Chapter 3 Radiation Integrals and Auxiliary Potential Functions

3.1 Introduction

→ Auxiliary functions or vector potentials

$\bar{A}$ – magnetic vector potential (Wb/m)

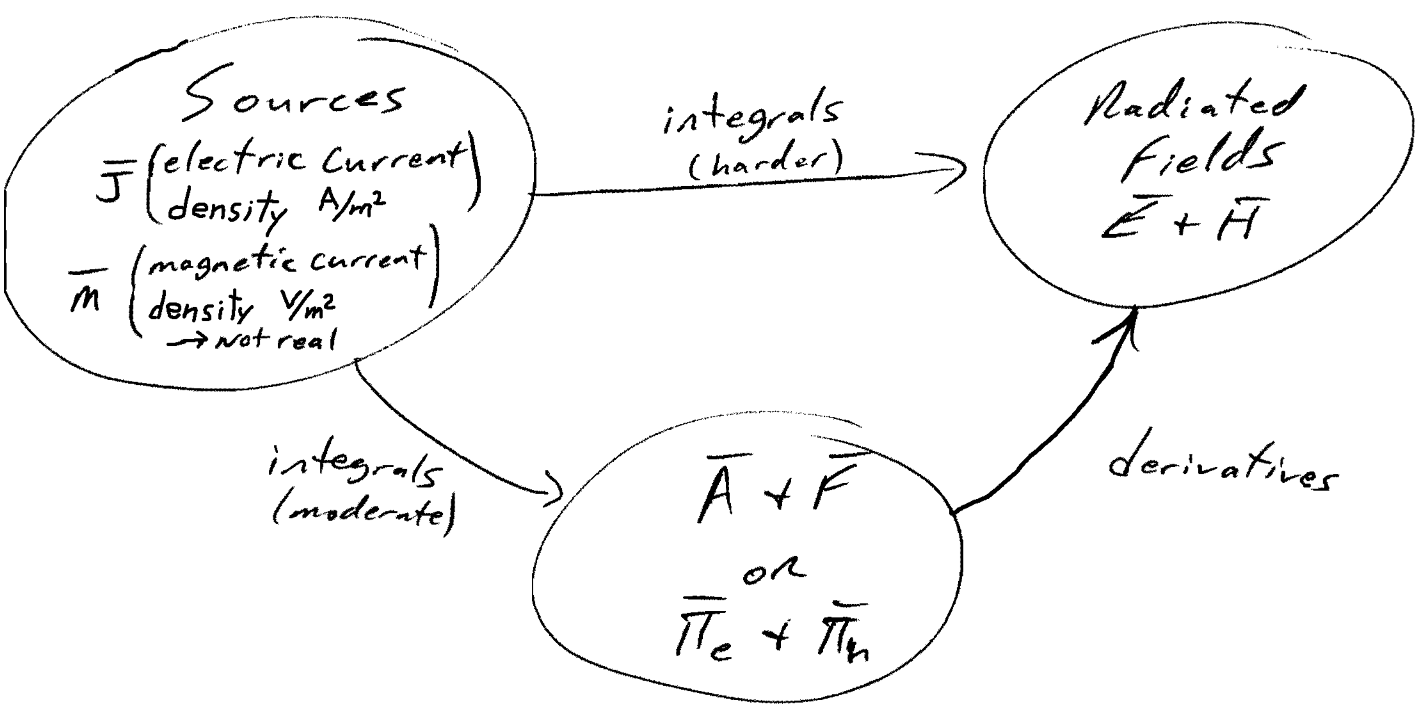
$\bar{F}$ – electric vector potential (C/m)

or

$\left.\begin{matrix}\bar{\Pi}_e \\ \bar{\Pi}_h\end{matrix}\right\}$ Hertz potentials

→ These are strictly mathematical tools, not measurable quantities (like $\bar{E}$, $\bar{H}$, + V)

→ Why? Make calculations for $\bar{E}$ + $\bar{H}$ easier



3.2 Magnetic Vector Potential $\bar{A}$ (wb/m) for Electric Current $\bar{J}$ (A/m²)

From Maxwell's Equations, $\bar{\nabla}\cdot\bar{B} = 0$

vector identity and constitutive relationship

$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = 0 \qquad \bar{B} = \mu\bar{H}$

$\Rightarrow \bar{B}_A = \bar{\nabla}\times\bar{A} = \mu\bar{H}_A$ — subscript indicates field quantity due to $\bar{A}$

☆ $\boxed{\bar{H}_A = \frac{1}{\mu}\bar{\nabla}\times\bar{A}}$ (A/m)

To get the corresponding electric field, one method is to use Ampere's Law

$$\bar{\nabla}\times\bar{H}_A = \bar{J} + j\omega\epsilon\bar{E}_A$$

w/ $\bar{J} = 0$ (typical for space surrounding an antenna).

This leads to: ☆ $\boxed{\bar{E}_A = \frac{1}{j\omega\epsilon}\bar{\nabla}\times\bar{H}_A}$ (V/m)

Another approach is to start w/ Faraday's Law

$\bar{\nabla}\times\bar{E}_A = -j\omega\mu\bar{H}_A = -j\omega\bar{\nabla}\times\bar{A} \Rightarrow \bar{\nabla}\times[\bar{E}_A + j\omega\bar{A}] = 0$,

apply vector identity $\bar{\nabla}\times(-\bar{\nabla}\phi_e) = 0$ to get

$$\bar{E}_A = -\bar{\nabla}\phi_e - j\omega\bar{A}$$

Now, $\bar{\nabla}\times\bar{B}_A = \bar{\nabla}\times\mu\bar{H}_A = \bar{\nabla}\times\bar{\nabla}\times\bar{A} = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \bar{\nabla}^2\bar{A}$

leads to the Lorentz condition:

$$\underline{\bar{\nabla}\cdot\bar{A} = -j\omega\epsilon\mu\phi_e} \Rightarrow \underline{\phi_e = \frac{-1}{j\omega\mu\epsilon}\bar{\nabla}\cdot\bar{A}}$$

3.2 cont.

and another method of computing $\bar{E}_A$ –

☆ $$\boxed{\bar{E}_A = -j\omega\bar{A} - j\frac{1}{\omega\mu\epsilon}\bar{\nabla}(\bar{\nabla}\cdot\bar{A})}$$ ☆

Inhomogeneous Helmholtz Eqn $\bar{\nabla}^2\bar{A} + k^2\bar{A} = -\mu\bar{J}$

3.3 Electric Vector Potential $\bar{F}\left(\frac{C}{m}\right)$ for Magnetic Current Source $\bar{M}\left(\frac{V}{m^2}\right)$

→ magnetic currents not physically realizable (no magnetic charge carriers), but can be used w/ equivalence theorems for mathematical convenience

→ $\bar{J} = 0$ & $\bar{M} \neq 0$ must satisfy Gauss' Law $\bar{\nabla}\cdot\bar{D} = 0$

⇓

☆ $$\boxed{\bar{E}_F = -\frac{1}{\epsilon}\bar{\nabla}\times\bar{F}\left(\frac{V}{m}\right)}$$

↓ using Maxwell's Eq'n $\bar{\nabla}\times\bar{H}_F = j\omega\epsilon\bar{E}_F$ & $\bar{\nabla}\times(-\bar{\nabla}\phi_m) = 0$

$\bar{H}_F = -\bar{\nabla}\phi_m - j\omega\bar{F}\left(\frac{A}{m}\right)$, ϕ_m is scalar magnetic potential

To get the corresponding magnetic field, one method is to use dual form of Faraday's Law

$$\bar{\nabla}\times\bar{E}_F = -\underbrace{\bar{M}}_{\to 0} - j\omega\mu\bar{H}_F \Rightarrow \boxed{\bar{H}_F = \frac{1}{-j\omega\mu}\bar{\nabla}\times\bar{E}_F}$$ ☆

3.3 cont.

Analogous to Lorentz condition, let

$$\bar{\nabla}\cdot\bar{F} = -j\omega\mu\epsilon\phi_m \Rightarrow \boxed{\phi_m = \frac{-1}{j\omega\mu\epsilon}\bar{\nabla}\cdot\bar{F}}$$

Another method to get the corresponding magnetic field is

$$\boxed{\bar{H}_F = -j\omega\bar{F} - \frac{j}{\omega\mu\epsilon}\bar{\nabla}(\bar{\nabla}\cdot\bar{F})}$$

Inhomogeneous Helmholtz Eq'n $\bar{\nabla}^2\bar{F} + k^2\bar{F} = -\epsilon\bar{M}$

3.4 Electric & Magnetic Fields for Electric ($\bar{J}$) and Magnetic ($\bar{M}$) Current Sources

→ To get $\bar{E}$ + $\bar{H}$, we'll need to be able to find $\bar{A}$ + $\bar{F}$

$$\boxed{\begin{aligned} &\rightarrow \bar{E} = \bar{E}_A + \bar{E}_F \\ &\rightarrow \bar{H} = \bar{H}_A + \bar{H}_F \end{aligned}}$$

Total or overall electric + magnetic fields

3.4 cont.

General Procedure

1. Specify (or find) $\bar{J}$ + $\bar{M}$

2. a)
$$\bar{A} = \frac{\mu}{4\pi}\iiint_V \bar{J}\,\frac{e^{-jkR}}{R}\,dv'$$

b)
$$\bar{F} = \frac{\epsilon}{4\pi}\iiint_V \bar{M}\,\frac{e^{-jkR}}{R}\,dv'$$

alternate forms for surface & filimentary current densities

where $k^2 = \omega^2\mu\epsilon$ (k is wave number)

R is distance from source to field point

Note! $k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}$

3. a) Find $\bar{H}_A = \frac{1}{\mu}\bar{\nabla}\times\bar{A}$

and $\bar{E}_A = -j\omega\bar{A} - j\frac{1}{\omega\mu\epsilon}\bar{\nabla}(\bar{\nabla}\cdot\bar{A})$

or use $\bar{\nabla}\times\bar{H}_A = j\omega\epsilon\bar{E}_A$ to find $\bar{E}_A$

b) Find $\bar{E}_F = -\frac{1}{\epsilon}\bar{\nabla}\times\bar{F}$

and $\bar{H}_F = -j\omega\bar{F} - j\frac{1}{\omega\mu\epsilon}\bar{\nabla}(\bar{\nabla}\cdot\bar{F})$

or use $\bar{\nabla}\times\bar{E}_F = -j\omega\mu\bar{H}_F$ to find $\bar{H}_F$

3.4 cont.

General Procedure cont.

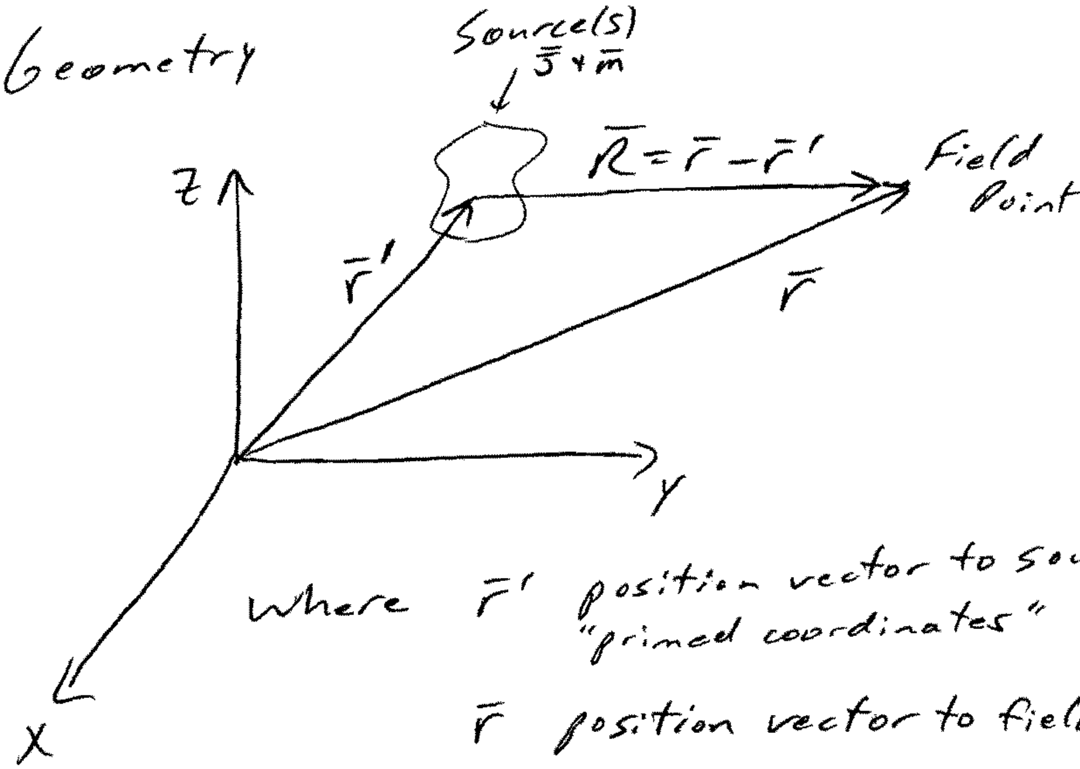
4. $\bar{E} = \bar{E}_A + \bar{E}_F$

$\bar{H} = \bar{H}_A + \bar{H}_F$

3.5 Solution of the Inhomogeneous Vector Potential Wave Equation

$$\left.\begin{aligned} \nabla^2 \bar{A} + k^2 \bar{A} &= -\mu \bar{J} \\ \nabla^2 \bar{F} + k^2 \bar{F} &= -\epsilon \bar{m} \end{aligned}\right\}$$ Inhomogeneous Helmholtz Eq'ns also Wave equations

Geometry



where $\bar{r}'$ position vector to source(s) "primed coordinates"

$\bar{r}$ position vector to field point

$\bar{R}$ distance vector from source to field point

3.5 cont.

$$\boxed{\bar{A} = \frac{\mu}{4\pi} \iiint_V \bar{J}(\bar{r}') \frac{e^{-jkR}}{R}\, dv'}$$

unprimed coordinates (field)

primed coordinates (source)

both primed & unprimed coordinates

$\bar{J} \rightarrow$ volume electric current density (A/m²)

$$\boxed{\bar{A} = \frac{\mu}{4\pi} \iint_S \bar{J}_s(\bar{r}') \frac{e^{-jkR}}{R}\, ds'}$$

surface electric current density (A/m)

$$\boxed{\bar{A} = \frac{\mu}{4\pi} \int_C \bar{I}_e(\bar{r}') \frac{e^{-jkR}}{R}\, dl'}$$

electric current (A)

Similarly

$$\bar{F} = \frac{\epsilon}{4\pi} \iiint_V \bar{M} \frac{e^{-jkR}}{R}\, dv' \quad \text{or}$$

$$= \frac{\epsilon}{4\pi} \iint_S \bar{M}_s \frac{e^{-jkR}}{R}\, ds' \quad \text{or}$$

$$= \frac{\epsilon}{4\pi} \int_C \bar{I}_m \frac{e^{-jkR}}{R}\, dl'$$

3.6 Far-Field Radiation

→ can neglect field components that decay as $\frac{1}{r^n}$ $n > 1$ (e.g., $\frac{1}{r^2}, \frac{1}{r^3}, \ldots$)

↓ For $\bar{A}$ contribution

$$E_r \simeq 0$$

$$\left.\begin{array}{l} E_\theta \simeq -j\omega A_\theta \\ E_\phi \simeq -j\omega A_\phi \end{array}\right\} \Rightarrow \boxed{\bar{E}_A \simeq -j\omega \bar{A}}$$

(only θ & ϕ components of $\bar{A}$)

$$H_r \simeq 0$$

$$H_\theta \simeq j\frac{\omega}{\eta} A_\phi = -\frac{E_\phi}{\eta}$$

$$H_\phi \simeq -j\frac{\omega}{\eta} A_\theta = \frac{E_\theta}{\eta}$$

$$\Rightarrow \bar{H}_A \simeq \frac{\hat{a}_r}{\eta} \times \bar{E}_A = -j\frac{\omega}{\eta}\hat{a}_r \times \bar{A}$$

(only θ & ϕ components)

⇑

Far-Field

3.6 cont.

For $\bar{M}$ contribution

$$H_r \simeq 0$$
$$H_\theta \simeq -j\omega F_\theta$$
$$H_\phi \simeq -j\omega F_\phi$$

$$\Rightarrow \boxed{\bar{H}_F \simeq -j\omega \bar{F}}$$

(Only θ + ϕ components of $\bar{F}$)

$$E_r \simeq 0$$
$$E_\theta \simeq -j\omega\eta F_\phi = \eta H_\phi$$
$$E_\phi \simeq j\omega\eta F_\theta = -\eta H_\theta$$

$$\Rightarrow \bar{E}_F = -\eta \hat{a}_r \times \bar{H}_F = j\omega\eta\, \hat{a}_r \times \bar{F}$$

(only θ + ϕ components of $\bar{F}$)

$\Uparrow$

Far-Field

→ $\bar{E}$ + $\bar{H}$ are orthogonal (TEM fields/waves)

→ Far-field $r > \frac{2D^2}{\lambda}$ (D is largest dimension of radiator)

3.6 cont.

ex. For some antenna, the electric vector potential is found to be

$$\bar{F} = \frac{\epsilon e^{-jkr}}{4\pi r}\left[\hat{a}_\theta E_0 \cos\theta\cos\phi - \hat{a}_\phi E_0 \sin\phi\right]$$

in the far-field. Determine the far-zone electric and magnetic fields.

$$\bar{H}_F = -j\omega\bar{F} \quad \text{(only } \theta + \phi \text{ components)}$$

$$\bar{H}_F = \bar{H} = \frac{-j\omega\epsilon E_0 e^{-jkr}}{4\pi r}\left[\hat{a}_\theta \cos\theta\cos\phi - \hat{a}_\phi \sin\phi\right]$$

$$\bar{E}_F = -\eta\hat{a}_r \times \bar{H}_F \quad \text{or} \quad \begin{aligned} E_\theta &= -j\omega\eta F_\phi \\ E_\phi &= j\omega\eta F_\theta \end{aligned}$$

$$E_\theta = \frac{-j\omega\eta\epsilon e^{-jkr}}{4\pi r}\left(-E_0 \sin\phi\right)$$

$$E_\phi = \frac{j\omega\eta\epsilon e^{-jkr}}{4\pi r}\left(E_0 \cos\theta\cos\phi\right)$$

$$\bar{E}_F = \bar{E} = \frac{j\omega\eta\epsilon E_0 e^{-jkr}}{4\pi r}\left(\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\theta\cos\phi\right)$$