Chapter 4 Linear Wire Antennas

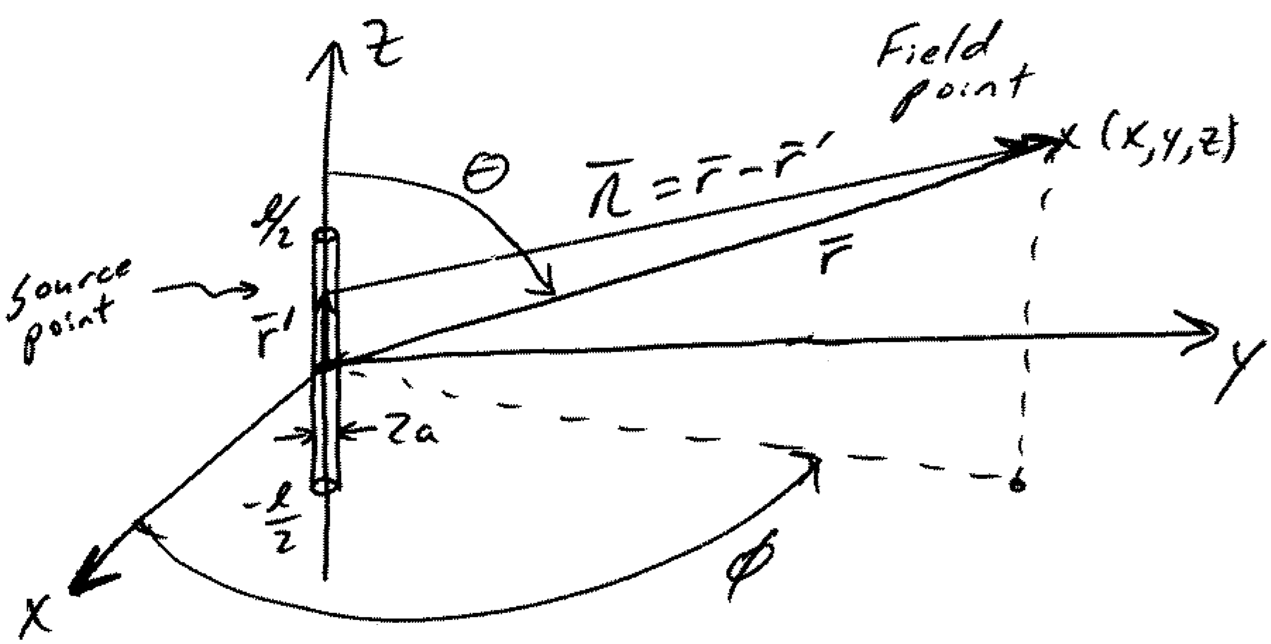
- Simple, robust, cheap, and give pretty good performance.
- Often used as base elements in antenna arrays

4.2 Infinitesimal Dipole

→ length $\ell \le \frac{\lambda}{50} << \lambda$ and radius $a << \lambda$

→ not practical in reality, but are good approx. to capacitor-loaded (top-hat) antennas

→ assume current is <u>constant</u> $\bar{I}(z') = \hat{a}_z I_0$



Geometry of infinitesimal dipole problem

4.2 cont.

Now to find $\bar{E}$ + $\bar{H}$ for this dipole

Sources: $\bar{I}(z') \rightarrow \bar{I}_e, \quad \bar{J} = \bar{J}_s = 0$

$$\bar{I}_m = \bar{M}_s = \bar{m} = 0 \Rightarrow \underline{\underline{\bar{F} = 0}}$$

Therefore, $\bar{H} = \bar{H}_A$ + $\bar{E} = \bar{E}_A$

$$\bar{A} = \frac{\mu}{4\pi} \int \bar{I}_e \frac{e^{-jkR}}{R} \, dl' \qquad dl' = dz'$$

$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

where $\bar{I}_e = \bar{I}(z') = \hat{a}_z I_0$

$\bar{r}' = z'\hat{a}_z$

$$\bar{R} = \bar{r} - \bar{r}' = x\hat{a}_x + y\hat{a}_y + (z - z')\hat{a}_z$$

$$R = |\bar{R}| = \sqrt{x^2 + y^2 + (z - z')^2}$$

$$\simeq \sqrt{x^2 + y^2 + z^2}$$

$$= r$$

for

$z' \ll z$ or

$z' \ll \sqrt{x^2 + y^2 + z^2}$

(better)

$$\bar{A} = \hat{a}_z \frac{\mu I_0}{4\pi} \frac{e^{-jkr}}{r} \int_{z'=-l/2}^{l/2} dz'$$

$$\underline{\underline{\bar{A} = \hat{a}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr}}}$$

$\rightarrow$ convert $\hat{a}_z A_z$ over to spherical coordinates

4.2 cont.

$$A_r = \cancelto{0}{A_x} \sin\theta \cos\phi + \cancelto{0}{A_y} \sin\theta \sin\phi + A_z \cos\theta$$

$$= \frac{\mu I_o l}{4\pi r} e^{-jkr} \cos\theta$$

$$A_\theta = \cancelto{0}{A_x} \cos\theta \cos\phi + \cancelto{0}{A_y} \cos\theta \sin\phi - A_z \sin\theta$$

$$= -\frac{\mu I_o l}{4\pi r} e^{-jkr} \sin\theta$$

$$A_\phi = -\cancelto{0}{A_x} \sin\phi + \cancelto{0}{A_y} \cos\phi = 0$$

$$\bar{A} = \left(\hat{a}_r \cos\theta - \hat{a}_\theta \sin\theta\right) \frac{\mu I_o l}{4\pi r} e^{-jkr}$$

$$\bar{H} = \frac{1}{\mu} \bar{\nabla} \times \bar{A} = \frac{1}{\mu r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{\mu I_o l}{4\pi r} e^{-jkr}\cos\theta & -r\frac{\mu I_o l}{4\pi r} e^{-jkr}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{\mu r^2 \sin\theta}\left[\hat{a}_r\, 0 - \hat{a}_r \cancelto{0}{\frac{\partial A_\theta}{\partial \phi}}\right] + \frac{1}{\mu r \sin\theta}\left[\hat{a}_\theta \cancelto{0}{\frac{\partial A_r}{\partial\phi}} - \hat{a}_\theta\, 0\right]$$

$$+ \frac{1}{\mu r}\left[\frac{\partial}{\partial r}\left(-\frac{\mu I_o l}{4\pi} e^{-jkr} \sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{\mu I_o l}{4\pi r} e^{-jkr}\cos\theta\right)\right]\hat{a}_\phi$$

$$= \frac{1}{\mu r}\left[+\frac{jk\mu I_o l}{4\pi} e^{-jkr}\sin\theta + \frac{\mu I_o l}{4\pi r} e^{-jkr}\sin\theta\right]\hat{a}_\phi$$

$$= \hat{a}_\phi \left[\frac{jk I_o l}{4\pi r} e^{-jkr}\sin\theta + \frac{I_o l}{4\pi r^2} e^{-jkr}\sin\theta\right]$$

$$\boxed{\bar{H} = \hat{a}_\phi \frac{jk I_o l \sin\theta}{4\pi r}\left(1 + \frac{1}{jkr}\right) e^{-jkr}}$$

4.2 cont.

$$(3\text{-}10)\quad \bar{\nabla}\times\bar{H} = \underbrace{\cancel{\bar{J}}}_{0\text{ (free space)}} + j\omega\epsilon\bar{E} \Rightarrow \bar{E} = \frac{1}{j\omega\epsilon}\bar{\nabla}\times\bar{H}$$

$$\bar{E} = \frac{1}{j\omega\epsilon r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & (r\sin\theta)\frac{jkI_o l\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr} \end{vmatrix}$$

$$= \frac{1}{j\omega\epsilon r^2\sin\theta}\hat{a}_r\frac{\partial}{\partial\theta}\left[\frac{jkI_o l\sin^2\theta}{4\pi}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]$$

$$-\frac{1}{j\omega\epsilon r^2\sin\theta}r\hat{a}_\theta\frac{\partial}{\partial r}\left[\frac{jkI_o l\sin^2\theta}{4\pi}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]$$

$$= \hat{a}_r\frac{1}{\cancel{j}\omega\epsilon r^2\cancel{\sin}\theta}\frac{\cancel{j}kI_o l\,2\cancel{\sin}\theta\cos\theta}{4\pi}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$

$$-\hat{a}_\theta\frac{1}{\cancel{j}\omega\epsilon r\sin\theta}\frac{\cancel{j}kI_o l\sin^2\theta}{4\pi}\left[-jke^{-jkr}\left(1+\frac{1}{jkr}\right)+\left(0-\frac{1}{jkr^2}\right)e^{-jkr}\right]$$

use $\frac{k}{\omega\epsilon} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\epsilon} = \sqrt{\mu/\epsilon} = \eta$

$$\boxed{\bar{E} = \hat{a}_r\frac{\eta I_o l\cos\theta}{2\pi r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr} + \hat{a}_\theta\frac{j\eta kI_o l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}}$$

$$\bar{W}_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left\{\bar{E}\times\bar{H}^*\right\} \quad (2\text{-}8)$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{(\hat{a}_r E_r + \hat{a}_\theta E_\theta)\times\hat{a}_\phi H_\phi^*\right\}$$

4.2 cont.

$$\overline{W}_{ave} = \tfrac{1}{2}\,\mathrm{Re}\left\{ -\hat{a}_\theta E_r H_\phi^* + \hat{a}_r E_\theta H_\phi^* \right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{ \hat{a}_r \frac{j\eta k I_0 \ell \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]\cancel{e^{-jkr}} \left(\frac{-jk I_0^* \ell \sin\theta}{4\pi r}\right)\left(1 + \frac{1}{-jkr}\right)\cancel{e^{+jkr}} \right.$$

$$\left. - \hat{a}_\theta \frac{\eta I_0 \ell \cos\theta}{2\pi r^2}\left(1 + \frac{1}{jkr}\right)\cancel{e^{-jkr}}\left(\frac{-jk I_0^* \ell \sin\theta}{4\pi r}\right)\left(1 + \frac{1}{-jkr}\right)\cancel{e^{+jkr}} \right\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{ \hat{a}_r \frac{-(-1)\eta k^2 |I_0|^2 \ell^2 \sin^2\theta}{(4\pi r)^2}\left(1 + \cancel{\frac{1}{jkr}} - \cancel{\frac{1}{(kr)^2}} - \cancel{\frac{1}{jkr}} + \cancel{\frac{1}{(kr)^2}} + \frac{1}{j(kr)^3}\right) \right.$$

$$\left. + \hat{a}_\theta \frac{j\eta k |I_0|^2 \ell^2 \cos\theta \sin\theta}{8\pi^2 r^3}\left(1 - \cancel{\frac{1}{jkr}} + \cancel{\frac{1}{jkr}} + \frac{1}{(kr)^2}\right) \right\}$$

use $k = 2\pi/\lambda$ in $\hat{a}_r$ term

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{ \hat{a}_r \frac{\eta}{4}\left|\frac{I_0 \ell}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}\left(1 + \frac{1}{j(kr)^3}\right) \right.$$

$$\left. + \hat{a}_\theta \frac{j\eta k |I_0 \ell|^2 \cos\theta \sin\theta}{8\pi^2 r^3}\left(1 + \frac{1}{(kr)^2}\right) \right\}$$

$$\boxed{\overline{W}_{ave} = \hat{a}_r \frac{\eta}{8}\left|\frac{I_0 \ell}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}}$$

$$P_{rad} = \oiint \overline{W}_{ave} \cdot d\overline{S}_r \qquad (2\text{-}9)$$

where $d\overline{S}_r = \hat{a}_r\, r^2 \sin\theta\, d\theta\, d\phi$

4.2 cont.

$$P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\eta}{8} \left| \frac{I_0 \ell}{\lambda} \right|^2 \sin^3\theta \, d\theta \, d\phi$$

$$= \frac{\eta}{8} \left| \frac{I_0 \ell}{\lambda} \right|^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{\eta}{8} \left| \frac{I_0 \ell}{\lambda} \right|^2 (2\pi)(4/3)$$

$$P_{rad} = \frac{\eta\pi}{3} \left| \frac{I_0 \ell}{\lambda} \right|^2$$

From (2-76) $P_{rad} = \frac{1}{2} |I_g|^2 R_r$ where $|I_0|^2 = |I_g|^2$

So $$R_r = \frac{\eta \, \pi/3 \left| \frac{I_0 \ell}{\lambda} \right|^2}{\frac{1}{2} |I_0|^2} = \eta \frac{2\pi}{3} \left(\frac{\ell}{\lambda} \right)^2 \ \Omega$$

For free-space $\eta_0 = 376.73 \, \Omega$

$$R_r = 789 \left(\ell/\lambda \right)^2 \ \Omega$$

To be infinitesimal $\ell/\lambda << 1$ (usually say $\ell/\lambda \lesssim 1/50$),

so $R_r \lesssim 0.3 \, \Omega$ (quite small)

→ Not practical, big mismatch problems w/ practical transmission lines

4.2 cont.

Since $\bar{I}_e = \hat{a}_z I_o$ ← constant current distribution

$$R_L = R_{hf} = \frac{\ell}{P}\sqrt{\frac{\omega\mu_0}{2\sigma}} \quad (2\text{-}90b)$$

The infinitesimal dipoles reactance will be negative (capacitive) → use analogy w/ open-ckt stub where $Z_{in} = -jZ_0/\tan(\beta\ell/2) \approx -j\frac{Z_0}{\beta\ell/2} = jX_A$

Since $\beta = 2\pi/\lambda \rightarrow \frac{\beta\ell}{2} = \pi\,\ell/\lambda << 1$

and $\tan(x) = x$ when $x << 1$

$$\boxed{\text{Radian distance} \equiv r = \lambda/2\pi \Rightarrow kr = 1}$$

Note: Real + Imaginary parts of $\bar{H}$ and E_r are equal in magnitude at the radian distance

$$\boxed{\text{Radian sphere} \rightarrow \text{sphere of radius } r = \lambda/2\pi}$$

→ $P_{reactive} > P_{rad}$ inside Radian Sphere

↑
Stored energy/power (the imaginary parts of $\bar{W}$)

4.2 cont.

Near-field Region ($kr \ll 1$)

$$\bar{E} = \hat{a}_r \frac{\eta I_0 l \cos\theta}{2\pi r^2}\left(1 + \underbrace{\frac{1}{jkr}}_{\text{big}}\right)e^{-jkr}$$

$$+\hat{a}_\theta \frac{j\eta k I_0 l \sin\theta}{4\pi r}\left[1 + \frac{1}{jkr} - \underbrace{\frac{1}{(kr)^2}}_{\text{big}}\right]e^{-jkr}$$

$$\bar{E}_{NF} \simeq \hat{a}_r(-j\eta)\frac{I_0 l \cos\theta\, e^{-jkr}}{2\pi k r^3} + \hat{a}_\theta(-j\eta)\frac{I_0 l \sin\theta\, e^{-jkr}}{4\pi k r^3}$$

$$\bar{H} = \hat{a}_\phi \frac{jk I_0 l \sin\theta}{4\pi r}\left(1 + \underbrace{\frac{1}{jkr}}_{\text{big}}\right)e^{-jkr}$$

$$\bar{H}_{NF} \simeq \hat{a}_\phi \frac{I_0 l \sin\theta}{4\pi r^2}e^{-jkr}$$

Note:

$$\overline{W}_{ave_{NF}} = \tfrac{1}{2}\operatorname{Re}\left\{\bar{E}_{NF} \times \bar{H}_{NF}^*\right\} = 0$$

$\bar{E}$ + $\bar{H}$ 90° out of phase

4.2 cont.

Far-field Region ($kr >> 1$)

$$\bar{E} = \hat{a}_r \frac{\eta I_o l \cos\theta}{2\pi \underbrace{r^2}_{\to 0}}\left(1 + \frac{1}{jkr}\right)e^{-jkr} +$$

$$+ \hat{a}_\theta \frac{j\eta k I_o l \sin\theta}{4\pi r}\left[1 + \underbrace{\frac{1}{jkr}}_{\to 0} - \underbrace{\frac{1}{(kr)^2}}_{\to 0}\right]e^{-jkr}$$

$$\boxed{\bar{E}_{FF} \simeq \hat{a}_\theta \frac{j\eta k I_o l \sin\theta}{4\pi r} e^{-jkr}}$$

$$\bar{H} = \hat{a}_\phi \frac{jk I_o l \sin\theta}{4\pi r}\left(1 + \underbrace{\frac{1}{jkr}}_{\to 0}\right)e^{-jkr}$$

$$\boxed{\bar{H}_{FF} \simeq \hat{a}_\phi \frac{jk I_o l \sin\theta}{4\pi r} e^{-jkr}}$$

Note that $Z_w = \frac{E_\theta}{H_\phi} = \eta \equiv$ wave impedance

Note: In far-field $\bar{E}$ & $\bar{H}$ are orthogonal & transverse to direction of propagation ($\hat{a}_r$) of wave

4.2 cont.

Directivity

we already found -

$$\overline{W_{ave}} = \bar{a}_r \frac{\eta}{8}\left|\frac{I_o \ell}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2} \quad \& \quad P_{rad} = \frac{\eta\pi}{3}\left|\frac{I_o \ell}{\lambda}\right|^2$$

use $\underline{U = r^2\left|\overline{W_{ave}}\right| = \frac{\eta}{8}\left|\frac{I_o \ell}{\lambda}\right|^2 \sin^2\theta}$ (2-12)

and
$$D = \frac{4\pi U}{P_{rad}} = \frac{\frac{4\cancel{\pi}\cancel{\eta}}{8}\cancel{\left|\frac{I_o \ell}{\lambda}\right|^2}\sin^2\theta}{\frac{\cancel{\eta}\cancel{\pi}}{3}\cancel{\left|\frac{I_o \ell}{\lambda}\right|^2}}$$

$$\underline{\underline{D(\theta) = 1.5\sin^2\theta}}$$

$$\underline{\underline{D_o = 1.5}} \ (@\ \theta = \pi/2)$$

$$\underline{\underline{D_o = 10\log_{10}(1.5) = 1.7609\ dB_i}}$$

ignoring losses $A_{em} = \left(\frac{\lambda^2}{4\pi}\right) D_o = 1.5\left(\frac{\lambda^2}{4\pi}\right) = \underline{\underline{\frac{3\lambda^2}{8\pi}}}$

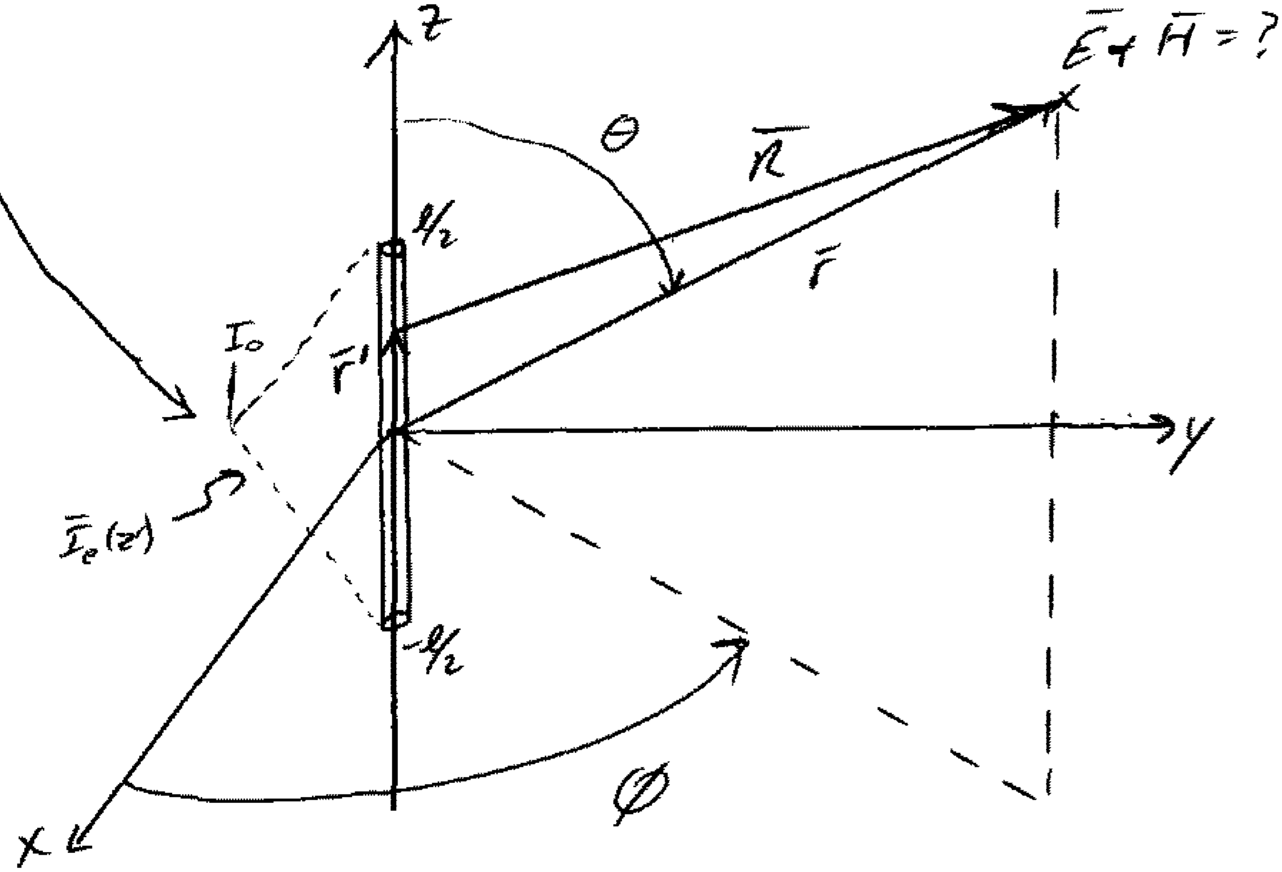
4.3 Small Dipole $\left(\lambda/50 < l \leq \lambda/10\right)$

→ Approximate the current distribution as

$$\bar{I}_e = \begin{cases} \hat{a}_z I_o \left(1 - \frac{z'}{(l/2)}\right) & 0 \leq z' \leq l/2 \\ \hat{a}_z I_o \left(1 + \frac{z'}{(l/2)}\right) & -l/2 \leq z' \leq 0 \end{cases}$$

$$= \hat{a}_z I_o \left(1 - \frac{|z'|}{l/2}\right) \quad -l/2 \leq z' \leq l/2$$

→ Triangular current distribution



Again, let $\bar{I}(z') = \bar{I}_o$ $\hookrightarrow \bar{A}$ $\quad \bar{J} = \bar{J}_s = \bar{I}_m = \bar{M}_s = \bar{m} = 0$ $\hookrightarrow \bar{F} = 0$

So, $\bar{H} = \bar{H}_A$ and $\bar{E} = \bar{E}_A$

4.3 cont.

$$\bar{A} = \frac{\mu}{4\pi}\left[\hat{a}_z \int_{z'=-\ell/2}^{0} I_0\left(1+\frac{2z'}{\ell}\right)\frac{e^{-jkR}}{R}\,dz' + \hat{a}_z \int_{z'=0}^{\ell/2} I_0\left(1-\frac{2z'}{\ell}\right)\frac{e^{-jkR}}{R}\,dz'\right]$$

again $R = |\bar{R}| = \sqrt{x^2+y^2+(z-z')^2} \simeq r$ only small phase results when r is small

$$\bar{A} = \frac{\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\hat{a}_z\left[\int_{-\ell/2}^{0}\left(1+\frac{2z'}{\ell}\right)dz' + \int_{0}^{\ell/2}\left(1-\frac{2z'}{\ell}\right)dz'\right]$$

$$= \hat{a}_z\frac{\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\left[\left(z'+\frac{z'^2}{\ell}\right)\Big|_{-\ell/2}^{0} + \left(z'-\frac{z'^2}{\ell}\right)\Big|_{0}^{\ell/2}\right]$$

$$= \hat{a}_z\frac{\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\left[\left(0+\ell/2-\frac{\ell}{4}\right)+\left(\ell/2-\ell/4-0+0\right)\right]$$

$$\bar{A} = \hat{a}_z \tfrac{1}{2}\left(\frac{\mu I_0 \ell e^{-jkr}}{4\pi r}\right)$$ $\leftarrow$ ½ result of infinitesimal dipole

Therefore:

$$\boxed{\begin{aligned}\bar{H}_{small} &= \tfrac{1}{2}\bar{H}_{inf.} = \hat{a}_\phi\frac{jkI_0\ell\sin\theta}{8\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}\\ \bar{E}_{small} &= \tfrac{1}{2}\bar{E}_{inf} = \hat{a}_r\frac{\eta I_0\ell\cos\theta}{4\pi r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}\\ &\quad+\hat{a}_\theta\frac{j\eta k I_0\ell\sin\theta}{8\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}\end{aligned}}$$

4.3 cont.

$\overline{W}_{ave,small} = \frac{1}{4}\overline{W}_{ave,inf}$ ← $(\frac{1}{2})^2$ from $\bar{E} + \bar{H}$ terms

$$\overline{W}_{ave,small} = \hat{a}_r \frac{\eta}{32}\left|\frac{I_o l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}$$

$$P_{rad,small} = \frac{1}{4}P_{rad,inf} = \frac{\eta\pi}{12}\left|\frac{I_o l}{\lambda}\right|^2$$

$$R_{r,small} = \frac{1}{4}R_{r,inf} = \eta\frac{2\pi}{12}\left(\frac{l}{\lambda}\right)^2 = \eta\,\frac{\pi}{6}\left(\frac{l}{\lambda}\right)^2$$

For free space $R_r = 197.25\left(\frac{l}{\lambda}\right)^2\ \Omega$

$$\underset{l=1/50}{0.08} \le R_{r,small} \le \underset{l=1/10}{2\ \Omega}$$ ← still hard to match

$R_L = \frac{R_{hf}}{3}$ due to triangular current distribution

Also $|\overline{W}_{ave}| = \frac{\eta}{32}\left|\frac{I_o l}{\lambda}\right|^2\frac{\sin^2\theta}{r^2}$

$$U = r^2|\overline{W}_{ave}| = \frac{\eta}{32}\left|\frac{I_o l}{\lambda}\right|^2 \sin^2\theta$$

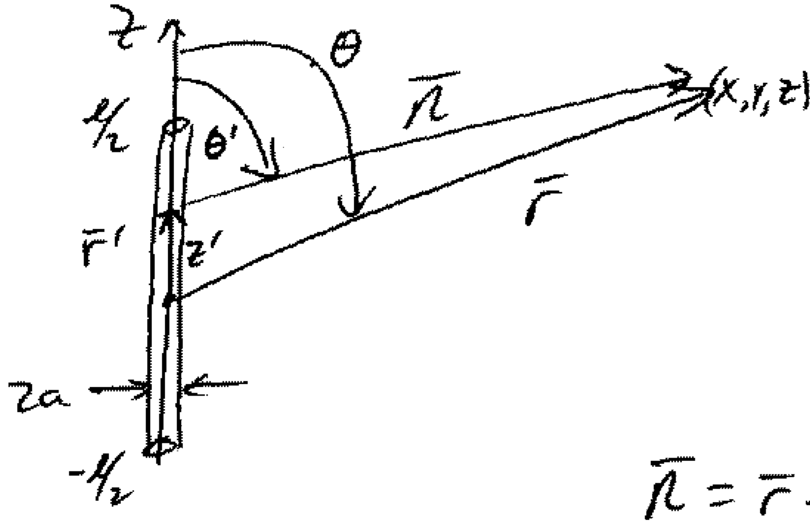
$$D(\theta) = \frac{4\pi U}{P_{rad}} = \frac{\cancel{4\pi\eta}/32\left|\cancel{\frac{I_o l}{\lambda}}\right|^2\sin^2\theta}{\cancel{\eta\pi}/12\left|\cancel{\frac{I_o l}{\lambda}}\right|^2}$$

$$D(\theta) = 1.5\sin^2\theta \quad \& \quad D_0 = 1.5 = 1.761\,\text{dBi}$$

4.4 Region Separation

On a finite length dipole on the z-axis

$$\bar{A} = \frac{\mu}{4\pi}\int \bar{I}_e(x',y',z')\frac{e^{-jkR}}{R}\,dl'$$



$dl' = dz'$

$\bar{R} = \bar{r} - \bar{r}'$ where

$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

$\bar{r}' = x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z$

$\simeq z'\hat{a}_z$ if $a \ll \lambda$

$$\bar{R} = \bar{r} - \bar{r}' = x\hat{a}_x + y\hat{a}_y + (z-z')\hat{a}_z$$

$$|\bar{R}| = R = \sqrt{x^2+y^2+(z-z')^2}$$

So,
$$\bar{A} = \frac{\mu}{4\pi}\int \bar{I}_e(x',y',z')\frac{e^{-jk\sqrt{x^2+y^2+(z-z')^2}}}{\sqrt{x^2+y^2+(z-z')^2}}\,dz'$$

can be difficult to evaluate

However, R can be expressed in terms of a binomial expansion

$$R = \sqrt{x^2+y^2+(z-z')^2} = \sqrt{x^2+y^2+z^2-2zz'+z'^2}$$

but $r^2 = x^2+y^2+z^2$ & $z = r\cos\theta$

4.4 cont.

So $R = \sqrt{r^2 - 2rz'\cos\theta + z'^2}$ ← still exact

a binomial series expansion $(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \dots$

with $n = \frac{1}{2}$, $a = r^2$, and $x = -2rz'\cos\theta + z'^2$ gives

$$R = r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right) + \frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right) + \dots$$

Far-field

as $r \gg z'$, we can drop the z'^n terms where $n > 2$

$$\Rightarrow \underline{R \approx r - z'\cos\theta}$$

What will be the worst case error?

$\Rightarrow$ when $\frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)$ is biggest (e.g. $\theta = \pi/2$)

$$Max\left\{\frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)\right\} = \frac{z'^2}{2r}$$

Note: all the other high order terms are zero when $\theta = \pi/2$ because they contain $\cos\theta$ arguments

Examining the $\frac{e^{-jkR}}{R}$ term of $\bar{A}$, we can see that small variations in $1/R$ will not have much effect. However, since e^{-jkR}

4.4 cont.

is periodic w/ $KR = n2\pi + KR \Rightarrow$ the phase error is more critical.

Researchers have found that a phase error $< 22.5°$ $(\pi/8)$ is usually acceptable

ex. $e^{j0} = 1$

$e^{\pm j\pi/8} = 0.9239 \pm j0.3827$ still 'fairly' close

$e^{\pm j\frac{\pi}{16}} = 0.9808 \pm j0.1951 \leftarrow$ better

So, the criteria is that $\frac{Kz'^2}{2r} \le \pi/8$

$\rightarrow$ worst case $z' = \pm \ell/2$

$$\frac{K\frac{\ell^2}{4}}{2r} \le \pi/8 \qquad (K = 2\pi/\lambda)$$

$$\frac{2\pi\ell^2}{8\lambda r} \le \pi/8$$

$$\downarrow$$

$$r \ge \frac{2\cancel{\pi}\ell^2}{\cancel{8}\lambda}\frac{\cancel{8}}{\cancel{\pi}} = \frac{2\ell^2}{\lambda}$$

but $\ell = D =$ max dimension of antenna

$$\underline{r \ge \frac{2D^2}{\lambda}} \qquad \text{(far-field!)}$$

4.4 cont.

Radiating Near-field (Fresnel) Region

Here, let $R \simeq r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)$

following a similar argument for phase error leads to

$$r \geq 0.62\sqrt{\ell^3/\lambda} \rightarrow 0.62\sqrt{D^3/\lambda}$$

Therefore, as discussed in Chapter 2

Near-field	$r < 0.62\sqrt{D^3/\lambda}$
Fresnel	$0.62\sqrt{D^3/\lambda} \leq r < 2D^2/\lambda$
Far-field	$r \geq 2D^2/\lambda$

Note: When converting these phasor results back to the time-domain, the e^{-jkR} term yields "retarded time"

e.g. $\mathrm{Re}\{E()\, e^{-jkR}\, e^{j\omega t}\} = \mathrm{Re}\{E()\, e^{j\omega(t - R/c)}\}$

"retarded" time
AKA: propagation time

since $k = \omega/c = 2\pi/\lambda$

→ have "retarded" potentials when we take this into account

4.5 Finite Length Dipole

→ use a better approximation that the current on a thin dipole is sinusoidally distributed when center-fed as:

$$\bar{I}_e(z') = \begin{cases} \hat{a}_z I_o \sin\left[k\left(\frac{l}{2} - z'\right)\right] & 0 \le z' \le \frac{l}{2} \\ \hat{a}_z I_o \sin\left[k\left(\frac{l}{2} + z'\right)\right] & -\frac{l}{2} \le z' \le 0 \end{cases}$$

→ Now the current $\underline{I_e(z' = \pm l/2) = 0}$

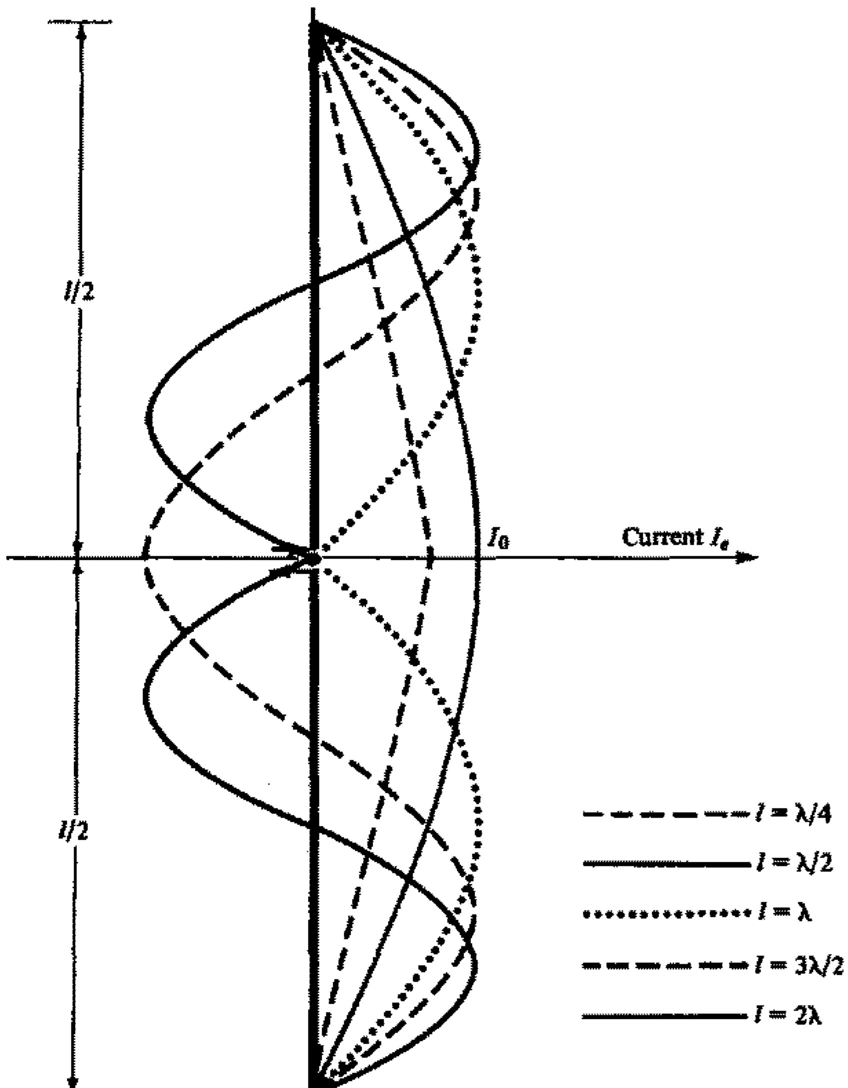


Figure 4.8 Current distributions along the length of a linear wire antenna.
(Balanis, Ant. Theory, 2nd Edn, p 156)

4.5 cont.

In the far-field, we have that

$$\left.\begin{aligned} E_\theta &\simeq j\eta \frac{kI_o \ell e^{-jkr}}{4\pi r}\sin\theta \\ H_\phi &\simeq j\frac{kI_o \ell e^{-jkr}}{4\pi r}\sin\theta \\ E_r &= E_\phi = H_r = H_\theta = 0 \end{aligned}\right\} \quad (4\text{-}26)$$

for an infinitesimal dipole of length ℓ located at the origin. A finite length dipole can be "built" from these.

To adapt this result to our finite length dipole, let $I_o \Rightarrow I_e(z')$

exponent: $r \Rightarrow R = r - z'\cos\theta$ ← accounts for infinitesimal dipole being off the origin

denominator: $\frac{1}{r} \Rightarrow \frac{1}{R} \Rightarrow \frac{1}{r}$

$\ell \Rightarrow dz'$

$E_\theta \Rightarrow dE_\theta$ ← contributions from each

$H_\phi \Rightarrow dH_\phi$ ← infinitesimal dipole

$$dE_\theta \simeq j\eta \frac{kI_e(z')\,dz'\,e^{-jk(r-z'\cos\theta)}}{4\pi r}\sin\theta$$

$$\boxed{\begin{aligned} dE_\theta &\simeq j\eta k \frac{e^{-jkr}}{4\pi r}\sin\theta\, I_e(z')\, e^{+jkz'\cos\theta}\,dz' \\ dH_\phi &\simeq jk \frac{e^{-jkr}}{4\pi r}\sin\theta\, I_e(z')\, e^{+jkz'\cos\theta}\,dz' \end{aligned}}$$

4.5 cont.

Now, if we "add-up"/integrate dE_θ & dH_ϕ over the length of our finite length dipoles

$$E_\theta = \int_{z'=-\ell/2}^{\ell/2} dE_\theta = \left(j\eta k \frac{e^{-jkr}}{4\pi r}\sin\theta\right)\left[\int_{z'=-\ell/2}^{\ell/2} I_e(z')\, e^{jkz'\cos\theta}\, dz'\right]$$

$$H_\phi = \int_{z'=-\ell/2}^{\ell/2} dH_\phi = \left(jk \frac{e^{-jkr}}{4\pi r}\sin\theta\right)\left[\int_{z'=-\ell/2}^{\ell/2} I_e(z')\, e^{jkz'\cos\theta}\, dz'\right]$$

Total fields = element factor × space factor

(same as infinitesimal dipole) (accounts for current distribution)

⇒ similar thing happens w/ antenna arrays

Substitute in $I_e(z')$ and use the indefinite integral

$$\int e^{\alpha x}\sin(\beta x + \gamma)\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\sin(\beta x+\gamma) - \beta\cos(\beta x+\gamma)\right]$$

w/ $\alpha = jk\cos\theta$ $\quad x \to z'$

$\beta = \pm k$

$\gamma = k\ell/2$

4.5 cont.

Then

$$\boxed{\begin{aligned} E_\theta &\simeq j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] \\ H_\phi &\simeq j \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] \\ &= \frac{E_\theta}{\eta} \end{aligned}}$$

$$\overline{W}_{ave} = \tfrac{1}{2}\,\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \tfrac{1}{2}\,\mathrm{Re}\{\hat{a}_\theta E_\theta \times \hat{a}_\phi H_\phi^*\}$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left\{\hat{a}_\theta E_\theta \times \hat{a}_\phi \frac{E_\theta^*}{\eta}\right\}$$

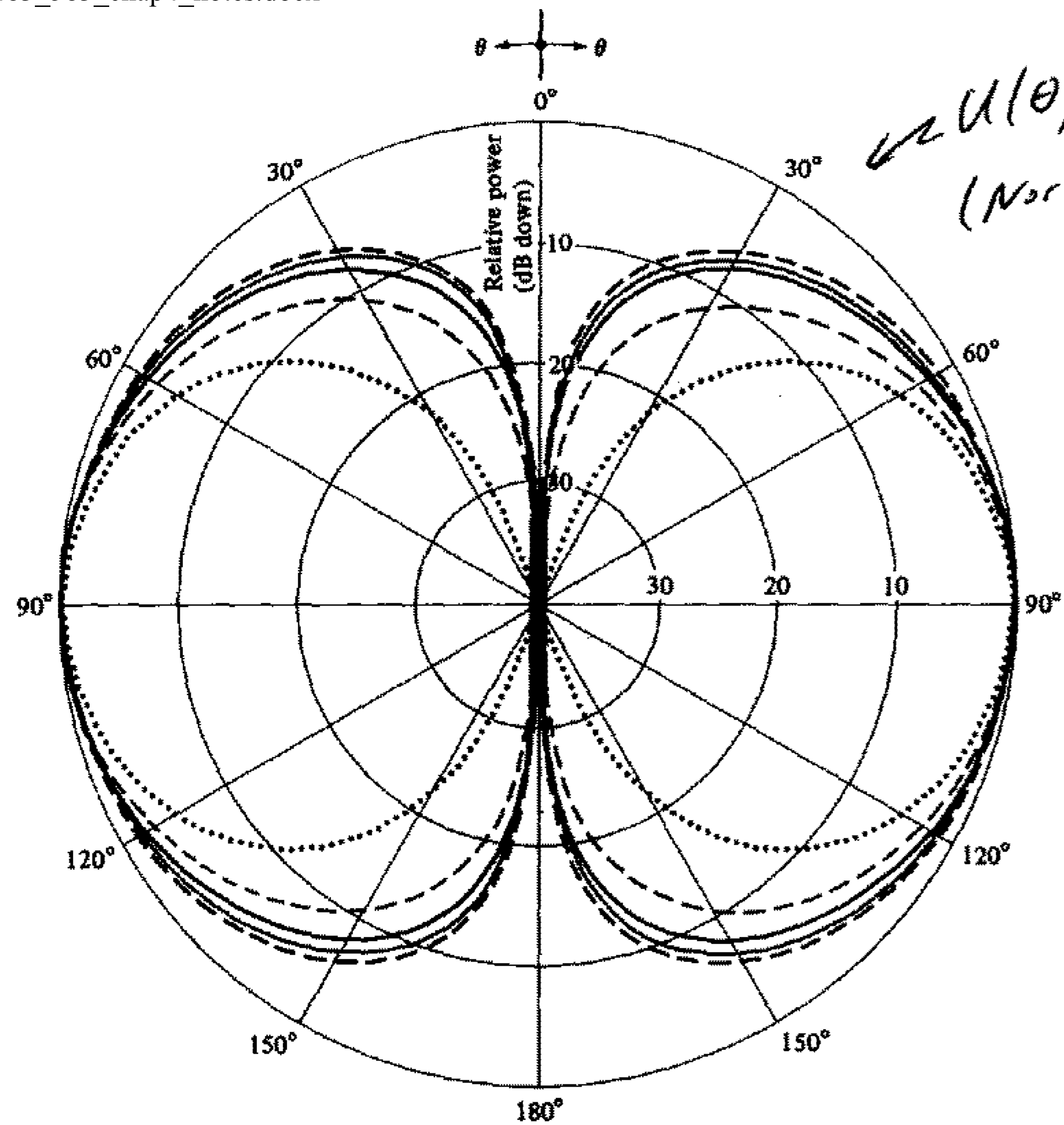
$$= \hat{a}_r \frac{1}{2\eta}|E_\theta|^2$$

$$\underline{\underline{\overline{W}_{ave} = \hat{a}_r \eta \frac{|I_0|^2}{8\pi^2 r^2}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos(kl/2)}{\sin\theta}\right]^2}}$$

$$\underline{\underline{U = r^2 W_{ave} = \eta\frac{|I_0|^2}{8\pi^2}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos(kl/2)}{\sin\theta}\right]^2}}$$

No ϕ dependence

$\swarrow U(\theta, \phi = 0, \pi)$ (Normalized)



- - - - $l \ll \lambda$ (3dB BW ~ 90°)
———— $l = \lambda/4$ (" 87°)
———— $l = \lambda/2$ (" 78°)
- - - - $l = 3\lambda/4$ (3dB BW ~64°)
·········· $l = \lambda$ (" 47.8°)

Figure 4.6 Elevation plane amplitude patterns for a thin dipole with sinusoidal current distribution ($l = \lambda/4, \lambda/2, 3\lambda/4, \lambda$).

[Balanis, Ant. Theory (2nd Ed), p.154]

Relative Power (Normalized) $= U(\theta, \phi)\,(\text{dB}_i) - \text{Max}\{U(\theta,\phi)(\text{dB}_i)\}$

$\rightarrow$ Makes it easier to compare the shapes of the radiation intensity.

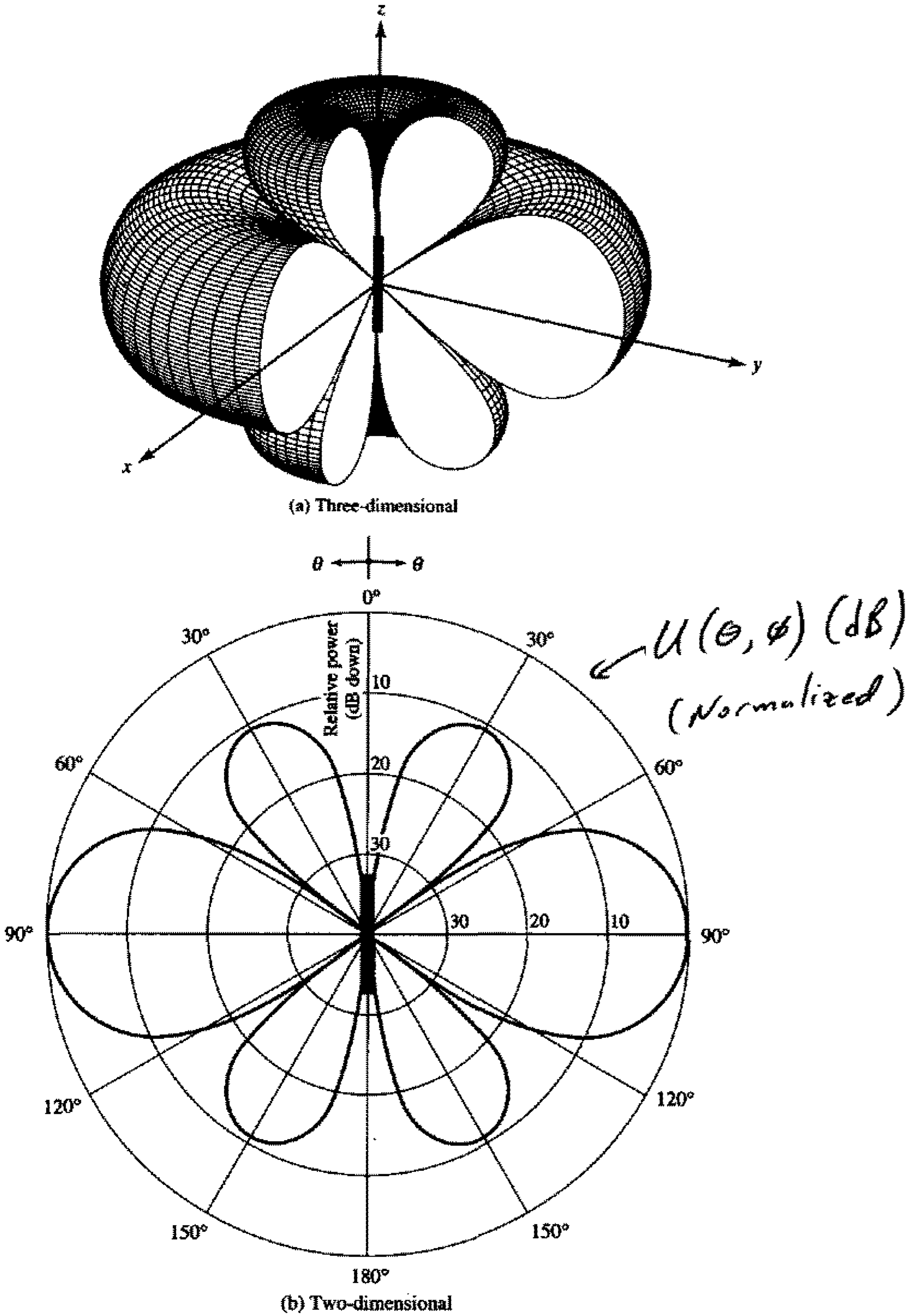


Figure 4.7 Three- and two-dimensional amplitude patterns for a thin dipole of $l = 1.25\lambda$ and sinusoidal current distribution.

[Balanis, Antenna Theory (2^{nd} Ed), p. 155]

4.5 cont.

Next, we'll determine the directivity + radiation resistance. To do so, we'll need P_{rad}

$$P_{rad} = \oiint_S \bar{W}_{ave} \cdot d\bar{s} \quad \sim \text{choose large sphere of radius } r$$

$$= \iint \cancel{\hat{a}_r} \frac{\eta |I_0|^2}{8\pi^2 \cancel{r^2}} \left[\frac{\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)}{\sin\theta} \right]^2 \cdot \cancel{\hat{a}_r} \cancel{r^2} \sin\theta \, d\theta \, d\phi$$

$$= \frac{\eta |I_0|^2}{8\pi^2} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \frac{\left[\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)\right]^2}{\sin\theta} d\theta$$

$$\boxed{P_{rad} = \frac{\eta |I_0|^2}{4\pi} \int_{\theta=0}^{\pi} \frac{\left[\cos\left(\frac{k\ell}{2}\cos\theta\right) - \cos\left(\frac{k\ell}{2}\right)\right]^2}{\sin\theta} d\theta}$$

Not bad numerically, but difficult analytically

OR

Euler's Constant

$$\boxed{\begin{aligned} P_{rad} = \eta \frac{|I_0|^2}{4\pi} \Big\{ & 0.5772 + \ln(k\ell) - C_i(k\ell) \\ & + \tfrac{1}{2}\sin(k\ell)\left[S_i(2k\ell) - 2S_i(k\ell)\right] \\ & + \tfrac{1}{2}\cos(k\ell)\left[0.5772 + \ln\left(\tfrac{k\ell}{2}\right) + C_i(2k\ell) - 2C_i(k\ell)\right] \Big\} \end{aligned}}$$

where $C_i(\,)$ + $S_i(\,)$ are the cosine + sine integrals

$$C_i(x) = -\int_x^{\infty} \frac{\cos y}{y} dy \quad + \quad S_i(x) = \int_0^x \frac{\sin y}{y} dy$$

Euler–Mascheroni constant (i.e, Euler's constant) = 0.5772156649

4.5 cont.

Now, we can get the radiation resistance

$$R_r = \frac{2P_{rad}}{|I_0|^2} = \frac{\eta}{2\pi}\int_{\theta=0}^{\pi} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta}\,d\theta$$

C or Euler's Constant

$$R_r = \frac{\eta}{2\pi}\Big\{0.5772 + \ln(kl) - C_i(kl) + \tfrac{1}{2}\sin(kl)\left[S_i(2kl) - 2S_i(kl)\right]$$
$$+ \tfrac{1}{2}\cos(kl)\left[0.5772 + \ln\left(\frac{kl}{2}\right) + C_i(2kl) - 2C_i(kl)\right]\Big\} \quad (4\text{-}70)$$

$$X_r = X_m = \frac{\eta}{4\pi}\Big\{2S_i(kl) + \cos(kl)\left[2S_i(kl) - S_i(2kl)\right]$$
$$-\sin(kl)\left[2C_i(kl) - C_i(2kl) - C_i\left(\frac{2ka^2}{l}\right)\right]\Big\} \quad (4\text{-}70a)$$

→ Remember $k = 2\pi/\lambda$, a, + l can be expressed in terms of λ. This makes it convenient to treat kl and $\frac{a}{l}$ as combined variables. E.g., $C_i\left(\frac{2ka^2}{l}\right) = C_i\left[2kl\left(\frac{a}{l}\right)^2\right]$

The directivity can be found using $D = \frac{4\pi U}{P_{rad}}$

$$D_{max} = D_0 = \frac{4\pi U_{max}}{P_{rad}} \quad \text{letting } U = \underbrace{B_0}_{\text{constant}} F(\theta,\phi) = \frac{\eta|I_0|^2}{8\pi^2}\underbrace{\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]^2}_{F(\theta,\phi)}$$

$$\text{So, } F(\theta) = \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]^2$$

$$D(\theta,\phi) = \frac{4\pi F(\theta,\phi)}{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} F(\theta,\phi)\sin\theta\,d\theta\,d\phi} \qquad (2\text{-}21)$$

4.5 cont.

This will lead to:

$$D(\theta,\phi) = D(\theta) = \frac{4\pi \overbrace{\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]^2}^{F(\theta)}}{\underbrace{\cancel{\int_{\phi=0}^{2\pi} d\phi}}_{2\pi} \underbrace{\int_{\theta=0}^{\pi} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta}\, d\theta}_{\text{Same integral again "Q"}}}$$

$$\boxed{D_0 = \frac{2\{F(\theta)\}_{max}}{\int_{\theta=0}^{\pi} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta}\, d\theta} = \frac{2\{F(\theta)\}_{max}}{Q}}$$

Where
$$Q = \Big\{ 0.5772 + \ln(kl) - C_i(kl) + \tfrac{1}{2}\sin(kl)\left[S_i(2kl) - 2S_i(kl)\right] + \tfrac{1}{2}\cos(kl)\left[0.5772 + \ln\left(\frac{kl}{2}\right) + C_i(2kl) - 2C_i(kl)\right]\Big\}$$

Again
$$\boxed{A_{em} = \frac{\lambda^2}{4\pi} D_0}$$

4.5 cont.

Input Resistance – with finite length dipoles the current maximum I_0 will not necessarily be at the input terminals (e.g. $\ell = 3\lambda/4$)

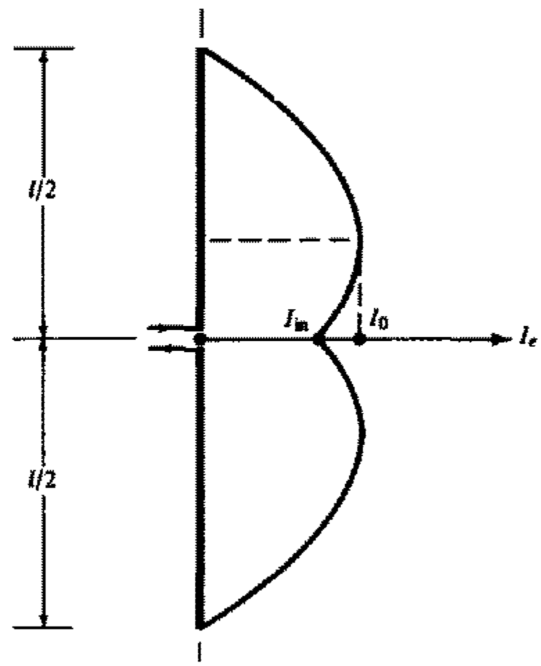


Figure 4.10 Current distribution of a linear wire antenna when current maximum does not occur at the input terminals. (Balanis, Ant Theory (2nd Ed), p. 160)

In this case, defining the radiation resistance as $P_{rad} = \frac{1}{2}|I_0|^2 R_r$ ← maximum current magnitude

will not give the radiation resistance at the input terminals. For a lossless antenna, however, $P_{in} = P_{rad}$

$$\frac{1}{2}|I_{in}|^2 R_{in} = \frac{1}{2}|I_0|^2 R_r$$

$$\hookrightarrow R_{in} = \left[\frac{|I_0|}{|I_{in}|}\right]^2 R_r$$

4.5 cont.

Now $I_{in} = I_0 \sin\left(\frac{kl}{2}\right)$, so

$$R_{in} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{R_r}{\sin^2(\pi l/\lambda)}$$

← Same for X_{in} & X_r

when $l/\lambda = 1, 2, 3, \ldots \Rightarrow \sin(n\pi) = 0$

so, in theory, $R_{in} \to \infty$ ($I_{in} = 0$).

Reality - There is a feed gap at the dipole center + dipoles have finite conductivity and radius → so, while R_{in} is very big (or I_{in} very small), it does not go to infinity

⇒ current distribution is not perfectly sinusoidal

⇒ these issues + how to model them analytically are still a topic of discussion. Usually, people use numerical models to analyze specific geometries.

⇒ one consequence is that the so called ½ dipole is actually slightly shorter as a result of these issues

4.6 Half-Wavelength Dipole

$R_r \approx 73\,\Omega$ (varies w/ wire radius & σ_{wire})

→ close to $75\,\Omega$ transmission line impedance (makes matching easy)

→ very widely used

w/ $\ell = \lambda/2$, the far fields are:

$$\boxed{\begin{aligned} E_\theta &\simeq j\eta \frac{I_o e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right] \\ H_\phi &\simeq j \frac{I_o e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right] \end{aligned}}$$

$$\boxed{\begin{aligned} \overline{W}_{ave} &= \eta \frac{|I_o|^2}{8\pi^2 r^2}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]^2 \hat{a}_r \\ &\simeq \eta \frac{|I_o|^2}{8\pi^2 r^2} \sin^3\theta \;\hat{a}_r \end{aligned}}$$

$$\boxed{\begin{aligned} U = r^2 W_{ave} &= \eta \frac{|I_o|^2}{8\pi^2}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]^2 \\ &\simeq \eta \frac{|I_o|^2}{8\pi^2} \sin^3\theta \end{aligned}}$$

4.6 cont.

get a 2π from $\int_0^{2\pi} d\phi$

$$P_{rad} = \oiint U \, d\Omega = \eta \frac{|I_0|^2}{4\pi} \left(\int_{\theta=0}^{\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} d\theta \right)$$

$\to 1.2188258$

$$= \eta \frac{|I_0|^2}{8\pi} \int_0^{2\pi} \left(\frac{1-\cos y}{y} \right) dy = \underline{\eta \frac{|I_0|^2}{8\pi} C_{in}(2\pi)}$$

where $C_{in}(2\pi) = 2.435$ ← book (2.43765339)

↖ TI-68 & MathCad

$$D_0 = 4\pi \frac{U_{max}}{P_{rad}} = 4\pi \frac{\eta \frac{|I_0|^2}{8\pi^2}}{\eta \frac{|I_0|^2}{8\pi} C_{in}(2\pi)}$$

$\theta = \pi/2$

$$\underline{\underline{D_0 = \frac{4}{C_{in}(2\pi)} = 1.6409 = 2.15088 \text{ dBi}}}$$

$$\underline{\underline{A_{em} = \frac{\lambda^2}{4\pi} D_0 \simeq 0.13058 \lambda^2}}$$

$$R_r = \frac{2 P_{rad}}{|I_0|^2} = \frac{\eta}{4\pi} C_{in}(2\pi) \simeq \underline{\underline{73.08 \Omega}}$$

$\underline{Z_{in} = Z_{ANT} \simeq 73 + j42.5 \Omega}$ when $\ell = \lambda/2 = 0.5\lambda$

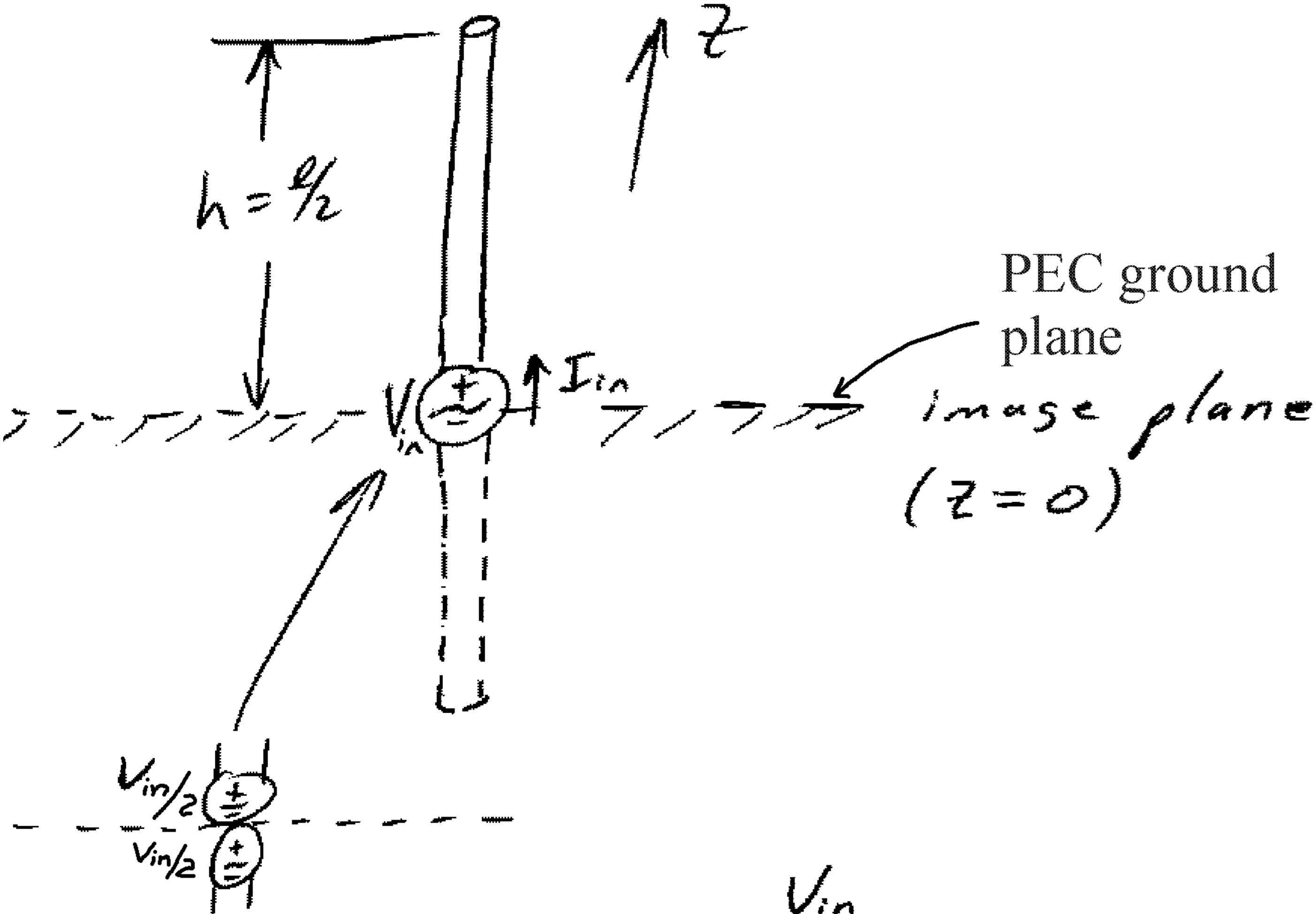
↳ due to non-sinusoidal current distribution actual first resonance occurs when

$$\underline{\underline{0.47\lambda < \ell < 0.48\lambda}}$$

where $X_A = 0$

bigger a ↗ ↖ small a (thin wire)

Monopoles (Whips)



$$Z_{in,dipole} = \frac{V_{in}}{I_{in}}$$

$$Z_{in,monopole} = \frac{V_{in}/2}{I_{in}} = \frac{1}{2}\,\frac{V_{in}}{I_{in}}$$

$$\boxed{Z_{in,monopole} = \frac{1}{2} Z_{in,dipole}}$$

Monopoles cont.

What about fields?

The electric + magnetic fields are identical in the positive half-space (i.e., $z>0$ or $0 \le \theta < 90°$) to those of the comparable dipole. This implies W_{ave} + $U(\theta,\phi)$ are the same.

What about directivity and gain?

If we have identical fields for the monopole + dipole, then $P_{rad,monopole} = 0.5\, P_{rad,dipole}$

since we are only doing positive half-space.

Then,
$$D_{monopole} = \frac{4\pi\, U_{monopole}}{P_{rad,monopole}}$$
$$= \frac{4\pi\, U_{dipole}}{0.5\, P_{rad,dipole}} = 2\, D_{dipole}\,!$$

$$D_{monopole}\,(dB) = D_{dipole}\,(dBi) + 3.01\,dB$$

$$\Rightarrow G_{monopole} = 2\, G_{monopole} = G_{dipole} + 3.01\,dB$$

Reality – Over earth (dirt), $G_{monopole} \simeq G_{dipole}$ due to losses (varies w/ soil params). Also, distorts fields to some extent.