If the vector electric potential for an antenna is $\overline{F} = \hat{a}_y C_0 \dfrac{e^{-jkr}}{r}$, find $\overline{E}$ and $\overline{H}$ everywhere. Give your answers in spherical coordinates. Assume $\overline{A} = 0$. Factor out common terms, e.g., $C_0 \dfrac{e^{-jkr}}{r}$.

Per (3-16), $\overline{E} = \overline{E}_F = \dfrac{-1}{\epsilon} \nabla \times \overline{F}$

First, convert $\hat{a}_y$ unit vector in $\overline{F}$ to spherical coordinates:

$$\overline{F} = C_0 \frac{e^{-jkr}}{r}\left[\sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi\right]$$

Now, compute $\overline{E}_F$

$$\overline{E} = \overline{E}_F = \frac{-1}{\epsilon}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial\phi}\right]\hat{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rF_\phi\right)\right]\hat{a}_\theta \right.$$
$$\left. + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial\theta}\right]\hat{a}_\phi\right\}$$

$$= -\frac{C_0}{\epsilon}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\cos\phi\,\frac{e^{-jkr}}{r}\sin\theta\right) - \frac{\partial}{\partial\phi}\left(\cos\theta\sin\phi\,\frac{e^{-jkr}}{r}\right)\right]\hat{a}_r\right.$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\,\frac{e^{-jkr}}{r}\right) - \frac{\partial}{\partial r}\left(\frac{r\,e^{-jkr}}{r}\cos\phi\right)\right]\hat{a}_\theta$$
$$\left. + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{r\,e^{-jkr}}{r}\cos\theta\sin\phi\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-jkr}}{r}\sin\theta\sin\phi\right)\right]\hat{a}_\phi\right\}$$

Cancel!

$$= -\frac{C_0}{\epsilon}\left\{\frac{1}{r\sin\theta}\left[\cos\theta\,\cancel{\cos\phi}\,\frac{e^{-jkr}}{r} - \cos\theta\,\cancel{\cos\phi}\,\frac{e^{-jkr}}{r}\right]\hat{a}_r\right.$$
$$+ \frac{1}{r}\left[\frac{1}{\cancel{\sin\theta}}\,\cancel{\sin\theta}\cos\phi\,\frac{e^{-jkr}}{r} - (-jk)e^{-jkr}\cos\phi\right]\hat{a}_\theta$$
$$\left. + \frac{1}{r}\left[(-jk)e^{-jkr}\cos\theta\sin\phi - \frac{e^{-jkr}}{r}\cos\theta\sin\phi\right]\hat{a}_\phi\right\}$$

$$\bar{E} = \bar{E}_F = \frac{-C_o}{\epsilon}\frac{e^{-jkr}}{r}\left[\cos\phi\left(\frac{1}{r} + jk\right)\hat{a}_\theta + \cos\theta\sin\phi\left(-jk - \frac{1}{r}\right)\hat{a}_\phi\right]$$

$$\underline{\underline{\bar{E} = \bar{E}_F = \frac{C_o}{\epsilon}\frac{e^{-jkr}}{r}\left(jk + \frac{1}{r}\right)\left[-\cos\phi\,\hat{a}_\theta + \cos\theta\sin\phi\,\hat{a}_\phi\right]}}$$

Use Faraday's Law (3-21) to find $\bar{H}_F$: $\bar{\nabla}\times\bar{E} = -\cancel{\bar{M}}^{\,0} - j\omega\mu\bar{H}$

$$\bar{H} = \bar{H}_F = \frac{\bar{\nabla}\times\bar{E}_F}{-j\omega\mu} = \frac{1}{-j\omega\mu}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right) - \frac{\partial E_\theta}{\partial\phi}\right]\hat{a}_r\right.$$
$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial \cancel{E_r}^{\,0}}{\partial\phi} - \frac{\partial}{\partial r}\left(rE_\phi\right)\right]\hat{a}_\theta$$
$$\left.+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial \cancel{E_r}^{\,0}}{\partial\theta}\right]\hat{a}_\phi\right\}$$

$$= \frac{-C_o}{j\omega\mu\epsilon}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{e^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\cos\theta\sin\theta\sin\phi\right)\right.\right.$$
$$\left.+\frac{\partial}{\partial\phi}\left(\frac{e^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\cos\phi\right)\right]\hat{a}_r$$
$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{re^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\cos\theta\sin\phi\right)\hat{a}_\theta$$
$$\left.-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{re^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\cos\phi\right)\hat{a}_\phi\right\}$$

$$= \frac{-C_o}{j\omega\mu\epsilon}\left\{\frac{1}{r\sin\theta}\left[\frac{e^{-jkr}}{r}\left(jk+\frac{1}{r}\right)\sin\phi\left(\cos^2\theta - \sin^2\theta\right) + \frac{e^{-jkr}}{r}\left(jk+\frac{1}{r}\right)(-\sin\phi)\right]\hat{a}_r\right.$$
$$-\frac{1}{r}\left[(-jk)e^{-jkr}\left(jk+\frac{1}{r}\right)\cos\theta\sin\phi + e^{-jkr}\left(0-\frac{1}{r^2}\right)\cos\theta\sin\phi\right]\hat{a}_\theta$$
$$\left.-\frac{1}{r}\left[(-jk)e^{-jkr}\left(jk+\frac{1}{r}\right)\cos\phi + e^{-jkr}\left(0-\frac{1}{r^2}\right)\cos\phi\right]\hat{a}_\phi\right\}$$

$$= \frac{-C_o}{j\omega\mu\epsilon}\frac{e^{-jkr}}{r}\left\{\left[\frac{\sin\phi}{r\sin\theta}\left(jk+\frac{1}{r}\right)\left(\cos^2\theta - \sin^2\theta - 1\right)\right]\hat{a}_r\right.$$
$$+\left[\cos\theta\sin\phi\left(-k^2 + \frac{jk}{r} + \frac{1}{r^2}\right)\right]\hat{a}_\theta$$
$$\left.+\left[\cos\phi\left(-k^2 + \frac{jk}{r} + \frac{1}{r^2}\right)\right]\hat{a}_\phi\right\}$$

Note: $\cos^2\theta - \sin^2\theta - 1 = \cos^2\theta - \sin^2\theta - (\cos^2\theta + \sin^2\theta) = -2\sin^2\theta$

$$\boxed{\bar{H} = \frac{C_0}{j\omega\mu\epsilon}\frac{e^{-jkr}}{r}\left[+2\sin\theta\sin\phi\left(\frac{jk}{r}+\frac{1}{r^2}\right)\hat{a}_r + \cos\theta\sin\phi\left(k^2 - \frac{jk}{r} - \frac{1}{r^2}\right)\hat{a}_\theta + \cos\phi\left(k^2 - \frac{jk}{r} - \frac{1}{r^2}\right)\hat{a}_\phi\right]}$$

An alternate approach to find $\bar{H} = \bar{H}_F$ is to use

(3-26) $\bar{H} = \bar{H}_F = -j\omega\bar{F} - \frac{j}{\omega\mu\epsilon}\bar{\nabla}(\bar{\nabla}\cdot\bar{F})$

$$\bar{\nabla}\cdot\bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(C_0 r e^{-jkr}\sin\theta\sin\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(C_0\frac{e^{-jkr}}{r}\cos\theta\sin\theta\sin\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(C_0\frac{e^{-jkr}}{r}\cos\phi\right)$$

$$= \frac{1}{r^2}C_0\sin\theta\sin\phi\left(e^{-jkr} + r(-jk)e^{-jkr}\right) + \frac{1}{r\sin\theta}C_0\frac{e^{-jkr}}{r}\sin\phi\left(-\sin^2\theta + \cos^2\theta\right) + \frac{1}{r\sin\theta}C_0\frac{e^{-jkr}}{r}(-\sin\phi)$$

$$= C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r^2}(1-jkr) + \frac{C_0}{r\sin\theta}\sin\phi\frac{e^{-jkr}}{r}\underbrace{(\cos^2\theta - \sin^2\theta - 1)}_{-2\sin^2\theta}$$

$$= C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r^2}(1-jkr) + C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r^2}(-2)$$

$$= C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r^2}(-1-jkr) = -C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r^2}(1+jkr)$$

Next, compute $\bar{\nabla}(\bar{\nabla}\cdot\bar{F}) = \hat{a}_r\frac{\partial(\bar{\nabla}\cdot\bar{F})}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial(\bar{\nabla}\cdot\bar{F})}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial(\bar{\nabla}\cdot\bar{F})}{\partial\phi}$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{F}) = -\hat{a}_r C_0 \sin\theta\sin\phi \frac{\partial}{\partial r}\left(e^{-jkr}\left(\frac{1}{r^2}+\frac{jk}{r}\right)\right)$$

$$-\hat{a}_\theta \frac{C_0\sin\phi}{r}\frac{e^{-jkr}}{r^2}(1+jkr)\frac{\partial(\sin\theta)}{\partial\theta}$$

$$-\hat{a}_\phi \frac{C_0\cancel{\sin\theta}}{r\cancel{\sin\theta}}\frac{e^{-jkr}}{r^2}(1+jkr)\frac{\partial(\sin\phi)}{\partial\phi}$$

$$= -\hat{a}_r C_0\sin\theta\sin\phi\left[(-jk)e^{-jkr}\left(\frac{1}{r^2}+\frac{jk}{r}\right)+e^{-jkr}\left(\frac{-2}{r^3}-\frac{jk}{r^2}\right)\right]$$

$$-\hat{a}_\theta C_0\cos\theta\sin\phi\frac{e^{-jkr}}{r^3}(1+jkr)$$

$$-\hat{a}_\phi C_0\cos\phi\frac{e^{-jkr}}{r^3}(1+jkr)$$

$$= -\hat{a}_r C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r^3}\left(k^2r^2-j2kr-2\right)$$

$$-\hat{a}_\theta C_0\cos\theta\sin\phi\frac{e^{-jkr}}{r^3}(1+jkr)$$

$$-\hat{a}_\phi C_0\cos\phi\frac{e^{-jkr}}{r^3}(1+jkr)$$

Now, substitute $\bar{F}$ and $\bar{\nabla}(\bar{\nabla}\cdot\bar{F})$ into (3-26)

$$\bar{H} = -j\omega C_0\frac{e^{-jkr}}{r}\left[\sin\theta\sin\phi\,\hat{a}_r+\cos\theta\sin\phi\,\hat{a}_\theta+\cos\phi\,\hat{a}_\phi\right]$$

$$+\hat{a}_r\frac{jC_0}{\omega\mu\epsilon}\sin\theta\sin\phi\frac{e^{-jkr}}{r}\left(k^2-\frac{j2k}{r}-\frac{2}{r^2}\right)$$

$$+\hat{a}_\theta\frac{jC_0}{\omega\mu\epsilon}\cos\theta\sin\phi\frac{e^{-jkr}}{r}\left(\frac{jk}{r}+\frac{1}{r^2}\right)$$

$$+\hat{a}_\phi\frac{jC_0}{\omega\mu\epsilon}\cos\phi\frac{e^{-jkr}}{r}\left(\frac{jk}{r}+\frac{1}{r^2}\right)$$

$$= \hat{a}_r C_0\sin\theta\sin\phi\frac{e^{-jkr}}{r}\left[-j\omega+\frac{jk^2}{\omega\mu\epsilon}+\frac{2k}{\omega\mu\epsilon r}-\frac{j2}{\omega\mu\epsilon r^2}\right]$$

$$+\hat{a}_\theta C_0\cos\theta\sin\phi\frac{e^{-jkr}}{r}\left[-j\omega-\frac{k}{\omega\mu\epsilon r}+\frac{j}{\omega\mu\epsilon r^2}\right]$$

$$+\hat{a}_\phi C_0\cos\phi\frac{e^{-jkr}}{r}\left[-j\omega-\frac{k}{\omega\mu\epsilon r}+\frac{j}{\omega\mu\epsilon r^2}\right]$$

Factor out $\dfrac{C_0}{j\omega\mu\epsilon}\dfrac{e^{-jkr}}{r}$ from all terms

$$
\begin{aligned}
\bar{H} = \frac{C_0}{j\omega\mu\epsilon}\frac{e^{-jkr}}{r}\Bigg\{ & \hat{a}_r \sin\theta\sin\phi\left[-j\omega(j\omega\mu\epsilon) + \frac{jk^2(j\omega\mu\epsilon)}{\omega\mu\epsilon} + \frac{2k(j\omega\mu\epsilon)}{\omega\mu\epsilon r} - \frac{j2(j\omega\mu\epsilon)}{\omega\mu\epsilon r^2}\right] \\
& + \hat{a}_\theta \cos\theta\sin\phi\left[-j\omega(j\omega\mu\epsilon) - \frac{kj\omega\mu\epsilon}{\omega\mu\epsilon r} + \frac{j(j\omega\mu\epsilon)}{\omega\mu\epsilon r^2}\right] \\
& + \hat{a}_\phi \cos\phi\left[-j\omega(j\omega\mu\epsilon) - \frac{kj\omega\mu\epsilon}{\omega\mu\epsilon r} + \frac{j(j\omega\mu\epsilon)}{\omega\mu\epsilon r^2}\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{C_0}{j\omega\mu\epsilon}\frac{e^{-jkr}}{r}\Bigg\{ & \hat{a}_r \sin\theta\sin\phi\left[\overbrace{\overset{k^2}{\cancel{\omega^2\mu\epsilon}} - k^2}^{\text{cancel}} + \frac{j2k}{r} + \frac{2}{r^2}\right] \\
& + \hat{a}_\theta \cos\theta\sin\phi\left[\overset{k^2}{\cancel{\omega^2\mu\epsilon}} - \frac{jk}{r} - \frac{1}{r^2}\right] \\
& + \hat{a}_\phi \cos\phi\left[\overset{k^2}{\cancel{\omega^2\mu\epsilon}} - \frac{jk}{r} - \frac{1}{r^2}\right]\Bigg\}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\bar{H} = \frac{C_0}{j\omega\mu\epsilon}\frac{e^{-jkr}}{r}\Bigg\{ & 2\sin\theta\sin\phi\left(\frac{jk}{r} + \frac{1}{r^2}\right)\hat{a}_r \\
& + \cos\theta\sin\phi\left(k^2 - \frac{jk}{r} - \frac{1}{r^2}\right)\hat{a}_\theta \\
& + \cos\phi\left(k^2 - \frac{jk}{r} - \frac{1}{r^2}\right)\hat{a}_\phi\Bigg\}
\end{aligned}
}
$$

same answer as was found using (3-21)!
(More work)