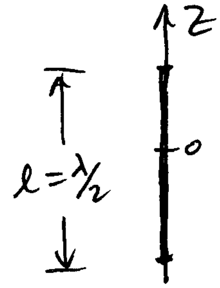
4.26 Write the far-zone electric and magnetic fields radiated by a magnetic dipole of $l = \lambda/2$ aligned with the z-axis. Assume a sinusoidal magnetic current with maximum value I_{m0}.



From problem 4.9, the magnetic field of an infinitesimal magnetic dipole of length ℓ is

$$\bar{H} = \hat{a}_r \frac{I_m \ell \cos\theta}{2\pi\eta} \frac{e^{-jkr}}{r^2}\left(1 + \frac{1}{jkr}\right) + \hat{a}_\theta \frac{jk I_m \ell \sin\theta}{4\pi\eta} \frac{e^{-jkr}}{r}\left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right)$$

For the far-field keep only term(s) $\propto \frac{1}{r}$ for inf. dipole

$$\bar{H}_{FF} = \hat{a}_\theta \frac{jk I_m \ell \sin\theta}{4\pi\eta} \frac{e^{-jkr}}{r}$$

Following the derivation in section 4.5.2, the contribution of each infinitesimal piece of the $\lambda/2$ magnetic dipole in the far-field is –

$$dH_\theta \simeq \frac{jk I_m(z')\, dz' \sin\theta}{4\pi\eta} \frac{e^{-jkR}}{R}$$

Note:
$I_m \to I_m(z')$
$\ell \to dz'$

Per (4-46), $R \simeq r - z'\cos\theta$ for phase
$R \simeq r$ for amplitude

Now

$$dH_\theta \simeq \frac{jk I_m(z') \sin\theta}{4\pi\eta} \frac{e^{-jkr}}{r} e^{+jkz'\cos\theta}\, dz'$$

Per an adaptation of (4-56)

$$I_M(z') = \begin{cases} I_{m0} \sin\left[k\left(\ell/2 - z'\right)\right] & 0 \le z' \le \ell/2 \\ I_{m0} \sin\left[k\left(\ell/2 + z'\right)\right] & -\ell/2 \le z' \le 0 \end{cases}$$

Per an adaptation of (4-58a), we get H_θ for our $\lambda/2$ magnetic dipole by integrating dH_θ

$$H_\theta \cong \int_{-\ell/2}^{\ell/2} dH_\theta = \frac{jk\sin\theta}{4\pi\eta}\,\frac{e^{-jkr}}{r}\int_{-\ell/2}^{\ell/2} I_m(z')\,e^{jkz'\cos\theta}\,dz'$$

$$\cong \frac{jI_{m0}k\sin\theta}{4\pi\eta}\,\frac{e^{-jkr}}{r}\left[\int_{0}^{\lambda/4}\sin\left[\frac{2\pi}{\lambda}\left(\lambda/4 - z'\right)\right]e^{jkz'\cos\theta}\,dz' + \int_{-\lambda/4}^{0}\sin\left[\frac{2\pi}{\lambda}\left(\lambda/4 + z'\right)\right]e^{jkz'\cos\theta}\,dz'\right]$$

Note: $\sin\left[\pi/2 \pm kz'\right] = \underbrace{\sin(\pi/2)}_{1}\cos kz' \pm \underbrace{\cos\pi/2}_{0}\sin kz' = \cos kz'$

$$H_\theta \simeq \frac{jI_{m0}k\sin\theta}{4\pi\eta}\,\frac{e^{-jkr}}{r}\int_{-\lambda/4}^{\lambda/4}\cos(kz')\,e^{jkz'\cos\theta}\,dz'$$

Use $\int \cos bx\, e^{ax}\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

where $b = k$, $a = jk\cos\theta$, & $x = z'$

$$H_\theta \simeq \frac{jI_{m0}k\sin\theta}{4\pi\eta}\,\frac{e^{-jkr}}{r}\left[\frac{e^{jkz'\cos\theta}}{-k^2\cos^2\theta + k^2}\left(jk\cos\theta\cos kz' + k\sin kz'\right)\right]\Bigg|_{z'=-\lambda/4}^{\lambda/4}$$

$$\simeq \frac{jI_{m0}k\sin\theta}{4\pi\eta}\,\frac{e^{-jkr}}{r}\left[\frac{e^{j\frac{\pi}{2}\cos\theta}}{k^2(1-\cos^2\theta)}\left(jk\cos\theta\underbrace{\cos\pi/2}_{0} + k\underbrace{\sin\pi/2}_{1}\right) - \frac{e^{-j\frac{\pi}{2}\cos\theta}}{k^2(1-\cos^2\theta)}\left(jk\cos\theta\underbrace{\cos(-\pi/2)}_{0} + k\underbrace{\sin(-\pi/2)}_{-1}\right)\right]$$

$$H_\theta \simeq \frac{j I_{mo} K \sin\theta}{4\pi\eta} \frac{e^{-jKr}}{r} \frac{K}{K^2 \sin^2\theta} \left[e^{j\frac{\pi}{2}\cos\theta} + e^{-j\frac{\pi}{2}\cos\theta} \right]$$

$$\simeq \frac{j I_{mo}}{4\pi\eta} \frac{e^{-jKr}}{r} \frac{1}{\sin\theta} \left[2\cos\left(\tfrac{\pi}{2}\cos\theta\right) \right]$$

$$\simeq j \frac{I_{mo}}{2\pi\eta} \frac{e^{-jKr}}{r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$\boxed{\bar{H}_{FF} = \hat{a}_\theta \, j \frac{I_{mo}}{2\pi\eta} \frac{e^{-jKr}}{r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}}$$

Per (3-59b),

$$E_r \simeq 0$$

$$E_\theta \simeq \eta \cancel{H_\phi}^{\,0} = 0$$

$$E_\phi \simeq -\eta H_\theta$$

$$\boxed{\bar{E}_{FF} = -\hat{a}_\phi \, j \frac{I_{mo}}{2\pi} \frac{e^{-jKr}}{r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}}$$