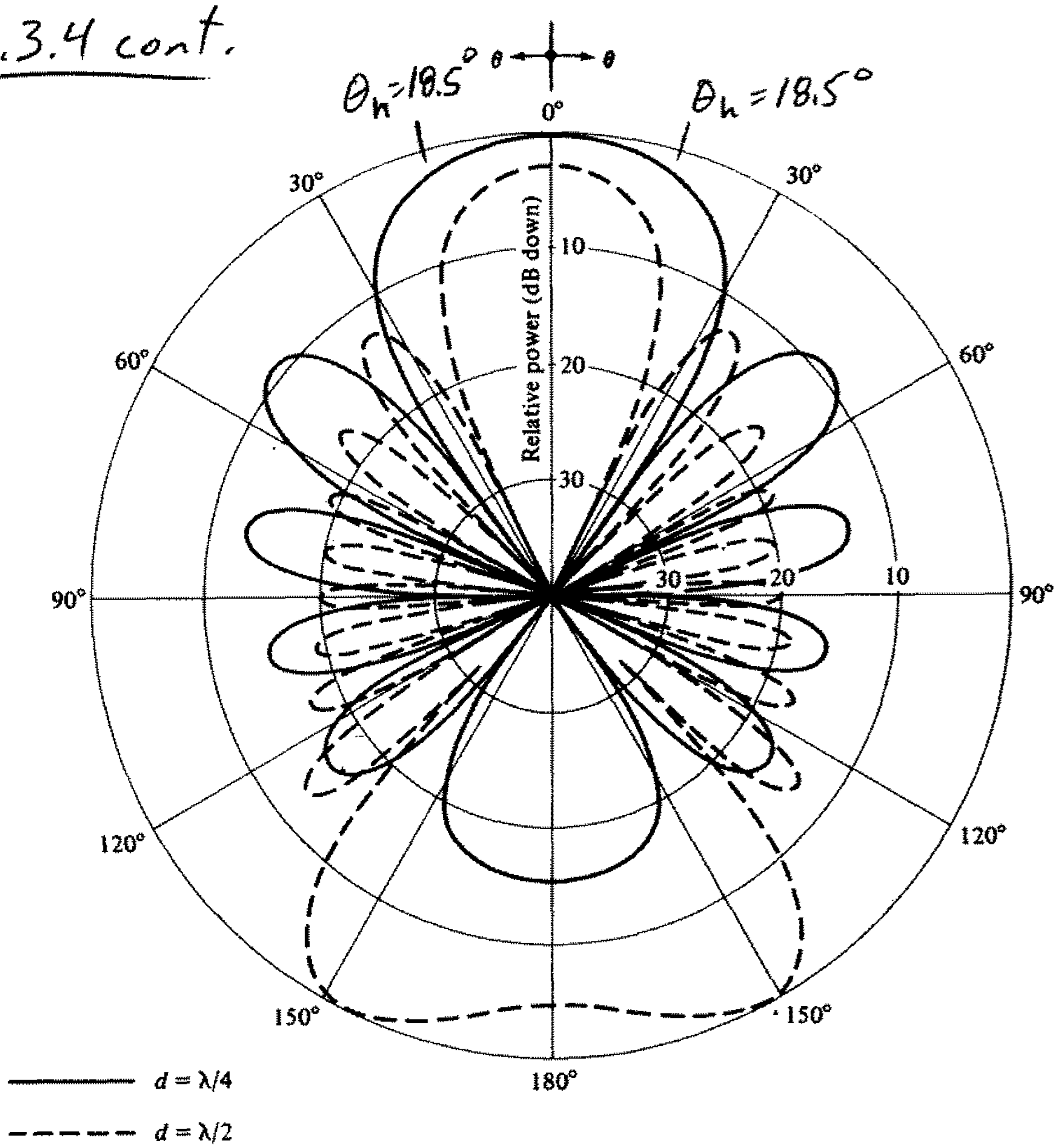


Figure 6.12 Array factor patterns of a 10-element uniform amplitude Hansen-Woodyard end-fire array [$N = 10$, $\beta = -(kd + \pi/N)$] [Antenna Theory (Second Edition) by Balanis]

* Above, for $\theta_{mB} = 0°$ & $N = 10$, compare the H-W array where $d = \lambda/4$ $(\beta = -3\pi/5)$ w/ an "H-W" array w/ the wrong spacing $d = \lambda/2$ $\left(\beta = \frac{-11\pi}{10}\right)$

$\Rightarrow$ backlobes bigger than "main beam"

6.3.4 cont.

* For further comparison, Fig 6.9 shows an ordinary end-fire array ($\beta = -kd = -\pi/2$) where $d = \lambda/4$

H-W array $d = \lambda/4$, $\beta = -\frac{3\pi}{5}$, HPBW = $\underline{37^\circ}$

$$D_{max} = 19 = \underline{12.788\ dBi}$$

Ordinary $d = \lambda/4$, $\beta = -\pi/2$, HPBW = $\underline{74^\circ}$

$$D_{max} = 11 = \underline{10.414\ dBi}$$

* H-W has about 2.4 dB higher gain/directivity

* Drawback of Hansen-Woodyard End-Fire Arrays $\Rightarrow$ higher side lobes (power has to go somewhere) by about 4 dB in this example. (compare Fig 6.12 + 6.9)

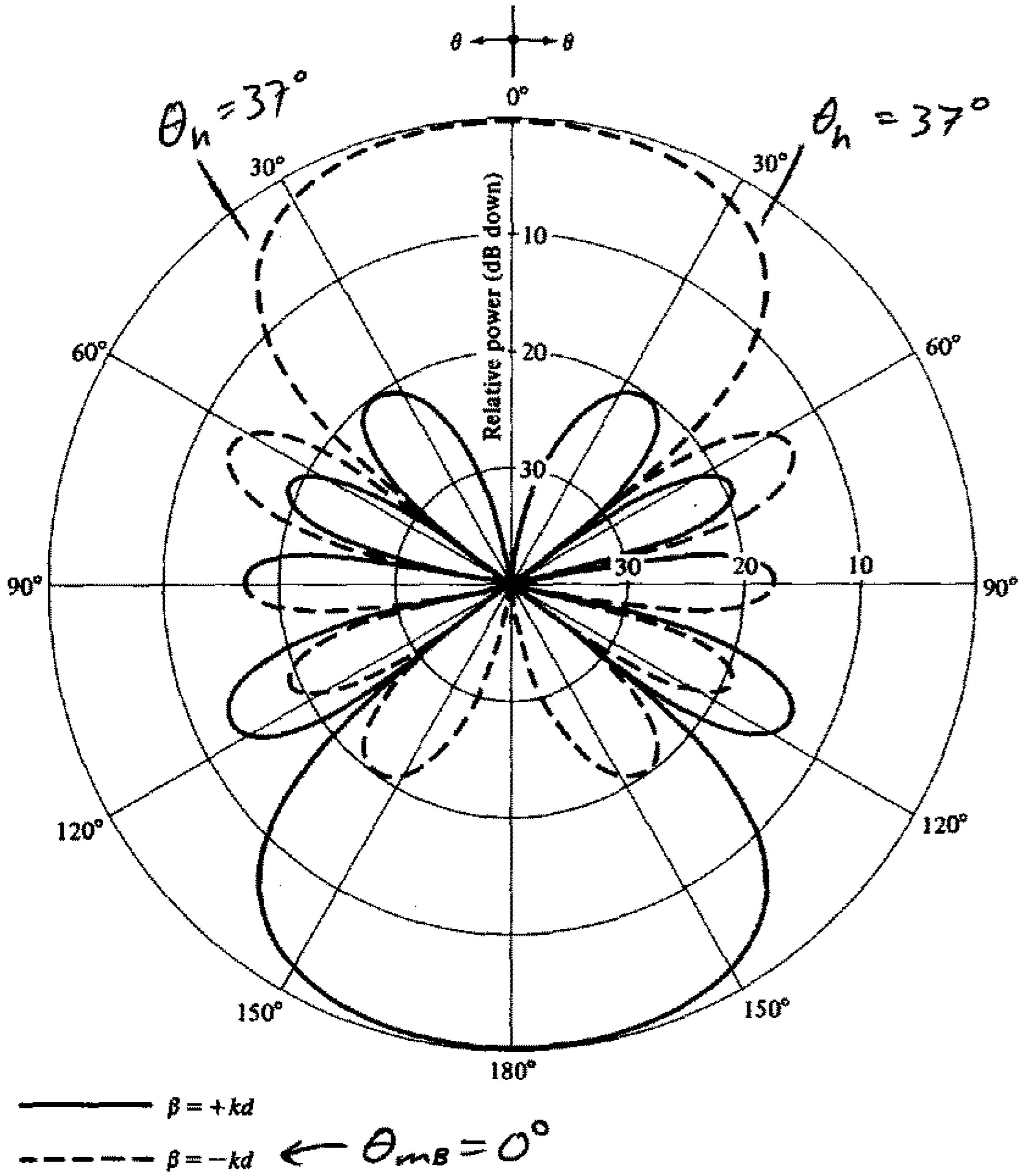


Figure 6.9 **Array factor patterns of a 10-element uniform amplitude end-fire array ($N = 10$, $d = \lambda/4$).** [Antenna Theory (Second Edition) by Balanis]

⇑

Ordinary End-Fire Array

6.4 N-Element Linear Array: Directivity

→ We'll find the directivity of the Array Factor (AF)

(2-12a) $U(\theta,\phi) = \frac{r^2}{2\eta}|E(r,\theta,\phi)|^2$

For Uniform arrays along the z-axis, there is no ϕ dependence.

$$U(\theta) \propto |AF|^2$$

can omit $|I_0|^2$ term as it divides out in directivity expression

$$= |I_0|^2 \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]^2 \qquad \psi = kd\cos\theta + \beta$$

$$P_{rad} = \oiint U\, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} U(\theta)\sin\theta\, d\theta\, d\phi$$

$$= 2\pi \int_{\theta=0}^{\pi} U(\theta)\sin\theta\, d\theta = 2\pi \int_{\theta=0}^{\pi} \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]^2 \sin\theta\, d\theta$$

$$\psi = kd\cos\theta + \beta$$

$$D(\theta) = \frac{U}{U_0} = \frac{4\pi U(\theta)}{P_{rad}} = \frac{2\,U(\theta)}{\int_{\theta=0}^{\pi} U(\theta)\sin\theta\, d\theta}$$

$U_0 = P_{rad}/4\pi$ (isotropic source)

6.4 cont.

Usually, when asked for directivity, the maximum directivity is what is required.

$$\boxed{D_{max} = D_0 = \frac{2U(\theta_m)}{\int_{\theta=0}^{\pi} U(\theta)\sin\theta\, d\theta}}$$

← Max of AF

$\rightarrow U(\theta_m) = |AF(\psi=0)|^2$ → N^2 for ordinary arrays; → $\neq N^2$ for Hansen-Woodyard

Sections 6.4.1 – 6.4.3 develop some approximate expressions for directivity (assume $d \ll L$) as listed below

TABLE 6.8 Directivities for Broadside and End-Fire Arrays

Array	Directivity	
BROADSIDE	$D_0 = 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 2\left(\frac{L}{\lambda}\right)$ $N\pi d/\lambda \to \infty, L \gg d$	
END-FIRE (ORDINARY)	$D_0 = 4N\left(\frac{d}{\lambda}\right) = 4\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 4\left(\frac{L}{\lambda}\right)$ $2N\pi d/\lambda \to \infty, L \gg d$	Only one maximum ($\theta_0 = 0°$ or $180°$)
	$D_0 = 2N\left(\frac{d}{\lambda}\right) = 2\left(1+\frac{L}{d}\right)\frac{d}{\lambda} \simeq 2\left(\frac{L}{\lambda}\right)$	Two maxima ($\theta_0 = 0°$ and $180°$)
END-FIRE (HANSEN-WOODYARD)	$D_0 = 1.805\left[4N\left(\frac{d}{\lambda}\right)\right] = 1.805\left[4\left(1+\frac{L}{d}\right)\frac{d}{\lambda}\right] = 1.805\left[4\left(\frac{L}{\lambda}\right)\right]$ $2N\pi d/\lambda \to \infty, L \gg d$	

[Antenna Theory (Third Edition) by Balanis]

Uniform Broadside Array Directivity example

Examine a uniform broadside array of 6 elements with quarterwave spacing.

$d\lambda := 0.25 \qquad kd := 2\cdot\pi\cdot d\lambda \qquad N := 6 \qquad I0 := 1 \qquad \beta := 0$

$\psi(\theta,\beta) := kd\cdot\cos(\theta) + \beta \qquad n := 0..180 \qquad \theta_n := \frac{\pi}{180}\cdot n - 0.0001$

From Table 6.8, find the estimated maximum AF directivity

$Dest := 2\cdot N\cdot d\lambda \qquad Dest\_dB := 10\cdot\log(Dest)$

Remembering that the $|AF|^2$ is proportional to the radiation intensity U and that the AF is independent of ϕ, calculate the "power radiated" as

$$Prad := 2\cdot\pi\cdot\int_0^{\pi}\left[\frac{\sin\left[\frac{N}{2}\cdot(kd\cdot\cos(\theta)+\beta)\right]}{\sin\left[\frac{1}{2}\cdot(kd\cdot\cos(\theta)+\beta)\right]}\right]^2\cdot\sin(\theta)\,d\theta \qquad Prad = 142.598$$

Remembering that the maximum of the $|AF| = N$, then U_{max} is proportional to N^2. Now, we can find the exact maximum directivity D_{exact}

$Dexact := \frac{4\cdot\pi\cdot N^2}{Prad} \qquad Dexact\_dB := 10\cdot\log(Dexact)$

Compare the Table 6.8 estimated AF directivity with the exact directivity

$\boxed{Dest = 3}$	$\boxed{Dest\_dB = 4.771}$	dBi
$\boxed{Dexact = 3.172}$	$\boxed{Dexact\_dB = 5.014}$	dBi

Very good agreement, especially considering the Table 6.8 requirements that $N\pi d/\lambda$ approach infinity and $L >> d$.

Uniform Broadside Array Directivity example cont.

Also, the directivity $D(\theta)$ as a function of angle θ is

$$D(\theta) := \frac{4\cdot\pi\cdot\left[\dfrac{\sin\left[\dfrac{N}{2}\cdot(kd\cdot\cos(\theta)+\beta)\right]}{\sin\left[\dfrac{1}{2}\cdot(kd\cdot\cos(\theta)+\beta)\right]}\right]^2}{Prad}$$

$Darray_n := D(\theta_n)$ $\qquad$ $D\_dB_n := 10\cdot\log(Darray_n)$

$Darray_{90} = 3.172$ $\qquad$ $D\_dB_{90} = 5.014$ dBi

Half power points occur when $D(\theta) = D_{max} / 2 = 1.586$ or $5.014 - 3.01 = 2$ dBi

$\theta1 := 72.61$ $\qquad$ $\theta2 := 107.39$

$D\left(\theta1\cdot\frac{\pi}{180}\right) = 1.586$ $\qquad$ $D\left(\theta2\cdot\frac{\pi}{180}\right) = 1.586$

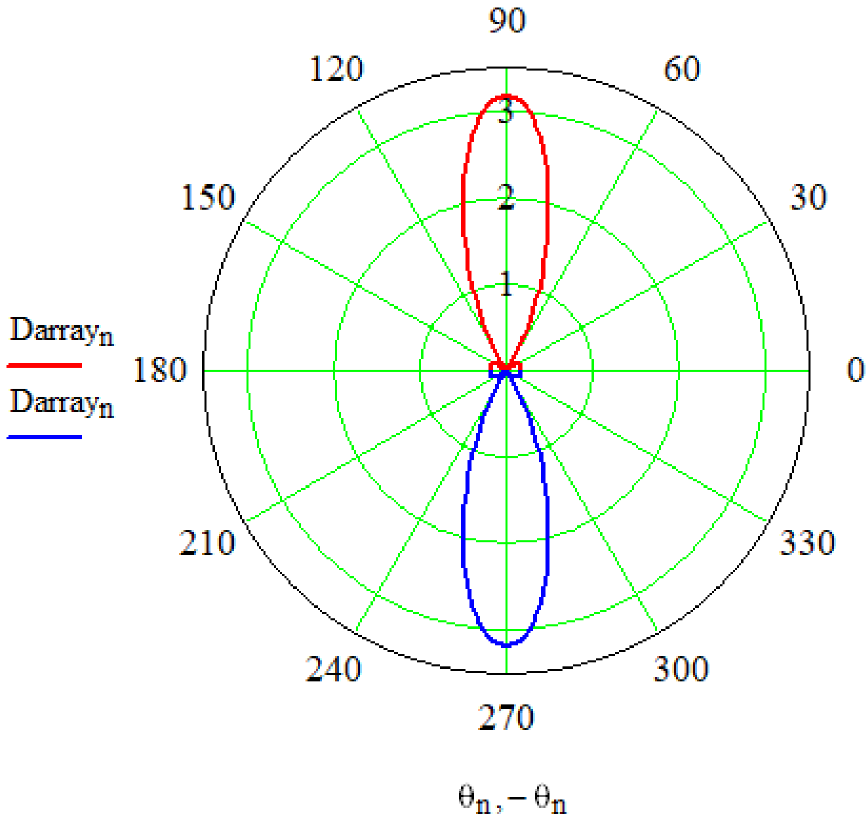
$HPBW := (\theta2 - \theta1)$ $\qquad$ $\boxed{HPBW = 34.78}$ deg

The Table 6.1 estimate of the HPBW for a broadside array is

$$HPBWest := 2\cdot\left(\frac{\pi}{2} - \text{acos}\left(\frac{1.391}{\pi\cdot N\cdot d\lambda}\right)\right)\cdot\frac{180}{\pi}$$ $\qquad$ $\boxed{HPBWest = 34.337}$ deg

Pretty good agreement.

Uniform Broadside Array Directivity example cont.



$$Dnorm\_dB_n := \text{if}\left(D\_dB_n - D\_dB_{90} + 40 < 0, 0, D\_dB_n - D\_dB_{90} + 40\right)$$