5.17 A very small circular loop of radius $a (a < \lambda/6\pi)$ and constant current I_0 is symmetrically placed about the origin at $x = 0$ and with the plane of its area parallel to the y-z plane. Find the

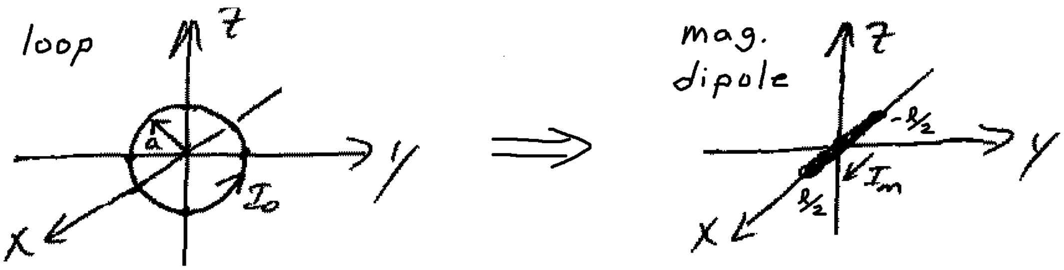
(a) spherical **E**- and **H**-field components radiated by the loop in the far zone

(b) directivity of the antenna

a) This would be an extremely difficult problem to set-up in Cartesian, cylindrical, or spherical coordinates. Instead, consider section 5.2.2 which discusses an infinitesimal magnetic dipole where

$$(5\text{-}21) \quad I_m \ell = jS\omega\mu I_0 = j\pi a^2 \omega\mu I_0$$

placed perpendicular to the plane of the loop



For the infinitesimal magnetic dipole, we evaluate the electric vector potential $\bar{F}$

$$(3\text{-}54) \quad \bar{F} = \frac{\epsilon}{4\pi}\int \bar{I}_m \frac{e^{-jkR}}{R}\, d\ell'$$

infinitesimal, so $R \simeq r$

$$= \frac{\epsilon}{4\pi}\int_{x'=-\ell/2}^{\ell/2} \hat{a}_x I_m \frac{e^{-jkr}}{r}\, dx'$$

$$= \hat{a}_x \frac{\epsilon I_m \ell}{4\pi} \frac{e^{-jkr}}{r}$$

sub in (5-21)

$$= \hat{a}_x \frac{\epsilon (j\pi a^2 \omega\mu I_0)}{4\pi} \frac{e^{-jkr}}{r}$$

a) cont.

Now $\hat{a}_x = \sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi$

$$\bar{F} = \left[\sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi\right] \frac{j a^2 \omega \mu \epsilon I_o}{4} \frac{e^{-jkr}}{r}$$

In the far-field, use (3-59a) & (3-59b) to get:

$$\bar{H} = -j\omega\left[\hat{a}_\theta F_\theta + \hat{a}_\phi F_\phi\right]$$

$$\bar{H} = \left(\hat{a}_\theta \cos\theta\cos\phi - \hat{a}_\phi \sin\phi\right) \frac{a^2\omega^2\mu\epsilon I_o}{4} \frac{e^{-jkr}}{r}$$

$$\bar{E} = j\omega\eta\left[-\hat{a}_\theta F_\phi + \hat{a}_\phi F_\theta\right]$$

$$\bar{E} = \left(\hat{a}_\theta \sin\phi + \hat{a}_\phi \cos\theta\cos\phi\right) \frac{-\eta a^2\omega^2\mu\epsilon I_o}{4} \frac{e^{-jkr}}{r}$$

b) $$U(\theta,\phi) = \frac{r^2}{2\eta}\left[|E_\theta|^2 + |E_\phi|^2\right] \quad (2\text{-}12a)$$

$$= \frac{\cancel{r^2}}{2\cancel{\eta}}\left[\frac{\eta^{\cancel{2}} a^4\omega^4\mu^2\epsilon^2 |I_o|^2}{16\,\cancel{r^2}}\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)\right]$$

$$U(\theta,\phi) = \frac{a^4\omega^4\mu^2\epsilon^2|I_o|^2}{32}\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)$$

(2-13) $$P_{rad} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{a^4\omega^4\mu^2\epsilon^2|I_0|^2}{32}\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)\sin\theta\, d\theta d\phi$$

$$= \underbrace{\frac{a^4\omega^4\mu^2\epsilon^2|I_0|^2}{32}}_{B_0} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\sin^2\phi\sin\theta + \cos^2\phi\sin\theta\cos^2\theta\right) d\theta d\phi$$

$$= B_0 \int_{\phi=0}^{2\pi} \sin^2\phi\left(-\cos\theta\right)\Big|_{\theta=0}^{\pi} + \cos^2\phi\left(\frac{-\cos^3\theta}{3}\right)\Big|_{\theta=0}^{\pi} d\phi$$

$$= B_0 \int_{\phi=0}^{2\pi}\left(2\sin^2\phi + \tfrac{2}{3}\cos^2\phi\right) d\phi$$

$$= B_0\left[\phi - \frac{\sin 2\phi}{2} + \frac{\phi}{3} + \frac{\sin 2\phi}{6}\right]\Big|_{\phi=0}^{2\pi}$$

$$= B_0\left[(2\pi - 0) - (0-0) + \left(\frac{2\pi}{3} - 0\right) + (0-0)\right]$$

$$P_{rad} = B_0\, \tfrac{8\pi}{3}$$

(2-16) $$D = \frac{4\pi U}{P_{rad}} = \frac{4\pi B_0\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)}{\tfrac{8\pi}{3} B_0}$$

$$\underline{\underline{D(\theta,\phi) = \tfrac{3}{2}\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)}}$$

$$\underline{\underline{D_0 = D_{max} = \tfrac{3}{2} = 1.7609\ dBi}}$$ (as expected)