If the vector <u>electric</u> potential for an antenna is $\bar{F} = \hat{a}_z F_0 \dfrac{e^{-jkr}}{r}$, find $\bar{E}$ and $\bar{H}$ **everywhere**. Give your answers in spherical coordinates. Assume $\bar{A} = 0$. Factor out common terms, e.g., $F_0 \dfrac{e^{-jkr}}{r}$.

First, convert $\hat{a}_z$ unit vector in $\bar{F}$ to spherical coordinates:

$$\bar{F} = (\hat{a}_r \cos\theta - \hat{a}_\theta \sin\theta) F_0 \frac{e^{-jkr}}{r}$$

Per (3-16), $\bar{E} = \bar{E}_F = \dfrac{-1}{\epsilon} \bar{\nabla} \times \bar{F}$

$$\bar{E}_F = \frac{-1}{\epsilon} \left\{ \frac{1}{r\sin\theta} \left[\frac{\partial (\overset{\to 0}{\cancel{F_\phi}} \sin\theta)}{\partial\theta} - \underbrace{\frac{\partial \cancel{F_\theta}}{\cancel{\partial\phi}}}_{\to 0} \right] \hat{a}_r + \frac{1}{r} \left[\frac{1}{\sin\theta} \underbrace{\frac{\partial \cancel{F_r}}{\cancel{\partial\phi}}}_{\to 0} - \frac{\partial (r \overset{0}{\cancel{F_\phi}})}{\partial r} \right] \hat{a}_\theta \right.$$

$$\left. + \frac{1}{r} \left[\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta} \right] \hat{a}_\phi \right\}$$

$$= \frac{-\hat{a}_\phi}{\epsilon r} \left[\frac{\partial}{\partial r} \left(-\sin\theta F_0 e^{-jkr} \right) - \frac{\partial}{\partial\theta} \left(\cos\theta F_0 \frac{e^{-jkr}}{r} \right) \right]$$

$$= \frac{-\hat{a}_\phi}{\epsilon r} \left[-\sin\theta F_0 (-jk) e^{-jkr} - (-\sin\theta) F_0 \frac{e^{-jkr}}{r} \right]$$

$$\boxed{\bar{E}_F = -\hat{a}_\phi \frac{F_0}{\epsilon} \sin\theta \left(\frac{e^{-jkr}}{r} \right) \left(jk + \frac{1}{r} \right)}$$

(1) Use Faraday's Law (3-21) to find $\bar{H}_F$

$$\bar{\nabla} \times \bar{E}_F = -\underbrace{\cancel{\bar{M}}}_{\to 0} - j\omega\mu \bar{H}_F$$

$$\bar{H}_F = \frac{\bar{\nabla} \times \bar{E}_F}{-j\omega\mu}$$

$$\bar{H}_F = \frac{1}{-j\omega\mu}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial(E_\phi \sin\theta)}{\partial\theta} - \frac{\partial \cancel{E_\theta}}{\partial\phi}\to 0\right]\hat{a}_r\right.$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial \cancel{E_r}}{\partial\phi}\to 0 - \frac{\partial(rE_\phi)}{\partial r}\right]\hat{a}_\theta$$

$$\left.+\frac{1}{r}\left[\frac{\partial(r\cancel{E_\theta}^{\,0})}{\partial r} - \frac{\partial \cancel{E_r}^{\,0}}{\partial\theta}\right]\hat{a}_\phi\right\}$$

$$= \frac{1}{-j\omega\mu}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{F_0}{\epsilon}\sin^2\theta\,\frac{e^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\right)\hat{a}_r\right.$$

$$\left.-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{-F_0}{\epsilon}\sin\theta\, e^{-jkr}\left(jk+\frac{1}{r}\right)\right)\hat{a}_\theta\right\}$$

$$= \frac{1}{-j\omega\mu}\left\{\left[\frac{1}{r\sin\theta}\frac{-F_0}{\epsilon}\,2\sin\theta\cos\theta\,\frac{e^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\right]\hat{a}_r\right.$$

$$\left.+\left[\frac{1}{r}\frac{F_0}{\epsilon}\sin\theta\left(e^{-jkr}(-jk)\left(jk+\frac{1}{r}\right)+e^{-jkr}\left(\frac{-1}{r^2}\right)\right)\right]\hat{a}_\theta\right\}$$

$$\boxed{\bar{H}_F = \frac{F_0}{j\omega\mu\epsilon}\left(\frac{e^{-jkr}}{r}\right)\left[\frac{2\cos\theta}{r}\left(jk+\frac{1}{r}\right)\hat{a}_r + \sin\theta\left(-k^2+\frac{jk}{r}+\frac{1}{r^2}\right)\hat{a}_\theta\right]}$$

(2) Find $\bar{H}_F$ using (3-26), as a check

$$\bar{H}_F = -j\omega\bar{F} - \frac{j}{\omega\mu\epsilon}\bar{\nabla}(\bar{\nabla}\cdot\bar{F})$$

$$\bar{\nabla}\cdot\bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial \cancel{F_\phi}}{\partial\phi}\to 0$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(F_0\cos\theta\, r\, e^{-jkr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta F_0\frac{e^{-jkr}}{r}\right)$$

$$= \frac{F_0\cos\theta}{r^2}\frac{\partial}{\partial r}\left(re^{-jkr}\right) + \frac{F_0}{r\sin\theta}\frac{e^{-jkr}}{r}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\right)$$

$$\bar{\nabla}\cdot\bar{F} = \frac{F_o\cos\theta}{r^2}\left[e^{-jkr} + r(-jk)e^{-jkr}\right] + \frac{F_o}{r^2 \cancel{\sin\theta}}\, e^{-jkr}\left(-2\cancel{\sin\theta}\cos\theta\right)$$

$$= \frac{F_o\cos\theta}{r^2}\, e^{-jkr}\left[1 - jkr - 2\right]$$

$$= -\frac{F_o\cos\theta}{r^2}\, e^{-jkr}\left(1 + jkr\right)$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{F}) = \hat{a}_r \frac{\partial}{\partial r}\left[-\frac{F_o\cos\theta}{r^2}\, e^{-jkr}(1+jkr)\right]$$

$$+ \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\frac{F_o\cos\theta}{r^2}\, e^{-jkr}(1+jkr)\right]$$

$$+ \hat{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\underbrace{\left[-\frac{F_o\cos\theta}{r^2}\, e^{-jkr}(1+jkr)\right]}_{\to 0}$$

$$= \hat{a}_r(-F_o\cos\theta)\frac{\partial}{\partial r}\left[\frac{e^{-jkr}}{r^2} + jk\frac{e^{-jkr}}{r}\right]$$

$$+ \hat{a}_\theta \frac{-F_o}{r^3}\, e^{-jkr}(1+jkr)\frac{\partial}{\partial\theta}(\cos\theta)$$

$$= \hat{a}_r(+F_o\cos\theta)\left[\frac{+2}{r^3}e^{-jkr} + \frac{+jk}{r^2}e^{-jkr} + \frac{+jk}{r^2}e^{-jkr} + \frac{(jk)^2}{r}e^{-jkr}\right]$$

$$+ \hat{a}_\theta \frac{+F_o}{r^3}\, e^{-jkr}(1+jkr)(+\sin\theta)$$

$$= \hat{a}_r F_o\cos\theta\left(\frac{e^{-jkr}}{r}\right)\left[\frac{2}{r^2} + \frac{2jk}{r} - k^2\right]$$

$$+ \hat{a}_\theta F_o\sin\theta\left(\frac{e^{-jkr}}{r}\right)\left(\frac{1}{r^2} + \frac{jk}{r}\right)$$

So,

$$\bar{H}_F = -j\omega F_o\left(\frac{e^{-jkr}}{r}\right)\left(\hat{a}_r\cos\theta - \hat{a}_\theta\sin\theta\right) + \hat{a}_r\left(\frac{-jF_o\cos\theta}{\omega\mu\epsilon}\right)\left(\frac{e^{-jk}}{r}\right)\left[\frac{2}{r^2} + \frac{2jk}{r} - k^2\right]$$

$$+ \hat{a}_\theta\left(\frac{-j}{\omega\mu\epsilon}\right)F_o\sin\theta\left(\frac{e^{-jkr}}{r}\right)\left(\frac{1}{r^2} + \frac{jk}{r}\right) \quad \text{Note: } -j = \frac{1}{j}$$

$$\bar{H}_F = \hat{a}_r \frac{F_o \cos\theta}{j}\left(\frac{e^{-jkr}}{r}\right)\left[\omega\left(\frac{\omega\mu\epsilon}{\omega\mu\epsilon}\right) + \frac{1}{\omega\mu\epsilon}\left(\frac{2}{r^2} + \frac{2jk}{r} - k^2\right)\right]$$

$$+ \hat{a}_\theta \frac{F_o \sin\theta}{j}\left(\frac{e^{-jkr}}{r}\right)\left[-\omega\left(\frac{\omega\mu\epsilon}{\omega\mu\epsilon}\right) + \frac{1}{\omega\mu\epsilon}\left(\frac{1}{r^2} + \frac{jk}{r}\right)\right]$$

$$= \frac{F_o}{j\omega\mu\epsilon}\left(\frac{e^{-jkr}}{r}\right)\left[\underbrace{\omega^2\mu\epsilon}_{\to k^2} + \frac{2}{r^2} + \frac{2jk}{r} - k^2\right]\cos\theta\,\hat{a}_r$$

$$+ \frac{F_o}{j\omega\mu\epsilon}\left(\frac{e^{-jkr}}{r}\right)\left[\underbrace{-\omega^2\mu\epsilon}_{\to -k^2} + \frac{1}{r^2} + \frac{jk}{r}\right]\sin\theta\,\hat{a}_\theta$$

Note: $\omega^2\mu\epsilon = \frac{\omega^2}{c^2} = k^2$ since $c = \frac{1}{\sqrt{\mu\epsilon}}$

$$\boxed{\bar{H}_F = \frac{F_o}{j\omega\mu\epsilon}\left(\frac{e^{-jkr}}{r}\right)\left[\frac{2\cos\theta}{r}\left(jk + \frac{1}{r}\right)\hat{a}_r + \sin\theta\left(-k^2 + \frac{jk}{r} + \frac{1}{r^2}\right)\hat{a}_\theta\right]}$$

Same answer!!

Repeat $\bar{E}_F$ to get on same page

$$\boxed{\bar{E}_F = -\hat{a}_\phi \frac{F_o}{\epsilon}\sin\theta\left(\frac{e^{-jkr}}{r}\right)\left(jk + \frac{1}{r}\right)}$$