2.30 The normalized far-zone field pattern of an antenna is given by

$$E = \begin{cases} (\sin\theta\cos^2\phi)^{1/2} & 0 \le \theta \le \pi \text{ and } 0 \le \phi \le \pi/2, 3\pi/2 \le \phi \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the directivity using

(a) the exact expression (b) Kraus' approximate formula

(c) Tai and Pereira's approximate formula

(d) the computer program **Directivity** of this chapter

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a) 'Normalized' electric field means that $\hat{a}\, E_0 \dfrac{e^{-jkr}}{r}$ term has been divided out. Based on (2-8) and notes, the normalized time-average Poynting vector is then

$$\overline{W}_{\text{ave,norm}} = \overline{W}_{\text{rad,norm}} = 0.5\,\text{Re}(\overline{E}\times\overline{H}^*) = \hat{a}_r \frac{|\overline{E}|^2}{2\eta} = \hat{a}_r \frac{\sin\theta\cos^2\phi}{2\eta},$$

and, based on (2-12), the radiation intensity is $U = r^2 W_{\text{rad}} = W_{\text{rad,norm}} = \dfrac{\sin\theta\cos^2\phi}{2\eta}$.

The maximum intensity $U_{\max} = \dfrac{1}{2\eta}$ occurs at $\theta = \pi/2$ and $\phi = 0$.

The maximum directivity is given by (2-16a) $D_{\max} = D_0 = 4\pi U_{\max} / P_{\text{rad}}$ where we can find the total radiated power per (2-13)

$$P_{\text{rad}} = \oiint_\Omega U\, d\Omega = \int_{\phi=1.5\pi}^{0.5\pi}\int_{\theta=0}^{\pi} \frac{\sin\theta\cos^2\phi}{2\eta}\sin\theta\, d\theta d\phi$$

$$= \frac{2}{2\eta}\int_{\phi=0}^{0.5\pi}\cos^2\phi\, d\phi \int_{\theta=0}^{\pi}\sin^2\theta\, d\theta = \frac{1}{\eta}\left[\frac{\phi}{2}+\frac{\sin 2\phi}{4}\right]_0^{0.5\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\pi}.$$

$$= \frac{1}{\eta}\left[\left(\frac{0.5\pi}{2}+\frac{\sin\pi}{4}\right)-\left(\frac{0}{2}+\frac{\sin 0}{4}\right)\right]\left[\left(\frac{\pi}{2}-\frac{\sin\pi}{4}\right)-\left(\frac{0}{2}-\frac{\sin 0}{4}\right)\right]$$

$$= \frac{\pi^2}{8\eta}$$

Therefore, we get $D_{\max} = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi(1/2\eta)}{\pi^2/8\eta} = \dfrac{16}{\pi}$ $\Rightarrow$ **$D_{\max}$ = 5.093 = 7.07 dBi.**

b) For Kraus' approximate formula, we need the HPBW in two orthogonal planes that include U_{max} at $\theta = \pi/2$ and $\phi = 0$. , i.e., the elevation plane wrt θ at $\phi = 0$ and the azimuthal plane wrt ϕ at $\theta = \pi/2$.

For the elevation plane, $U(\theta) = \dfrac{\sin\theta\cos^2 0}{2\eta} = \dfrac{\sin\theta}{2\eta}$, we need the angles where

$$\frac{U(\theta)}{U_{\max}} = \frac{\sin\theta / 2\eta}{2\eta} = \sin\theta_h = 0.5 \Rightarrow \theta_h = \sin^{-1} 0.5 = 30° \text{ or } 150° \Rightarrow \theta_{1d} = 150 - 30 = 120°.$$

For the azimuthal plane, $U(\phi) = \dfrac{\sin 0.5\pi\cos^2\phi}{2\eta} = \dfrac{\cos^2\phi}{2\eta}$, we need the angles where

$$\frac{U(\phi)}{U_{\max}} = \frac{\cos^2\phi / 2\eta}{2\eta} = \cos^2\phi_h = 0.5 \Rightarrow \phi_h = \cos^{-1}\sqrt{0.5} = 45° \text{ or } 315° \Rightarrow \theta_{2d} = 2(45) = 90°.$$

Per (2-27), $D_0 \simeq \dfrac{4\pi(180/\pi)^2}{\theta_{1d}\,\theta_{2d}} = \dfrac{4\pi(180/\pi)^2}{(120)\,90}$ $\Rightarrow$ **$D_{0,\text{Kraus}} = 3.8197 = 5.82$ dBi**.

c) For Tai and Perreia's approximate formula, we again need the HPBW in two orthogonal planes that include U_{max} at $\theta = \pi/2$ and $\phi = 0$. , i.e., the elevation plane wrt θ at $\phi = 0$ and the azimuthal plane wrt ϕ at $\theta = \pi/2$.

Per (2-30), $D_0 \simeq \dfrac{32\ln 2(180/\pi)^2}{\theta_{1d}^2 + \theta_{2d}^2} = \dfrac{32\ln 2(180/\pi)^2}{120^2 + 90^2}$ $\Rightarrow$ **$D_{0,\text{T-P}} = 3.2362 = 5.10$ dBi**.

(d) First, define the radiation intensity in m-file U.m (below) using the Matlab editor.

```
% m-file U.m
function y = U(THETA, PHI)
y = sin(THETA).*(cos(PHI)).^2/2/376.7303;
```

Then, open DIRECTIVITY.m (modified version of Directivity.m w/ 0.1 deg step sizes) in the Matlab editor and press <F5>. Matlab command window:

```
!!!WARNING: Make sure you define radiation intensity in file U.m !!!
Output device option
                    Option (1): Screen
                    Option (2): File
Output device=1
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 180
The lower bound of phi in degrees = -90    % 'Trick' Matlab w/ phi angles.
The upper bound of phi in degrees = 90

Input parameters:
-----------------
The lower bound of theta in degrees =   0
The upper bound of theta in degrees = 180
The lower bound of phi in degrees = -90
The upper bound of phi in degrees =  90

Output parameters:
------------------
Radiated power (watts) = 0.00327
Directivity (dimensionless) = 5.09296
Directivity (dB)  = 7.06970
```

Numerically, we calculated $\Rightarrow$ $\boxed{\boldsymbol{D_0 = 5.09296 = 7.06970\ \text{dBi}}}$.

This result agrees with exact calculation. Not very good agreement with Kraus' or Tai-Perreia's aprroximate formulas as this is **NOT** a very directive radiation pattern.