3.1 If $\mathbf{H}_e = j\omega\varepsilon\, \nabla \times \mathbf{\Pi}_e$, where $\mathbf{\Pi}_e$ is the electric Hertzian potential, show that

(a) $\nabla^2\mathbf{\Pi}_e + k^2\mathbf{\Pi}_e = j\dfrac{1}{\omega\varepsilon}\mathbf{J}$ (b) $\mathbf{E}_e = k^2\mathbf{\Pi}_e + \nabla(\nabla \cdot \mathbf{\Pi}_e)$

(c) $\mathbf{\Pi}_e = -j\dfrac{1}{\omega\mu\varepsilon}\mathbf{A}$

a) Start w/ Faraday's Law $\bar{\nabla}\times\bar{E}_e = -j\omega\mu\bar{H}_e$

and substitute $j\omega\epsilon\bar{\nabla}\times\bar{\Pi}_e$ for $\bar{H}_e$

$$\bar{\nabla}\times\bar{E}_e = -j\omega\mu(j\omega\epsilon\bar{\nabla}\times\bar{\Pi}_e) = \omega^2\underbrace{\mu\epsilon}_{k^2}\bar{\nabla}\times\bar{\Pi}_e$$

$$\bar{\nabla}\times\left[\bar{E}_e - k^2\bar{\Pi}_e\right] = 0$$

Use the vector identity $\bar{\nabla}\times(-\bar{\nabla}\phi_e) = 0$, to say that $\bar{E}_e - k^2\bar{\Pi}_e = -\bar{\nabla}\phi_e$

$$\hookrightarrow \underline{\bar{E}_e = -\bar{\nabla}\phi_e + k^2\bar{\Pi}_e}$$

Next, take the curl of $\bar{H}_e$ above

$$\bar{\nabla}\times\bar{H}_e = \bar{\nabla}\times\left[j\omega\epsilon\bar{\nabla}\times\bar{\Pi}_e\right] = j\omega\epsilon(\bar{\nabla}\times\bar{\nabla}\times\bar{\Pi}_e)$$

Use the vector identity $\bar{\nabla}\times\bar{\nabla}\times\bar{A} = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \bar{\nabla}^2\bar{A}$

$$\bar{\nabla}\times\bar{H}_e = j\omega\epsilon\bar{\nabla}(\bar{\nabla}\cdot\bar{\Pi}_e) - j\omega\epsilon\bar{\nabla}^2\bar{\Pi}_e$$

However, by Ampere's Law, $\bar{\nabla}\times\bar{H}_e = \bar{J} + j\omega\epsilon\bar{E}_e$, so

$$\bar{J} + j\omega\epsilon\bar{E}_e = j\omega\epsilon\bar{\nabla}(\bar{\nabla}\cdot\bar{\Pi}_e) - j\omega\epsilon\bar{\nabla}^2\bar{\Pi}_e$$

Now substitute $-\bar{\nabla}\phi_e + k^2\bar{\Pi}_e$ for $\bar{E}_e$

$$\bar{J} - j\omega\epsilon\bar{\nabla}\phi_e + j\omega\epsilon k^2\bar{\Pi}_e = j\omega\epsilon\bar{\nabla}(\bar{\nabla}\cdot\bar{\Pi}_e) - j\omega\epsilon\bar{\nabla}^2\bar{\Pi}_e$$

Re-arrange to get

$$+j\omega\epsilon\bar{\nabla}^2\bar{\Pi}_e + j\omega\epsilon k^2\bar{\Pi}_e = -\bar{J} + j\omega\epsilon\bar{\nabla}\phi_e + j\omega\epsilon\bar{\nabla}(\bar{\nabla}\cdot\bar{\Pi}_e)$$

$$\bar{\nabla}^2\bar{\Pi}_e + k^2\bar{\Pi}_e = \frac{-1}{j\omega\epsilon}\bar{J} + \bar{\nabla}(\phi_e + \bar{\nabla}\cdot\bar{\Pi}_e)$$

If we define $\underline{\underline{\phi_e = -\bar{\nabla}\cdot\bar{\Pi}_e}}$, then

$$\underline{\underline{\bar{\nabla}^2\bar{\Pi}_e + k^2\bar{\Pi}_e = \frac{j}{\omega\epsilon}\bar{J}}} \quad \therefore$$

b) substitute $\phi_e = -\bar{\nabla}\cdot\bar{\Pi}_e$ into $\bar{E}_e = -\bar{\nabla}\phi_e + k^2\bar{\Pi}_e$

to get $\bar{E}_e = \bar{\nabla}(\bar{\nabla}\cdot\bar{\Pi}_e) + k^2\bar{\Pi}_e$

$$\underline{\underline{\bar{E}_e = k^2\bar{\Pi}_e + \bar{\nabla}(\bar{\nabla}\cdot\bar{\Pi}_e)}} \quad \therefore$$

c) Looking at (3-14) $\bar{\nabla}^2\bar{A} + k^2\bar{A} = -\mu\bar{J}$ and part a)

$$\bar{J} = \frac{\omega\epsilon}{j}\left[\bar{\nabla}^2\bar{\Pi}_e + k^2\bar{\Pi}_e\right] = \frac{-1}{\mu}\left[\bar{\nabla}^2\bar{A} + k^2\bar{A}\right]$$

Therefore, $\frac{\omega\epsilon}{j}\bar{\Pi}_e = \frac{-1}{\mu}\bar{A}$ for the eqn to hold

$$\Rightarrow \underline{\underline{\bar{\Pi}_e = \frac{-j}{\omega\mu\epsilon}\bar{A}}} \quad \therefore$$