Chapter 2 Transmission Line Theory

At higher frequencies, we can NOT ignore causality, i.e., it takes time for signals to propagate along interconnections between components in circuits.

This gives rise to transmission line theory where we recognize that currents + voltages are waves that can change with location *and* time.

2.1 Lumped-element circuit model for a transmission line (TL)

→ applies to circuits where the physical dimensions are an appreciable fraction of a wavelength (or more)

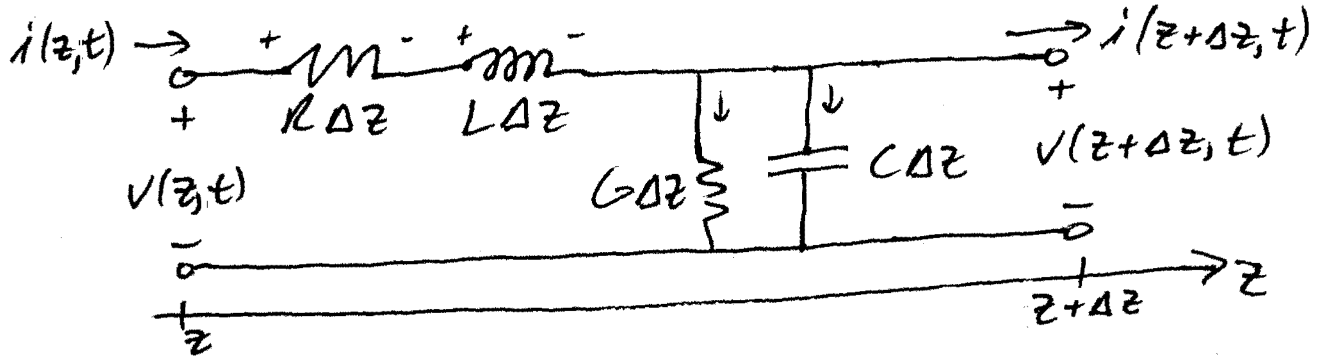
→ Transmission line theory only truly applies to TLs where we have transverse electromagnetic (TEM) waves. Lossy lines where we have voltage drops in the direction of wave propagation are *NOT* true TEM (only get approximate results). Fortunately, most TLs are *low* loss.

2.1 cont.

→ Why TEM? Here, voltage has a unique relationship w/ the electric field and current has a unique relationship w/ the magnetic field.

→ TEM waves are only possible for structures w/ at least two conductors. (One conductor → waveguide)

Lumped-element circuit model for TL



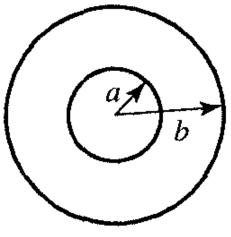
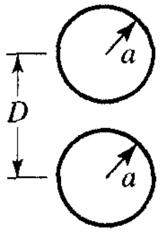
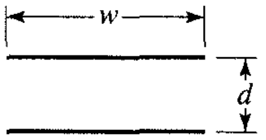
$R \equiv$ Series resistance per unit length (Ω/m)
→ imperfect conductors ($\sigma < \infty$)

$L \equiv$ series inductance per unit length (H/m)
→ energy stored in magnetic field from current flow on TL

$G \equiv$ shunt conductance per unit length (S/m)
→ imperfect insulators ($\sigma > 0$)

$C \equiv$ shunt capacitance per unit length (F/m)
→ energy stored in electric field between charges on conductors

TABLE 2.1 Transmission Line Parameters for Some Common Lines

	COAX	TWO-WIRE	PARALLEL PLATE
			
L	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$	$\frac{\mu d}{w}$
C	$\frac{2\pi\epsilon'}{\ln b/a}$	$\frac{\pi\epsilon'}{\cosh^{-1}(D/2a)}$	$\frac{\epsilon' w}{d}$
R	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_s}{\pi a}$	$\frac{2R_s}{w}$
G	$\frac{2\pi\omega\epsilon''}{\ln b/a}$	$\frac{\pi\omega\epsilon''}{\cosh^{-1}(D/2a)}$	$\frac{\omega\epsilon'' w}{d}$

Microwave Engineering (Fourth Edition), Pozar, Wiley, 2012, ISBN 978-0-470-63155-3.

Notes: 1) $R_S = \frac{1}{\sigma\delta_s}$ where $\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}}$

2) All material parameters for R have to do with conductors.

3) All material parameters for L, C, & G have to do with dielectric material/media surrounding conductors. Note the parameters are expressed in terms of the complex permittivity as defined $\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon(1 - j\sigma/\omega\varepsilon) = \varepsilon(1 - j\tan\delta)$.

TABLE 11.1 Distributed Line Parameters at High Frequencies*

Parameters	Coaxial Line	Two-Wire Line	Planar Line
R (Ω/m)	$\dfrac{1}{2\pi\delta\sigma_c}\left[\dfrac{1}{a}+\dfrac{1}{b}\right]$ $(\delta \ll a, c - b)$	$\dfrac{1}{\pi a\delta\sigma_c}$ $(\delta \ll a)$	$\dfrac{2}{w\delta\sigma_c}$ $(\delta \ll t)$
L (H/m)	$\dfrac{\mu}{2\pi}\ln\dfrac{b}{a}$	$\dfrac{\mu}{\pi}\cosh^{-1}\dfrac{d}{2a}$	$\dfrac{\mu d}{w}$
G (S/m)	$\dfrac{2\pi\sigma}{\ln\dfrac{b}{a}}$	$\dfrac{\pi\sigma}{\cosh^{-1}\dfrac{d}{2a}}$	$\dfrac{\sigma w}{d}$
C (F/m)	$\dfrac{2\pi\varepsilon}{\ln\dfrac{b}{a}}$	$\dfrac{\pi\varepsilon}{\cosh^{-1}\dfrac{d}{2a}}$	$\dfrac{\varepsilon w}{d}$ $(w \gg d)$

$*\delta = \dfrac{1}{\sqrt{\pi f \mu_c \sigma_c}}$ = skin depth of the conductor; $\cosh^{-1}\dfrac{d}{2a} \simeq \ln\dfrac{d}{a}$ if $\left[\dfrac{d}{2a}\right]^2 \gg 1$.

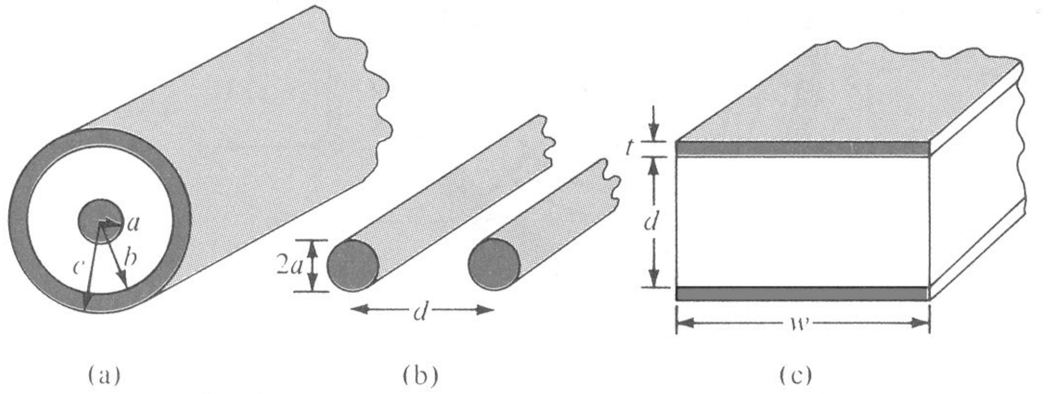


Figure 11.2 Common transmission lines: **(a)** coaxial line, **(b)** two-wire line, **(c)** planar line.

Elements of Electromagnetics (Sixth Edition), Sadiku, Oxford, 2015, ISBN 978-0-19-932138-4.

Notes:

1) All material parameters for R have to do with conductors.
2) All material parameters for L, C, & G have to do with the dielectric material/media surrounding conductors.
3) For the **dielectric**, we are using an effective σ that encompasses both conductive and electric dipole losses and assuming ε is real.

2.1 cont.

Assuming $\Delta z << \lambda$, we can neglect causality and apply circuit theory.

By KVL (CW around the TL circuit model)

$$-v(z,t) + R\Delta z\, i(z,t) + L\Delta z \frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t) = 0$$

re-arrange

$$\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Letting $\Delta z \to 0$, the LHS becomes a derivative!

Telegrapher Eq'n $$\boxed{\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}}$$

By KCL (@ top right node)

$$i(z,t) - G\Delta z\, v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

re-arrange

$$\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G\, v(z+\Delta z,t) - C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Letting $\Delta z \to 0$, the LHS becomes a derivative!

Telegrapher Eq'n $$\boxed{\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}}$$

2.1 cont.

Assuming time-harmonic (i.e., sinusoidal) signals, the phasor form of the Telegrapher Eqn's become

$$\frac{d\,V(z)}{dz} = -(R + j\omega L)\,I(z) \qquad \textcircled{A}$$

$$\frac{d\,I(z)}{dz} = -(G + j\omega C)\,V(z) \qquad \textcircled{B}$$

where $I(z)$ & $V(z)$ are the <u>phasor</u> current and voltage (still depend on location).

$i(z,t) = \text{Re}\{I(z)\,e^{j\omega t}\}$ & $v(z,t) = \text{Re}\{V(z)\,e^{j\omega t}\}$

Unfortunately, we have equations w/ <u>two</u> variables (current & voltage). To get single variable equations, we take another derivative wrt z.

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\,\frac{dI(z)}{dz} \qquad \textcircled{C}$$

$$\frac{d^2I(z)}{dz^2} = -(G + j\omega C)\,\frac{dV(z)}{dz} \qquad \textcircled{D}$$

Next, substitute RHS of Ⓑ into Ⓒ and substitute RHS of Ⓐ into Ⓓ.

2.1 cont.

This yields-

$$\frac{d^2V(z)}{dz^2} = +(R+j\omega L)(G+j\omega C)V(z)$$

$$\frac{d^2I(z)}{dz^2} = +(R+j\omega L)(G+j\omega C)I(z)$$

or

$$\boxed{\begin{aligned} \frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \\ \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \end{aligned}} \Bigg\} \text{ Wave Equations!}$$

where $\gamma^2 = (R+j\omega L)(G+j\omega C)$

$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} \equiv$ propagation constant $\left(\frac{1}{m}\right)$

$= \alpha + j\beta \; (1/m)$

$\alpha = \mathrm{Re}\{\gamma\} \equiv$ attenuation constant (Np/m)

$\beta = \mathrm{Im}\{\gamma\} \equiv$ phase constant (rad/m)

Define:

characteristic impedance $\equiv Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} \; (\Omega)$

wavelength $\equiv \lambda = 2\pi/\beta \; (m)$

phase velocity $\equiv v_p = \omega/\beta = f\lambda \; (m/s)$

2.1 Cont.

The solutions to the lossy TL wave equations are:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \underset{\substack{\uparrow \\ \text{fwd prop.} \\ \text{waves}}}{I_0^+ e^{-\gamma z}} + \underset{\substack{\uparrow \\ \text{bwd prop.} \\ \text{waves}}}{I_0^- e^{+\gamma z}}$$

Using Z_0, we can write the phasor current

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

w/ $V_0^+ = |V_0^+| e^{j\phi^+}$ & $V_0^- = |V_0^-| e^{j\phi^-}$, we can write the time-domain solutions as:

$$\boxed{\begin{aligned} v(z,t) &= |V_0^+| \cos(\omega t - \beta z + \phi^+) e^{-\alpha z} \\ &\quad + |V_0^-| \cos(\omega t + \beta z + \phi^-) e^{+\alpha z} \\ i(z,t) &= \frac{|V_0^+|}{|Z_0|} \cos(\omega t - \beta z + \phi^+ - \phi_{Z_0}) e^{-\alpha z} \\ &\quad - \frac{|V_0^+|}{|Z_0|} \cos(\omega t + \beta z + \phi^- - \phi_{Z_0}) e^{+\alpha z} \end{aligned}}$$

Note: $Z_0 = |Z_0| \underline{/\phi_{Z_0}} = |Z_0| e^{j\phi_{Z_0}}$

2.1 cont.

Many times for microwave circuits, we will be dealing w/ relatively short low loss TLs where we can neglect losses, i.e., $R \to 0$ and $G \to 0$. The lossless TL !!

Lossless Transmission Line Eqns

$$\gamma = j\omega\sqrt{LC} \Rightarrow \begin{array}{l} \alpha = 0 \\ \beta = \omega\sqrt{LC} = \frac{\omega}{v_p} \end{array}$$

$$Z_0 = \sqrt{L/C} \sim \text{real \#!}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{1}{f\sqrt{LC}} = v_p/f$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

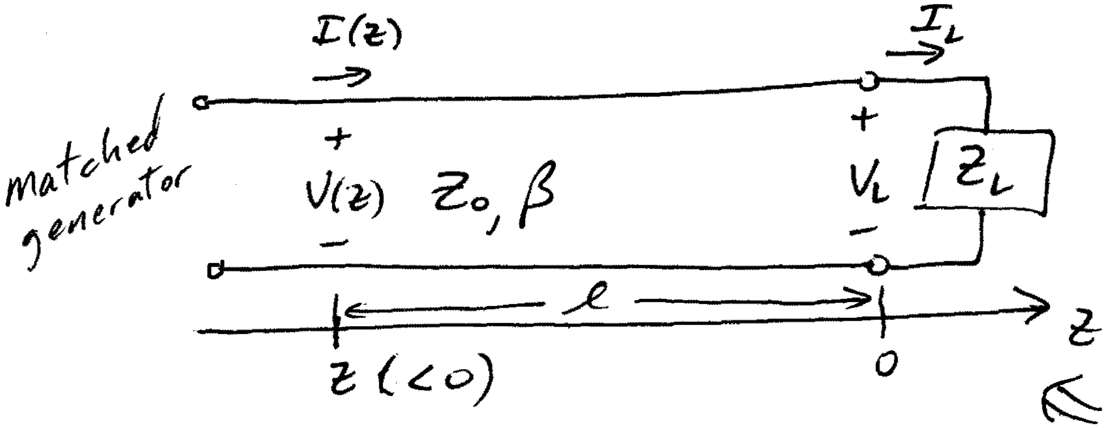
$$\left.\begin{array}{l} V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z} \end{array}\right\} \text{lossless TL phasor solutions}$$

$$\left.\begin{array}{l} V(z,t) = |V_0^+| \cos(\omega t - \beta z + \phi^+) \\ \qquad + |V_0^-| \cos(\omega t + \beta z + \phi^-) \\ i(z,t) = \frac{|V_0^+|}{Z_0} \cos(\omega t - \beta z + \phi^+) \\ \qquad - \frac{|V_0^-|}{Z_0} \cos(\omega t + \beta z + \phi^-) \end{array}\right\} \text{lossless TL time-domain solutions}$$

2.2 Field Analysis of Transmission Lines

→ skip

2.3 The Terminated Lossless Transmission Line



⇐ Different than EE 381/382!!

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

@ $z = 0$ (@ load)

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} (Z_0)$$

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

Define

voltage reflection coefficient @ load $\equiv \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_L$

2.3 cont.

Using Γ, we can write the phasor voltage + current as

$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[\underset{\substack{\uparrow \\ \text{Incident} \\ \text{waves}}}{e^{-j\beta z}} - \underset{\substack{\uparrow \\ \text{Reflected} \\ \text{waves}}}{\Gamma e^{j\beta z}}\right]$$

⇒ An implication of having both incident + reflected waves is that we will have constructive + destructive inteference leading to <u>standing waves</u> whenever $\Gamma \neq 0$

⇒ $\Gamma = \Gamma_L = 0$ only when $Z_L = Z_0$ (matched load)

What about power?

$$P_{avg} = \frac{1}{2}\operatorname{Re}\left\{V(z)\,\overset{\swarrow\text{complex conjugate}}{I^*(z)}\right\}$$

$$= \frac{1}{2}\operatorname{Re}\left\{\left(V_0^+\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]\right)\left(\frac{V_0^{+*}}{Z_0}\left[e^{+j\beta z} - \Gamma^* e^{-j\beta z}\right]\right)\right\}$$

$$= \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\operatorname{Re}\left\{1 - \Gamma^* e^{-j2\beta z} + \Gamma e^{j2\beta z} - |\Gamma|^2\right\}$$

* Used complex number identity $AA^* = |A|^2$ ← magnitude squared

* Now, note the complex number identity $A - A^* = 2j\,\operatorname{Im}(A)$ applies to the middle two terms. After the Re{ } operation,

$$\boxed{P_{avg} = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma|^2\right)}$$

2.3 cont.

Notes: 1) For a lossless TL, P_{avg} = constant, i.e., power is the same all along the TL so $P_{ave}(z=0) = P_{ave}(z<0)$

2) $P_{avg,inc} = P_{avg^+} = \frac{1}{2}\frac{|V_0^+|^2}{Z_0} \equiv$ incident pwr

3) $P_{avg,ref} = P_{avg^-} = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}|\Gamma|^2 \equiv$ reflected power

So, when $|\Gamma| > 0$ (mismatched load) not all of the incident/available from the generator is delivered to the load. To characterize this 'lost' power, define

$$\text{Return Loss} = RL = -20\log_{10}|\Gamma| \text{ (dB)}$$

Here, a RL that is large is usually considered to be good. E.g., $\Gamma = 0$ (matched load)

$RL = -20\log_{10} 0 \rightarrow \infty$ all inc. pwr 'lost'

E.g., $\Gamma = 1$ (open circuit)

$RL = -20\log_{10} 1 = 0$ dB No inc. pwr 'lost' all reflected

ex. For $Z_L = 80\Omega$ and $Z_0 = 50\Omega$, find Γ + RL

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{80-50}{80+50} = \underline{\underline{0.2308}}$$

$$RL = -20\log_{10} 0.2308 = \underline{\underline{12.736 \text{ dB}}}$$

2.3 cont.

Consider the magnitude of the phasor voltage

$$|V(z)| = |V_0^+|\,|e^{-j\beta z} + \Gamma e^{j\beta z}| \quad \text{use } |e^{jA}| = 1$$

$$= |V_0^+|\,|1 + \Gamma e^{j2\beta z}| \quad \swarrow \text{ Next, } \Gamma = |\Gamma| e^{j\theta}$$

$+$ let $\ell = -z$ be the positive distance from load

$$|V(z)| = |V_0^+|\,|1 + |\Gamma| e^{j(\theta - 2\beta\ell)}|$$

Find maximum voltage magnitude by letting $e^{j(\theta - 2\beta\ell)} = 1$, resulting in

$$V_{max} = Max\{|V(z)|\} = |V_0^+|(1 + |\Gamma|)$$

Find minimum voltage magnitude by letting $e^{j(\theta - 2\beta\ell)} = -1$, resulting in

$$V_{min} = Min\{|V(z)|\} = |V_0^+|(1 - |\Gamma|)$$

Define a measure of how large the voltage standing wave is along the TL

$$\text{Standing wave ratio} \equiv SWR = \frac{V_{max}}{V_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

(AKA: VSWR)

$$1 \le SWR < \infty$$

$\uparrow$ matched load $Z_0 = Z_L \rightarrow \Gamma = 0$

$\uparrow$ purely reactive load $|\Gamma| = 1$

2.3 cont.

* From the $e^{j(\theta-2\beta\ell)} = e^{j\theta}e^{-j2\beta\ell}$ term in the $|V_s(z)|$ expression, we can deduce that V_{max} + V_{min} repeat every $\Delta\ell = \lambda/2$ since $e^{jA} = e^{j(A+2\pi)}$ and $2\beta\Delta\ell = 2\frac{2\pi}{\lambda}\lambda/2 = 2\pi$.

* Further, the distance between $e^{j(\theta-2\beta\ell)} = 1$ and $e^{j(\theta-2\beta\ell)} = -1$ is $\Delta\ell = \lambda/4$

Next, let's find the reflection coefficient and input impedance @ some distance $\ell = -z$ from the load

$$\Gamma(\ell) = \frac{V_0^- e^{-j\beta\ell}}{V_0^+ e^{j\beta\ell}} = \overset{\Gamma \text{ or } \Gamma_L \atop \downarrow}{\Gamma(0)}\, e^{-j2\beta\ell} = \Gamma_{in}$$

Note: $|\Gamma(\ell)| = |\Gamma(0)| = |\Gamma|$ for lossless TL!

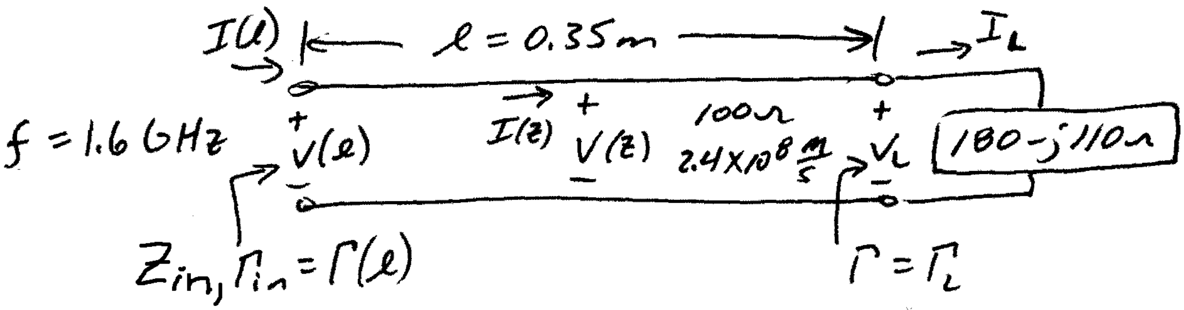
Using Ohm's Law

$$Z_{in} = \frac{V(z)}{I(z)} = \frac{V(-\ell)}{I(-\ell)} = \frac{V_0^+\left(e^{j\beta\ell} + \Gamma e^{-j\beta\ell}\right)}{\frac{V_0^+}{Z_0}\left(e^{j\beta\ell} - \Gamma e^{-j\beta\ell}\right)}$$

$$\boxed{\begin{aligned} Z_{in} &= Z_0\left[\frac{1+\Gamma e^{-j2\beta\ell}}{1-\Gamma e^{-j2\beta\ell}}\right] = Z_0\left[\frac{1+\Gamma(\ell)}{1-\Gamma(\ell)}\right] \\ &= Z_0\left[\frac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}\right] \end{aligned}}$$

2.3 cont.

ex. Calculate various quantities for a lossless TL ($Z_o = 100\Omega$, $v_p = 2.4 \times 10^8 \, m/s$) of length $\ell = 0.35\,m$ terminated in a load $Z_L = 180 - j110\,\Omega$ operating @ 1.6 GHz w/ $V_o^+ = 10\angle 40^\circ$ V.



$$\lambda = v_p/f = \frac{2.4\times10^8}{1.6\times10^9} = \underline{0.15\,m}$$

$$\ell/\lambda = \frac{0.35}{0.15} = \underline{2.\overline{33}} \leftarrow \text{TL length in terms of } \lambda$$

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{v_p} = \frac{2\pi(1.6\times10^9)}{2.4\times10^8} = \underline{41.8879 \,\tfrac{rad}{m}}$$

$$\Gamma = \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{(180-j110)-100}{(180-j110)+100} = \underline{0.4521\angle{-32.525^\circ}}$$

$$\Gamma(\ell) = \Gamma_{in} = \Gamma e^{-j2\beta\ell} = (0.4521\angle{-32.525^\circ})\, e^{-j2(41.89)0.35}$$

$$\underline{\Gamma(\ell) = \Gamma_{in} = 0.4521\angle 87.475^\circ}$$

$$Z_{in} = Z_o\left(\frac{1+\Gamma_{in}}{1-\Gamma_{in}}\right) = 100\left(\frac{1+0.452\angle 87.5^\circ}{1-0.452\angle 87.5^\circ}\right) = \underline{68.3145 + j77.57\,\Omega}$$

$$RL = -20\log_{10} 0.4521 = \underline{6.895\,dB}$$

$$SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.4521}{1-0.4521} = \underline{2.6505}$$

} Not a good match

2.3 Cont., ex. cont.

Find various voltage-related items

$$V_0^- = V_0^+ \Gamma = (10\angle 40^\circ)(0.4521\angle -32.525^\circ) = \underline{4.5213\angle 7.475^\circ\,V}$$

$$V_L = V_0^+ \underset{\to 1}{e^{-j\beta(0)}} + V_0^- \underset{\to 1}{e^{+j\beta(0)}} = (10\angle 40^\circ) + (4.5213\angle 7.475^\circ)$$

$$\underline{V_L = 14.0244\angle 30.018^\circ\,V}$$

$$V_{max} = |V_0^+|(1+|\Gamma|) = \overset{10}{\left|10\cancel{\angle 40^\circ}\right|}(1+0.4521) = \underline{14.521\,V}$$

$$V_{min} = |V_0^+|(1-|\Gamma|) = 10(1-0.4521) = \underline{5.479V}$$

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$\underline{V(z) = (10\angle 40^\circ)\,e^{-j41.89z} + (4.52\angle 7.5^\circ)e^{+j41.89z}\quad V}$$

$$\underline{\text{for } -0.35m \le z \le 0}$$

Find various current-related items

$$I_0^+ = \frac{V_0^+}{Z_0} = \frac{10\angle 40^\circ}{100} = \underline{0.1\angle 40^\circ A}$$

$$I_0^- = -\frac{V_0^-}{Z_0} = \frac{-(4.52\angle 7.5^\circ)}{100} = \underline{0.0452\angle -172.525^\circ A}$$

$$I_L = I_0^+ e^0 + I_0^- e^0 = (0.1\angle 40^\circ) + (0.0452\angle -172.525^\circ)$$

$$\underline{I_L = 0.0665\angle 61.448^\circ A} \quad \left(\text{or } I_L = V_L/Z_L\right)$$

$$I_{max} = \frac{|V_0^+|}{Z_0}(1+|\Gamma|) = |I_0^+|(1+|\Gamma|) = 0.1(1+0.452) = \underline{0.1452A}$$

$$I_{min} = |I_0^-|(1-|\Gamma|) = 0.0452(1-0.452) = \underline{0.0548A}$$

$$\underline{I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} = (0.1\angle 40^\circ)e^{-j41.89z} + (0.0452\angle -172.5^\circ)e^{j41.89z}\quad A}$$

$$\underline{-0.35\,m \le z \le 0}$$

2.3 cont.

ex. cont. **Find some power-related quantities**

$$P_{ave,L} = \tfrac{1}{2}\,\mathrm{Re}\{V_L I_L^*\} = 0.5\,\mathrm{Re}\{(14.02\angle 30^\circ)(0.0665\angle -61.45^\circ)\}$$

$$\underline{P_{ave,L} = 0.3978\text{ W}}$$

$$P_{avg} = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\left(1-|\Gamma|^2\right) = \frac{1}{2}\frac{10^2}{100}\left(1-0.4521^2\right)$$

$\underline{P_{avg} = 0.3978\text{ W}}$ same!

$$P_{avg,inc} = P_{avg}^+ = \frac{1}{2}\frac{|V_0^+|^2}{Z_0} = \frac{1}{2}\frac{10^2}{100} = \underline{0.5\text{ W}}$$

$$P_{avg,ref} = P_{avg}^- = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}|\Gamma|^2 = \frac{1}{2}\frac{10^2}{100}\,0.4521^2 = \underline{0.1022\text{ W}}$$

⇒ see following MathCad pages for confirmation of numbers/calculations as well as plots of $|V(z)|$, $|I(z)|$, and $P_{ave}(z)$ for $-0.35\text{ m} \le z \le 0$.

Find location of V_{max} (& I_{min}) closest to load

For V_{max} (& I_{min}), $\Gamma e^{j2\beta z} = |\Gamma|(1) \Rightarrow 1\angle -32.525^\circ\, e^{j2\beta z} = 1$

$$e^{j(-32.525\,\pi/180 + 2(41.888)z)} = e^{j0}$$

$$\hookrightarrow -32.525\,\pi/180 + 2(41.888)\,z = 0$$

$z = 0.006776\text{ m}$ ← NOT possible

$$z = 0.006776 - \lambda/2 = 0.006776 - \frac{0.15}{2}$$

$$\underline{\underline{z = -0.068224\text{ m}}}$$

Enter given information

$$V0p := 10 \cdot e^{j \cdot 40 \cdot \frac{\pi}{180}} \quad V \qquad f := 1.6 \cdot 10^{9} \quad Hz \qquad ZL := 180 - j \cdot 110 \quad \Omega$$

$$l := 0.35 \quad m \qquad vp := 2.4 \cdot 10^{8} \quad m/s \qquad Z0 := 100 \quad \Omega$$

Calculate variables related to transmission line

$$\omega := 2 \cdot \pi \cdot f \qquad \lambda := \frac{vp}{f} \qquad \boxed{\lambda = 0.15} \quad m \qquad l\lambda := \frac{l}{\lambda} \qquad \boxed{l\lambda = 2.333}$$

$$\beta := \frac{\omega}{vp} \qquad \boxed{\beta = 41.8879} \quad rad/m \qquad n := 0 .. 466 \qquad z_n := \frac{-n}{466} \cdot l$$

Calculate reflection coefficients, return loss, SWR, & input impedance

$$\Gamma := \frac{ZL - Z0}{ZL + Z0} \qquad \boxed{|\Gamma| = 0.4521} \qquad \boxed{\arg(\Gamma) \cdot \frac{180}{\pi} = -32.525} \quad deg$$

$$RL := -20 \cdot \log(|\Gamma|) \qquad \boxed{RL = 6.895} \quad dB$$

$$\Gamma l := \Gamma \cdot e^{-j \cdot 2 \cdot \beta \cdot l} \qquad \boxed{|\Gamma l| = 0.4521} \qquad \boxed{\arg(\Gamma l) \cdot \frac{180}{\pi} = 87.475} \quad deg$$

$$Zin := Z0 \cdot \frac{(1 + \Gamma l)}{(1 - \Gamma l)} \qquad \boxed{Zin = 68.3145 + 77.5709i} \quad \Omega$$

$$SWR := \frac{1 + |\Gamma|}{1 - |\Gamma|} \qquad \boxed{SWR = 2.6505}$$

Calculate V0m, V_L, V_{max}, V_{min}, & phasor voltage

$$V0m := V0p \cdot \Gamma \qquad \boxed{|V0m| = 4.5213} \quad V \qquad \boxed{\arg(V0m) \cdot \frac{180}{\pi} = 7.475} \quad deg$$

$$VL := V0p + V0m \qquad \boxed{|VL| = 14.0244} \quad V \qquad \boxed{\arg(VL) \cdot \frac{180}{\pi} = 30.018} \quad deg$$

$$V_n := V0p \cdot e^{-j \cdot \beta \cdot z_n} + V0m \cdot e^{j \cdot \beta \cdot z_n}$$

Phasor voltage versus position along TL.

$$Vmax := |V0p| \cdot (1 + |\Gamma|) \qquad \boxed{Vmax = 14.5213} \quad V$$

$$Vmin := |V0p| \cdot (1 - |\Gamma|) \qquad \boxed{Vmin = 5.4787} \quad V \qquad \frac{Vmax}{Vmin} = 2.6505$$

Calculate I0p, I0m, I_L, I_{max}, I_{min}, & phasor current

$$I0p := \frac{V0p}{Z0} \qquad \boxed{\left|I0p\right| = 0.1} \quad \text{A} \qquad \boxed{\arg(I0p)\cdot\frac{180}{\pi} = 40} \quad \text{deg}$$

$$I0m := \frac{-V0m}{Z0} \qquad \boxed{\left|I0m\right| = 0.0452} \quad \text{A} \qquad \boxed{\arg(I0m)\cdot\frac{180}{\pi} = -172.525} \quad \text{deg}$$

$$IL := I0p + I0m \qquad \boxed{\left|IL\right| = 0.0665} \quad \text{A} \qquad \boxed{\arg(IL)\cdot\frac{180}{\pi} = 61.448} \quad \text{deg}$$

$$ILalt := \frac{VL}{ZL} \qquad \boxed{\left|ILalt\right| = 0.0665} \quad \text{A} \qquad \boxed{\arg(ILalt)\cdot\frac{180}{\pi} = 61.448} \quad \text{deg}$$

$$I_n := \frac{V0p}{Z0}\cdot e^{-j\cdot\beta\cdot z_n} - \frac{V0m}{Z0}\cdot e^{j\cdot\beta\cdot z_n}$$

Phasor current versus position along TL.

$$Imax := \frac{\left|V0p\right|}{Z0}\cdot\left(1 + \left|\Gamma\right|\right) \qquad \boxed{Imax = 0.1452} \quad \text{A}$$

$$Imin := \frac{\left|V0p\right|}{Z0}\cdot\left(1 - \left|\Gamma\right|\right) \qquad \boxed{Imin = 0.0548} \quad \text{A}$$

Calculate time-average total, incident, & reflected powers

$$Pavg1 := 0.5\cdot\mathrm{Re}\left(VL\cdot\overline{IL}\right) \qquad \boxed{Pavg1 = 0.3978} \quad \text{W}$$

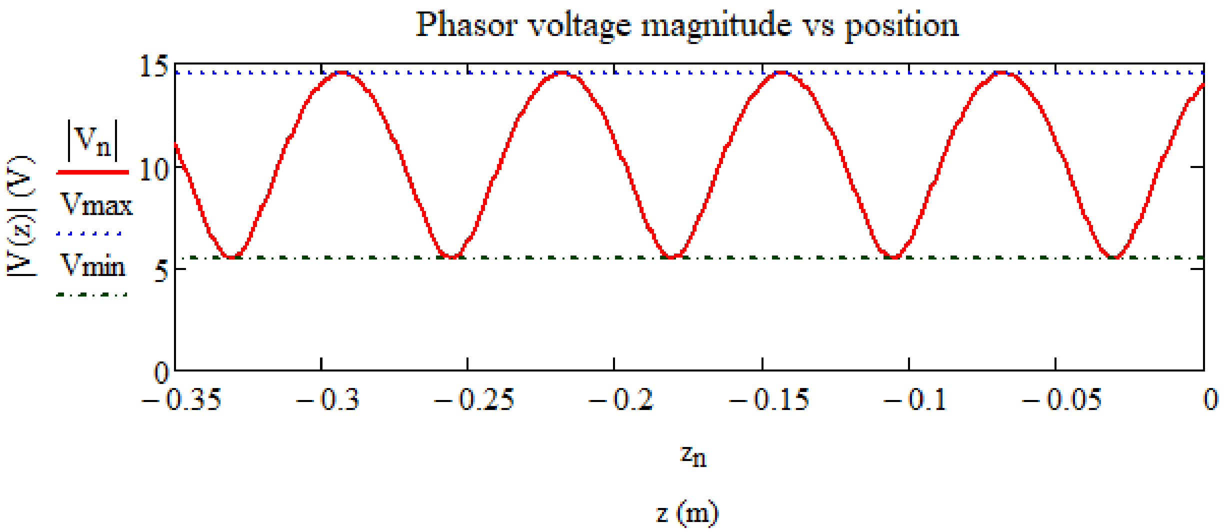
$$Pavg := 0.5\cdot\frac{\left(\left|V0p\right|\right)^2}{Z0}\cdot\left[1 - \left(\left|\Gamma\right|\right)^2\right] \qquad \boxed{Pavg = 0.3978} \quad \text{W}$$

$$Pavg\_inc := 0.5\cdot\frac{\left(\left|V0p\right|\right)^2}{Z0} \qquad \boxed{Pavg\_inc = 0.5} \quad \text{W}$$

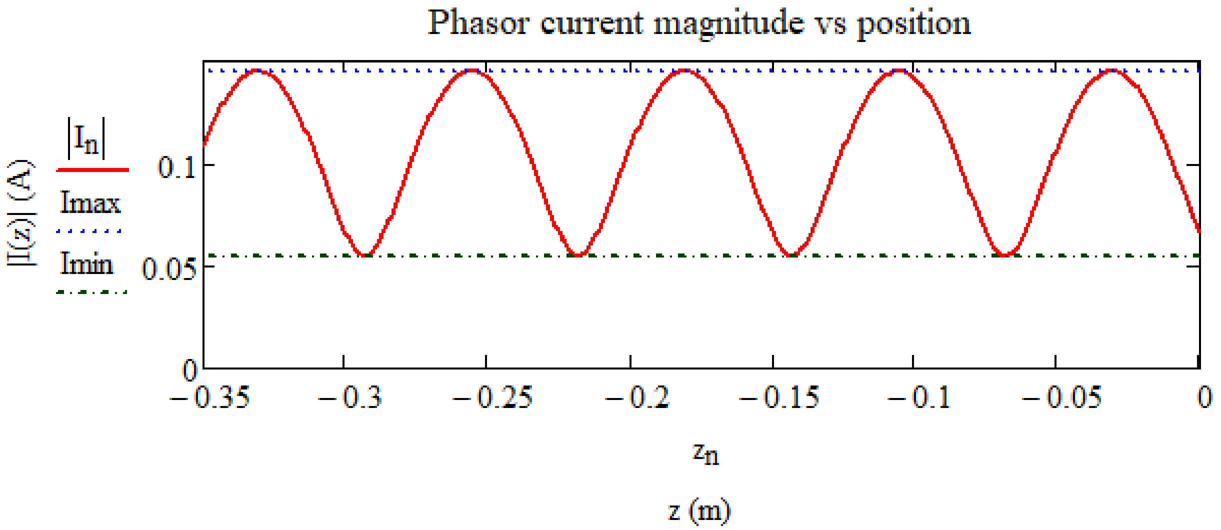
$$Pavg\_ref := 0.5\cdot\frac{\left(\left|V0p\right|\right)^2}{Z0}\cdot\left(\left|\Gamma\right|\right)^2 \qquad \boxed{Pavg\_ref = 0.1022} \quad \text{W}$$

$$Pave_n := 0.5\cdot\mathrm{Re}\left(V_n\cdot\overline{I_n}\right)$$

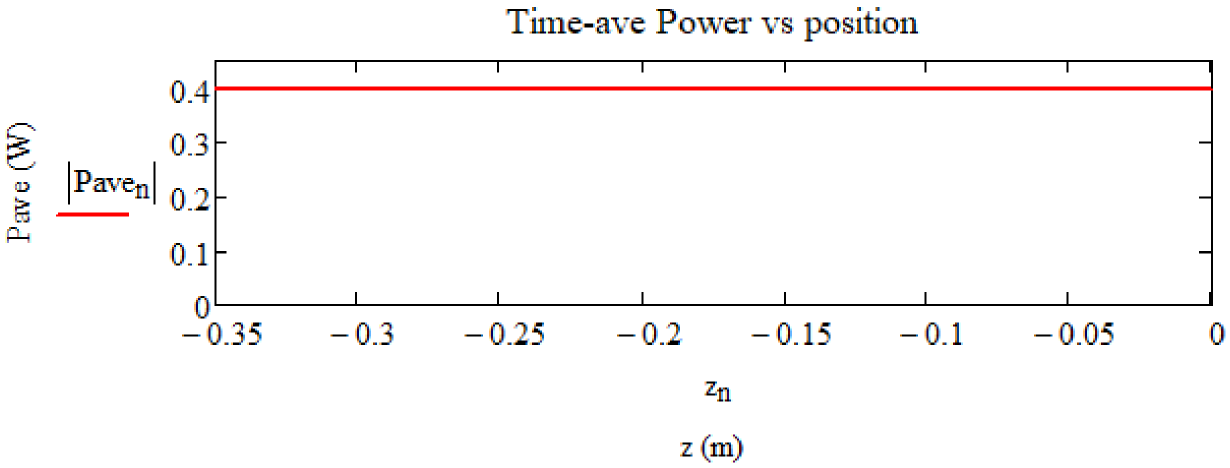
Check to see if power is really constant.



$|V_{91}| = 14.521$ V $z_{91} = -0.0683$ m $|V_{41}| = 5.479$ V $z_{41} = -0.0308$ m



$|I_{41}| = 0.145$ A $z_{41} = -0.0308$ m $|I_{91}| = 0.0548$ A $z_{91} = -0.0683$ m



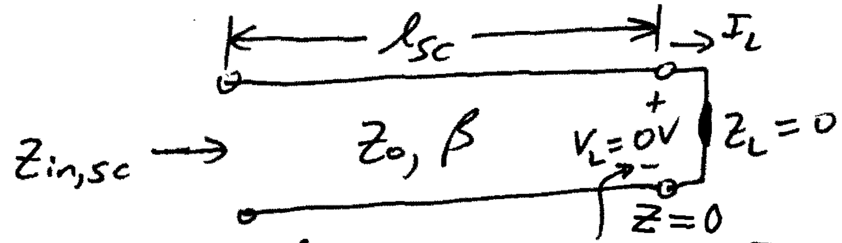
2.3 cont.

Special Cases of Lossless Terminated Lines

(AKA: Open & short circuited stubs)

* Given the difficulty of fabricating high quality lumped capacitors + inductors at microwave frequencies as well as the expense, stubs w/ equivalent input reactances/impedances are often used.

Short circuit stub ($Z_L = 0$)



RL = 0 dB

$$\Gamma_{sc} = \frac{0 - Z_0}{0 + Z_0} = -1, \quad VSWR \rightarrow \infty$$

$$V_{sc}(z) = V_0^+ \left(e^{-j\beta z} - e^{j\beta z}\right) = -j2V_0^+ \sin(\beta z)$$

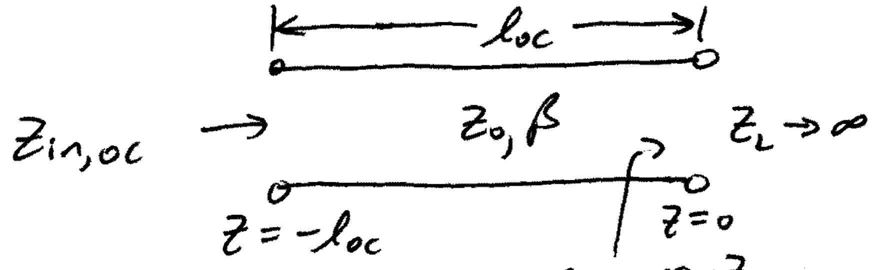
$$I_{sc}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} + e^{j\beta z}\right) = \frac{2V_0^+}{Z_0}\cos(\beta z)$$

$$Z_{in,sc} = \frac{V_{sc}(-\ell_{sc})}{I_{sc}(-\ell_{sc})} = jZ_0 \tan(\beta \ell_{sc})$$

↳ By changing ℓ_{sc}, we can get any reactive impedance ($\pm jX$), 0, or ∞.

2.3 cont.

Open Circuit Stub ($Z_L \to \infty$)



RL = 0 dB

$$\Gamma_{oc} = \frac{\infty - Z_0}{\infty + Z_0} = 1, \quad \text{VSWR} \to \infty$$

$$V_{oc}(z) = V_0^+ \left(e^{-j\beta z} + e^{j\beta z} \right) = 2 V_0^+ \cos(\beta z)$$

$$I_{oc}(z) = \frac{V_0^+}{Z_0} \left(e^{-j\beta z} - e^{j\beta z} \right) = -j \frac{2 V_0^+}{Z_0} \sin(\beta z)$$

$$\underline{Z_{in,oc} = -j Z_0 \cot(\beta \ell_{oc}) = \frac{-j Z_0}{\tan(\beta \ell_{oc})}}$$

Again by changing ℓ_{oc}, we can get any $\pm jX$, 0, or ∞ for $Z_{in,oc}$.

Notes: 1) Calculators will sometimes give ℓ_{sc} or ℓ_{oc} values that are negative, i.e., NOT physically realizable.

2) Everything repeats w/ $\underline{n\lambda/2}$. So, you can always add integer multiples of $\lambda/2$ to any ℓ_{sc} or ℓ_{oc}.

see plots on following page

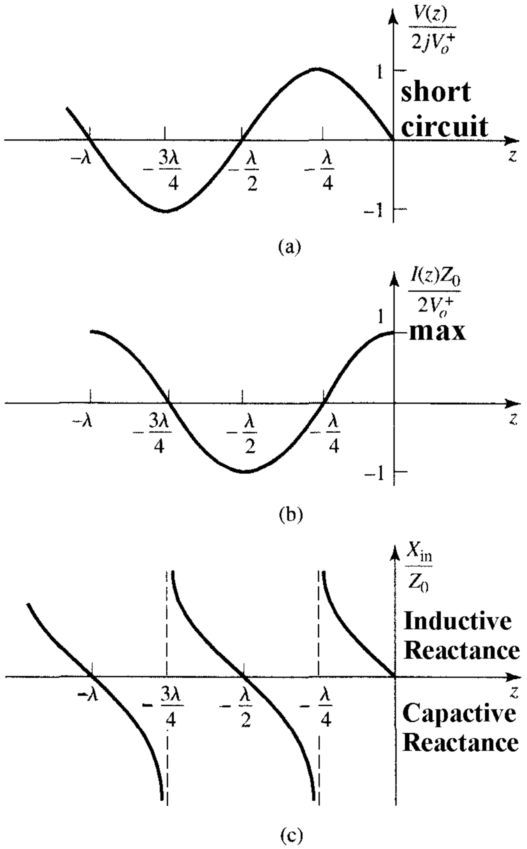


FIGURE 2.6 (a) Voltage, (b) current, and (c) impedance ($R_{in} = 0$ or ∞) variation along a short-circuited transmission line.

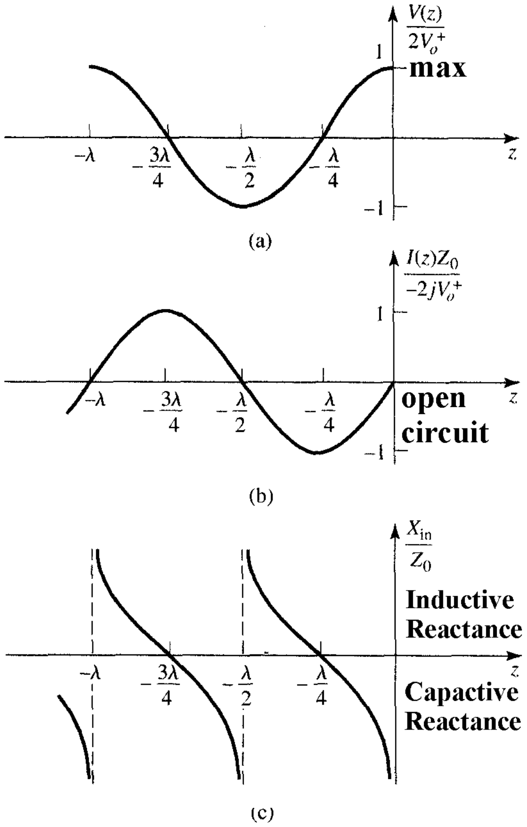
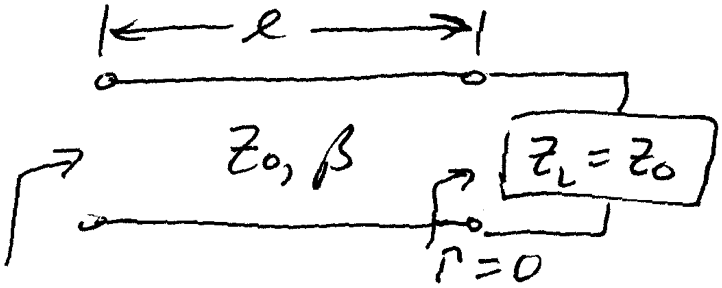


FIGURE 2.8 (a) Voltage, (b) current, and (c) impedance ($R_{in} = 0$ or ∞) variation along an open-circuited transmission line.

2.3 cont.

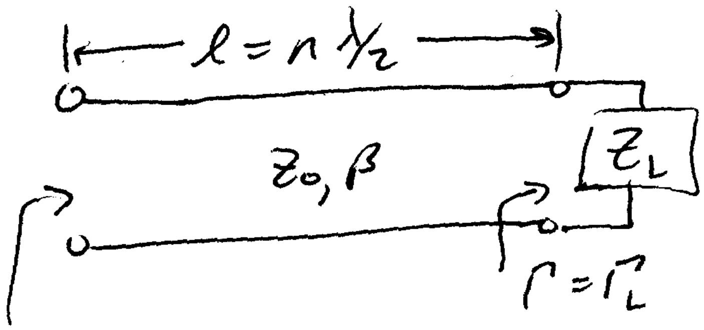
Matched Load



$Z_{in,ML} = Z_0$ independent of ℓ!

& $\Gamma(\ell) = 0$ ⇒ VSWR = 1 & RL = ∞

Halfwave TL ($n\lambda/2$)

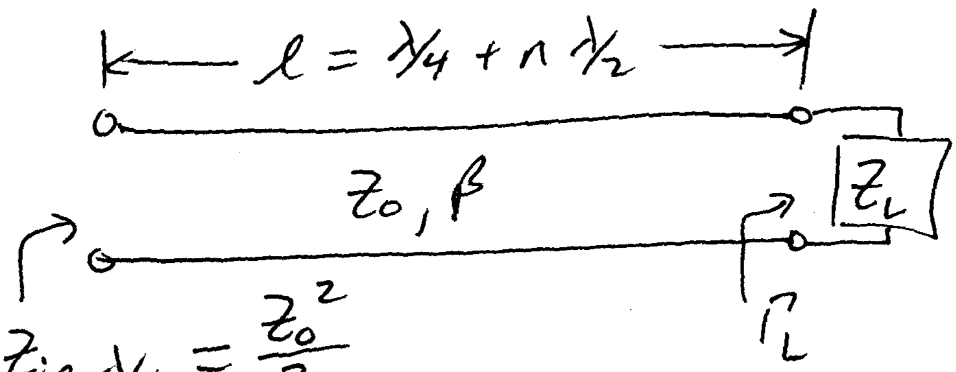


$Z_{in} = Z_L$

& $\Gamma(\ell) = \Gamma$

⇒ Z_{in} & Γ repeat @ $n\lambda/2$ intervals along lossless TLs

Quarterwave TL (we'll revisit this later)



$$Z_{in,\lambda/4} = \frac{Z_0^2}{Z_L}$$

2.3 cont.

ex. A capacitive reactance of $-j66\,\Omega$ is desired using a microstrip TL where $V_p = 2.45 \times 10^8$ m/s and $Z_o = 52\,\Omega$ operating at 1.2 GHz. We desire the shortest open or short circuit stub of length greater than 13 mm.

Short circuit $\quad Z_{in,sc} = -j66 = j\,Z_o \tan \beta l_{sc}$

Open circuit $\quad Z_{in,oc} = -j66 = -j\,Z_o \cot \beta l_{oc}$

where $Z_o = 52\,\Omega$ and $\beta = \frac{\omega}{V_p} = \frac{2\pi(1.2 \times 10^9)}{2.45 \times 10^8} = 30.775 \frac{rad}{m}$

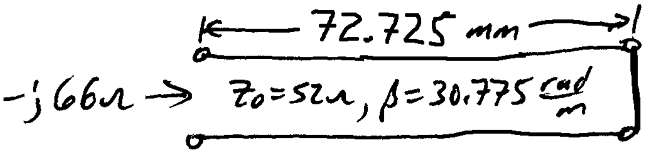
solving $\quad l_{sc} = \frac{1}{30.775} \tan^{-1}\left(\frac{-66}{52}\right) = -0.029358\text{ m}$

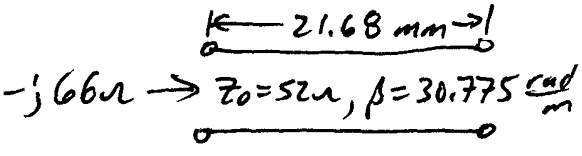
$$l_{oc} = \frac{1}{30.775} \tan^{-1}\left(52/66\right) = 0.0216835\text{ m}$$

Obviously, l_{sc} sol'n is unrealizable. So, we will need to add $n\lambda/2$ until we get a positive solution. $\lambda = \frac{V_p}{f} = \frac{2.45 \times 10^8}{1.2 \times 10^9} = 0.2041\overline{6}\text{ m}$

$$l_{sc} = -0.02936 + \frac{0.2041\overline{6}}{2} = 0.0727252\text{ m}$$

$l_{sc} = 72.725\text{ mm}$ vs. $\underline{\underline{l_{oc} = 21.6835\text{ mm}}}$

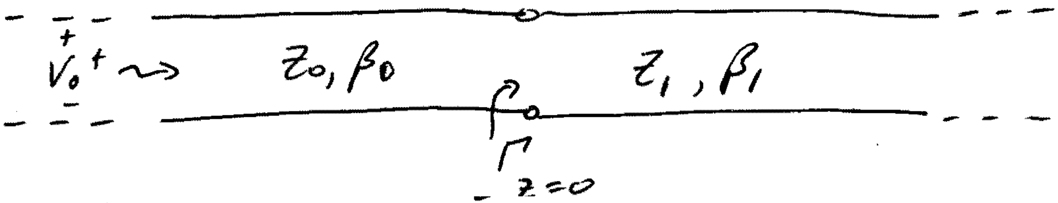




2.3 cont.

Transmission Coefficient + Insertion Loss

Consider the junction between two TLs



→ Assume we have an incident wave from the left which results in a reflection @ $z=0$, but there is no backward traveling wave for $z>0$.

@ interface $\underline{\Gamma = \dfrac{Z_1 - Z_0}{Z_1 + Z_0}}$

$z<0$, $V(z) = V_0^+ \left(e^{-j\beta_0 z} + \Gamma e^{j\beta_0 z} \right)$

$z>0$, $V(z) = V_1^+ e^{-j\beta_1 z}$

Since V_1^+ is due to V_0^+, define

Transmission coefficient $\equiv T = \dfrac{V_1^+}{V_0^+}$ @ $z=0$

Now, $z>0$, $V(z) = T V_0^+ e^{-j\beta_1 z}$

@ $z=0$, the two voltages must be equal

$$V_0^+ (1+\Gamma) = T V_0^+$$

$$\hookrightarrow \underline{T = 1 + \Gamma = \frac{2Z_1}{Z_1 + Z_0}}$$

2.3 cont.

To characterize how much of the power in the incident wave makes it onto, i.e., is transmitted, the RH TL we define

$$\text{Insertion Loss} \equiv IL = -20\log_{10}|T|$$

As a reminder, $RL = -20\log_{10}|\Gamma|$

What are 'good' + 'bad' for IL?

If $\Gamma = 0$, i.e., no power is reflected + all power is transmitted ($T = 1+\Gamma = 1$)

$$IL = -20\log_{10}|1+0| = \underline{0\ dB}$$

$$RL = -20\log_{10} 0 \rightarrow \underline{\infty\ dB}$$

If $\Gamma = -1$, i.e., all power is reflected + no power is transmitted ($T = 1+\Gamma = 0$)

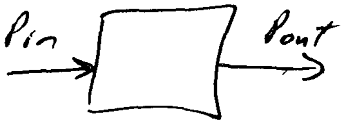
$$IL = -20\log_{10}|1-1| \rightarrow \underline{\infty}\ dB$$

$$RL = -20\log_{10}|-1| = \underline{0\ dB}$$

2.3 cont.

Power & Gains in decibels and Nepers

Often the gain (or loss) of a microwave component is expressed in decibels defined as



$$\text{Gain}(dB) = 10 \log_{10} \frac{P_{out}}{P_{in}}$$

ex. The power input into a 12dB attenuator is 10W. How much power is output?

$$G = -12\,dB = 10 \log_{10} \frac{P_{out}}{10W}$$

$$\hookrightarrow P_{out} = (10W) 10^{-12/10} = \underline{\underline{0.631\,W}}$$

$$\uparrow$$

$$\underset{(\text{unitless})}{G} = \frac{P_{out}}{P_{in}} = 10^{-1.2} = \underline{0.063}$$

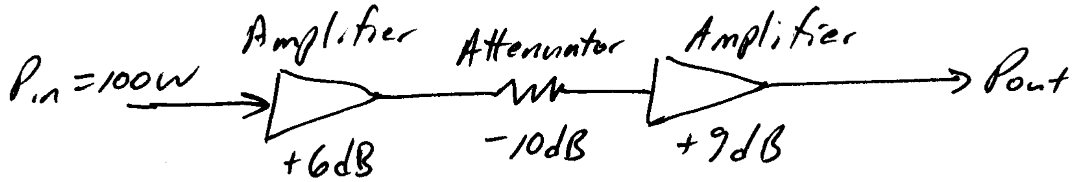
It is often easier to do all power calculations in decibels since we can <u>add</u> terms. However, to do so, requires we express the input (& output) power in decibels. We do this by comparing these absolute powers to a reference, e.g., 1mW or 1W.

2.3 cont.

$$\underline{P(dBm) = 10\log_{10}\frac{P}{10^{-3}W}}$$ ← decibels wrt 1 mW (most common)

$$\underline{P(dBW) = 10\log_{10}\frac{P}{1W}}$$ ← decibels wrt 1W

ex. Using decibels, find the power out of the following microwave circuit



$$P_{in}(dBm) = 10\log_{10}\frac{100}{10^{-3}} = 50dBm$$

$$P_{out}(dBm) = 50dBm + 6dB - 10dB + 9dB$$

$$\underline{P_{out} = 55dBm}$$

or $$P_{out} = (1mW)10^{55/10} = (10^{-3}W)10^{5.5} = \underline{316.23W}$$

For current or voltage ratios/gains <u>with constant load resistance</u>

$$G = 20\log_{10}\frac{|V_{out}|}{|V_{in}|} = 20\log_{10}\frac{|I_{out}|}{|I_{in}|}\ (dB)$$

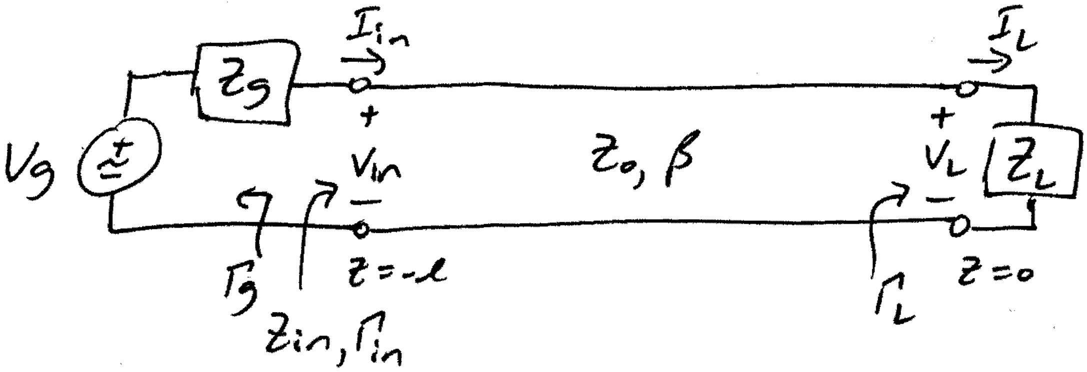
Since $P \propto V^2$ or I^2

Often attenuation constants (i.e., $e^{-\alpha z}$) are expressed in Nepers/m or dB/m where

$$\boxed{\begin{aligned} 1\ \text{Neper} = 1\ Np = 10\log_{10}e^2 = 20\log_{10}e \\ = 8.68588964\,dB \end{aligned}}$$

2.6 Generator and Load Mismatches

In section 2.3, we just assumed a V_0^+ was present at the load, implicitly assuming a matched source. Now, we'll connect a generator (source) Thevenin equivalent to our lossless TL terminated in a load.



We have already found $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ and

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z} \right).$$

How can we find V_0^+ (fwd wave @ load) from the info present in the above ckt?

From circuit theory (voltage division),

$$V_{in} = V_g \left(\frac{Z_{in}}{Z_g + Z_{in}} \right) = V(z=-l) = V_0^+ \left(e^{+j\beta l} + \Gamma_L e^{-j\beta l} \right)$$

Solving for V_0^+, we get

$$\underline{V_0^+ = V_g \left(\frac{Z_{in}}{Z_g + Z_{in}} \right) \frac{1}{e^{j\beta l} + \Gamma_L e^{-j\beta l}} = \frac{V_{in}}{e^{j\beta l} + \Gamma_L e^{-j\beta l}}}$$

2.6 cont.

where we know

$$Z_{in} = Z_0\left[\frac{1+\Gamma(-\ell)}{1-\Gamma(-\ell)}\right] = Z_0\,\frac{1+\Gamma_L e^{-j2\beta\ell}}{1-\Gamma_L e^{j2\beta\ell}}$$

$$= Z_0\left[\frac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}\right]$$

This can be substituted into the V_0^+ eq'n to yield (after much algebra)

$$\boxed{V_0^+ = V_g\left(\frac{Z_0}{Z_0+Z_g}\right)\frac{e^{-j\beta\ell}}{1-\Gamma_L\Gamma_g e^{-j2\beta\ell}} \quad \text{where } \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}}$$

What about power?

For a <u>lossless TL</u>, $P_{avg} = P_L = P_{in} = P$

$$P = \tfrac{1}{2}\,\mathrm{Re}\{V_{in} I_{in}^*\} = \tfrac{1}{2}\,\mathrm{Re}\left\{V_{in}\frac{V_{in}^*}{Z_{in}^*}\right\}$$

$$= \frac{|V_{in}|^2}{2}\,\mathrm{Re}\left\{\frac{1}{Z_{in}^*}\right\} = \frac{1}{2}|V_g|^2\left|\frac{Z_{in}}{Z_{in}+Z_g}\right|^2\mathrm{Re}\left\{\frac{1}{Z_{in}^*}\right\}$$

[Note: (2.74) of text omited complex conj. of Z_{in}]

Putting $Z_{in} = R_{in} + jX_{in}$ and $Z_g = R_g + jX_g$ in rectangular form and doing some algebra + complex numbers,

2.6 cont.

$$P = \frac{1}{2}|V_g|^2 \frac{R_{in}}{(R_{in}+R_g)^2 + (X_{in}+X_g)^2} \quad (2.75)$$

How can we utilize this result?

1) Matched Load $(Z_L = Z_0)$ $\Rightarrow \Gamma_L = 0$ & $Z_{in} = Z_0$
VSWR = 1

$$P_1 = \frac{1}{2}|V_g|^2 \frac{Z_0}{(Z_0+R_g)^2 + X_g^2} \quad (2.76)$$

If we further make $Z_g = Z_0$ (quite common),
we get: $P_{1_{Z_0}} = \frac{1}{2}|V_g|^2 \frac{Z_0}{(Z_0+Z_0)^2+0} = \frac{|V_g|^2}{8Z_0}$

2) Match Generator to Z_{in}, i.e., $Z_g = Z_{in}$

Here Z_L may or may NOT equal Z_0 $(\Gamma_L \neq 0)$,
However, by either manipulating Z_g or
the combination of βl, Z_0 & Z_L, we make
$Z_g = Z_{in}$. Therefore,

$$(2.78) \quad P_2 = \frac{1}{2}|V_g|^2 \frac{R_g}{(R_g+R_g)^2 + (X_g+X_g)^2} = \frac{1}{8}|V_g|^2 \frac{R_g}{R_g^2 + X_g^2}$$

Again, if we make $Z_g = Z_{in} = Z_0$, we get

$$P_{2_{Z_0}} = \frac{|V_g|^2}{8Z_0}$$

2.6 cont.

3) Complex Conjugate Match

→ Go back to Circuits I and select (or make)

$$Z_{in} = Z_g^* \Longrightarrow R_{in} = R_g \text{ \& } X_{in} = -X_g$$

$$P_3 = \frac{1}{2}|V_g|^2 \frac{R_g}{(R_g+R_g)^2 + (\underbrace{-X_g + X_g}_{\to 0})^2}$$

$$\underline{P_3 = \frac{|V_g|^2}{8R_g}}$$

→ comparing P_3 w/ P_1 + P_2, we see that P_3 is greater than P_1 + P_2 in general (equal in a few exceptional case like $X_g = 0$ for P_2 or $Z_g = Z_0$ for P_1).

[If we make $Z_g = Z_0$, $P_1 > P_3$ when $R_g > Z_0$ for complex conj. match]

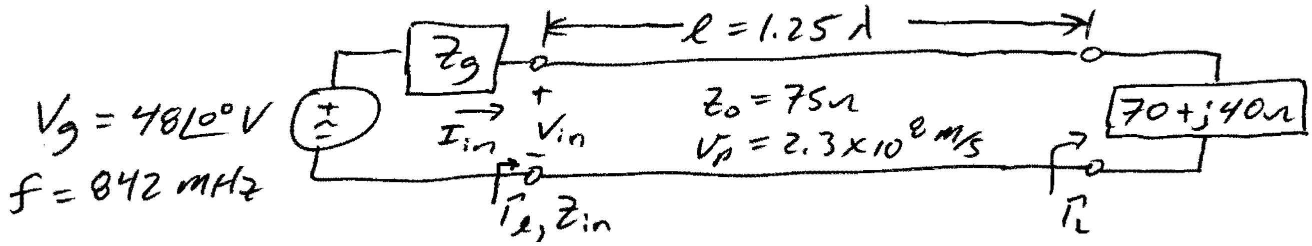
→ P_3 is also the maximum available power from the generator per circuit theory

→ Drawback, we could have a larger VSWR since Γ_L can be non-zero and power is reflected back into generator (in general)

Note: Efficiency, i.e., % of power to load from total output of generator, is NOT the same as maximum power to <u>load</u>.

2.6 cont. **ee481_581_lossless_TL_example_02.xmcd**

ex. Find powers and related quantities for various match conditions for the circuit shown.



$Vg := 48 \cdot e^{j \cdot 0 \cdot \frac{\pi}{180}}$ V $\quad f := 842 \cdot 10^6$ Hz $\quad ZL := 70 + j \cdot 40$ Ω

$l\lambda := 1.25$ $\quad vp := 2.3 \cdot 10^8$ m/s $\quad Z0 := 75$ Ω

Calculate variables related to transmission line

$\omega := 2 \cdot \pi \cdot f$ $\quad \lambda := \frac{vp}{f}$ $\quad \boxed{\lambda = 0.273}$ m $\quad l := l\lambda \cdot \lambda$ $\quad \boxed{l = 0.3414}$ m

$\beta := \frac{\omega}{vp}$ $\quad \boxed{\beta = 23.0019}$ rad/m

Calculate reflection coefficients & input impedance

$\Gamma L := \frac{ZL - Z0}{ZL + Z0}$ $\quad \boxed{|\Gamma L| = 0.268}$ $\quad \boxed{\arg(\Gamma L) \cdot \frac{180}{\pi} = 81.703}$ deg

$\Gamma l := \Gamma L \cdot e^{-j \cdot 2 \cdot \beta \cdot l}$ $\quad \boxed{|\Gamma l| = 0.268}$ $\quad \boxed{\arg(\Gamma l) \cdot \frac{180}{\pi} = -98.297}$ deg

$Zin := Z0 \cdot \frac{(1 + \Gamma l)}{(1 - \Gamma l)}$ $\quad \boxed{Zin = 60.5769 - 34.6154i}$ Ω

1) Assume $Z_g = Z_{in} = Z_0$ and $\Gamma_{L1} = 0$, i.e., used matching network on load.

$\Gamma L1 := 0$ $\quad Zin1 := Z0$ $\quad Zg1 := Z0$

$Vin1 := Vg \cdot \left(\frac{Zin1}{Zin1 + Zg1}\right)$ $\quad \boxed{|Vin1| = 24}$ V $\quad \boxed{\arg(Vin1) \cdot \frac{180}{\pi} = 0}$ deg

$V0p1 := \frac{Vin1}{e^{j \cdot \beta \cdot l} + \Gamma L1 \cdot e^{-j \cdot \beta \cdot l}}$ $\quad \boxed{|V0p1| = 24}$ V $\quad \boxed{\arg(V0p1) \cdot \frac{180}{\pi} = -90}$ deg

$Iin1 := \frac{Vin1}{Zin1}$ $\quad \boxed{|Iin1| = 0.32}$ A $\quad \boxed{\arg(Iin1) \cdot \frac{180}{\pi} = 0}$ deg

$$Pin1 := 0.5 \cdot \mathrm{Re}\left(Vin1 \cdot \overline{Iin1}\right) \qquad Pavg1 := 0.5 \cdot \frac{(|V0p1|)^2}{Z0} \cdot \left[1 - (|\Gamma L1|)^2\right] \qquad P1 := \frac{(|Vg|)^2}{8 \cdot Z0}$$

$\boxed{Pin1 = 3.84}$ W $\boxed{Pavg1 = 3.84}$ W $\boxed{P1 = 3.84}$ W

$$Pavg\_inc1 := 0.5 \cdot \frac{(|V0p1|)^2}{Z0}$$ $\boxed{Pavg\_inc1 = 3.84}$ W

$$Pavg\_refl1 := 0.5 \cdot \frac{(|V0p1|)^2}{Z0} \cdot (|\Gamma L1|)^2$$ $\boxed{Pavg\_refl1 = 0}$ W

$$PVg1 := 0.5 \cdot \mathrm{Re}\left(Vg \cdot \overline{Iin1}\right)$$ $\boxed{PVg1 = 7.68}$ W

$$VSWR1 := \frac{1 + |\Gamma L1|}{1 - |\Gamma L1|}$$ $\boxed{VSWR1 = 1}$

$RL1 = 20\log(0) = \infty$ $$\eta 1 := \frac{P1}{PVg1}$$ $\boxed{\eta 1 \cdot 100 = 50}$ %

2) Choose $Z_g = Z_{in}$. Note, load is NOT matched.

$Zg2 := Zin$ $\boxed{Zg2 = 60.577 - 34.615i}$ Ω $Rg2 := \mathrm{Re}(Zg2)$ $Xg2 := \mathrm{Im}(Zg2)$

$$Vin2 := Vg \cdot \left(\frac{Zin}{Zin + Zg2}\right)$$ $\boxed{|Vin2| = 24}$ V $\boxed{\arg(Vin2) \cdot \frac{180}{\pi} = 0}$ deg

$$V0p2 := \frac{Vin2}{e^{j\cdot\beta\cdot l} + \Gamma L \cdot e^{-j\cdot\beta\cdot l}}$$ $\boxed{|V0p2| = 24.067}$ V $\boxed{\arg(V0p2) \cdot \frac{180}{\pi} = -74.58}$ deg

$$Iin2 := \frac{Vin2}{Zin}$$ $\boxed{|Iin2| = 0.344}$ A $\boxed{\arg(Iin2) \cdot \frac{180}{\pi} = 29.745}$ deg

$$Pin2 := 0.5 \cdot \mathrm{Re}\left(Vin2 \cdot \overline{Iin2}\right) \qquad P2 := \frac{(|Vg|)^2}{8} \frac{Rg2}{Rg2^2 + Xg2^2}$$

$\boxed{Pin2 = 3.584}$ W $\boxed{P2 = 3.584}$ W

$$Pavg\_inc2 := 0.5 \cdot \frac{(|V0p2|)^2}{Z0}$$ $\boxed{Pavg\_inc2 = 3.8613}$ W

$$Pavg\_refl2 := 0.5 \cdot \frac{(|V0p2|)^2}{Z0} \cdot (|\Gamma L|)^2$$ $\boxed{Pavg\_refl2 = 0.2773}$ W

$$PVg2 := 0.5 \cdot \mathrm{Re}\left(Vg \cdot \overline{Iin2}\right)$$ $\boxed{PVg2 = 7.168}$ W

$$VSWR2 := \frac{1+|\Gamma L|}{1-|\Gamma L|} \qquad \boxed{VSWR2 = 1.732}$$

$$RL2 := 20 \cdot \log(|\Gamma L|) \qquad \boxed{RL2 = -11.437} \ \text{dB}$$

$$\eta 2 := \frac{P2}{PVg2} \qquad \boxed{\eta 2 \cdot 100 = 50} \ \%$$

3) Choose Zg = Zin*.

$$Zg3 := \overline{Zin} \qquad \boxed{Zg3 = 60.577 + 34.615i} \ \Omega \qquad Rg3 := \mathrm{Re}(Zg3) \qquad Xg3 := \mathrm{Im}(Zg3)$$

$$Vin3 := Vg \cdot \left(\frac{Zin}{Zin + Zg3}\right) \qquad \boxed{|Vin3| = 27.642} \ \text{V} \qquad \boxed{\arg(Vin3) \cdot \frac{180}{\pi} = -29.74} \ \text{deg}$$

$$V0p3 := \frac{Vin3}{e^{j \cdot \beta \cdot l} + \Gamma L \cdot e^{-j \cdot \beta \cdot l}} \qquad \boxed{|V0p3| = 27.719} \ \text{V} \qquad \boxed{\arg(V0p3) \cdot \frac{180}{\pi} = -104.32} \ \text{deg}$$

$$Iin3 := \frac{Vin3}{Zin} \qquad \boxed{|Iin3| = 0.3962} \ \text{A} \qquad \boxed{\arg(Iin3) \cdot \frac{180}{\pi} = 0} \ \text{deg}$$

$$Pin3 := 0.5 \cdot \mathrm{Re}\left(Vin3 \cdot \overline{Iin3}\right) \qquad P3 := \frac{(|Vg|)^2}{8 \cdot Rg3}$$

$$\boxed{Pin3 = 4.7543} \ \text{W} \qquad \boxed{P3 = 4.7543} \ \text{W}$$

$$Pavg\_inc3 := 0.5 \cdot \frac{(|V0p3|)^2}{Z0} \qquad \boxed{Pavg\_inc3 = 5.1222} \ \text{W}$$

$$Pavg\_refl3 := 0.5 \cdot \frac{(|V0p3|)^2}{Z0} \cdot (|\Gamma L|)^2 \qquad \boxed{Pavg\_refl3 = 0.3679} \ \text{W}$$

$$PVg3 := 0.5 \cdot \mathrm{Re}\left(Vg \cdot \overline{Iin3}\right) \qquad \boxed{PVg3 = 9.5086} \ \text{W}$$

$$VSWR3 := \frac{1+|\Gamma L|}{1-|\Gamma L|} \qquad \boxed{VSWR3 = 1.732}$$

$$RL3 := 20 \cdot \log(|\Gamma L|) \qquad \boxed{RL3 = -11.437} \ \text{dB}$$

$$\eta 3 := \frac{P3}{PVg3} \qquad \boxed{\eta 3 \cdot 100 = 50} \ \%$$

$P_3 = 4.754\,W > P_1 = 3.84\,W > P_2 = 3.584\,W$

VSWR = 1.73 VSWR = 1 VSWR = 1.73

$\eta = 50\%$ in all cases.

2.4 The Smith Chart

→ Our hero is Phillip H. Smith who developed this graphical tool while @ Bell Telephone Laboratories. He published –

P.H. Smith, "Transmission Line Calculator," Electronics, Jan 1939, vol. 12, No. 1, pp. 29-31. with a follow-up in 1944.

What is a Smith Chart?

It is a graphical representation of the reflection coefficient (Γ) with related quantities superimposed (e.g., impedance, admittance, VSWR, RL, ...). This allows many TL problems to be solved w/out using calculators/computers to do complex numbers. It also (with practice) gives the user a more intuitive grasp of the problem(s).

* Despite the tremendous growth of CAD tools, Smith Charts are still extensively used and show up as display options for S/W packages and test instruments (e.g. vector network analyzers).

2.4 cont.

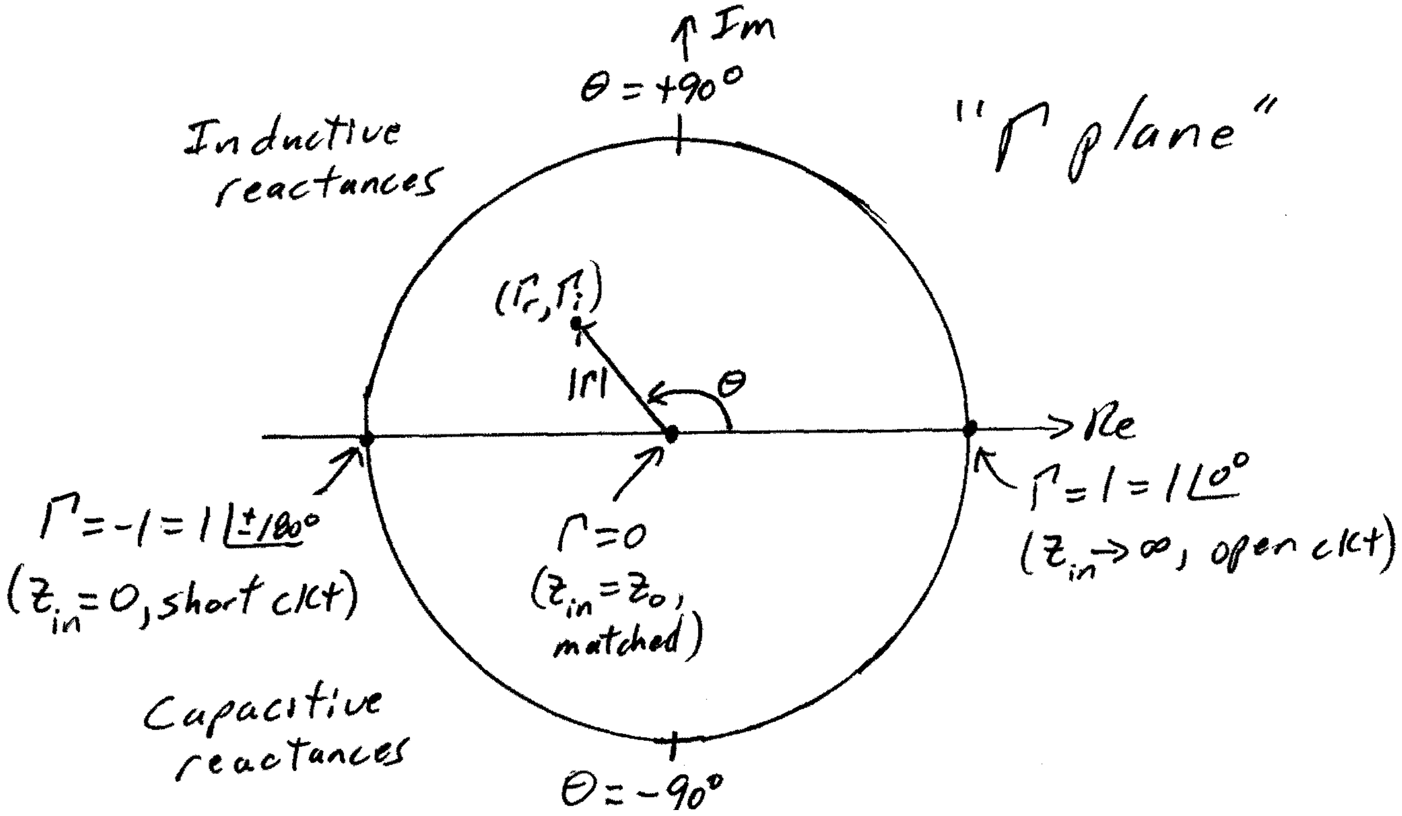
Let's consider a reflection coefficient @ some point along our lossless TL (Z_0)

$$\Gamma(z) = \Gamma_L e^{j2\beta z} = \frac{Z_{in}(z) - Z_0}{Z_{in}(z) + Z_0}$$

For our discussion, we will only consider passive loads $\Rightarrow |\Gamma_L| = |\Gamma| \leq 1$. This limits us to a circle of radius $|\Gamma| = 1$ on the complex plane where we can write

$$\Gamma(z) = |\Gamma| e^{j\theta} = |\Gamma| \angle \theta = \Gamma_r + j\Gamma_i$$

(polar) (rectangular)



2.4 cont.

How can we relate $\Gamma(z)$ to impedances?

Go back to

$$\Gamma(z) = \frac{Z_{in}(z) - Z_0}{Z_{in}(z) + Z_0} = \frac{\frac{Z_{in}(z)}{Z_0} - 1}{\frac{Z_{in}(z)}{Z_0} + 1}$$

where we will define a normalized impedance

$$z_{in}(z) = \frac{Z_{in}(z)}{Z_0} = r + jx \left(\frac{\Omega}{\Omega}\right)$$

So, $\Gamma(z) = \Gamma_r + j\Gamma_i = \dfrac{(r+jx) - 1}{(r+jx) + 1}$

$\Downarrow$ lot's of complex algebra & equating real parts + imaginary parts separately leads to

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \quad (2.56a)$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \quad (2.56b)$$

Equations of circles!

* (2.56a) gives circles centered on the horizontal axis that always pass thru $\Gamma = 1 = 1\angle 0^\circ$ for different values of $\underline{\underline{r}}$.
* (2.56b) gives circles for different $\underline{\underline{x}}$, centered on $\Gamma_r = 1$.

2.4 cont.

ex. For $r=1$ in (2.56a), we get

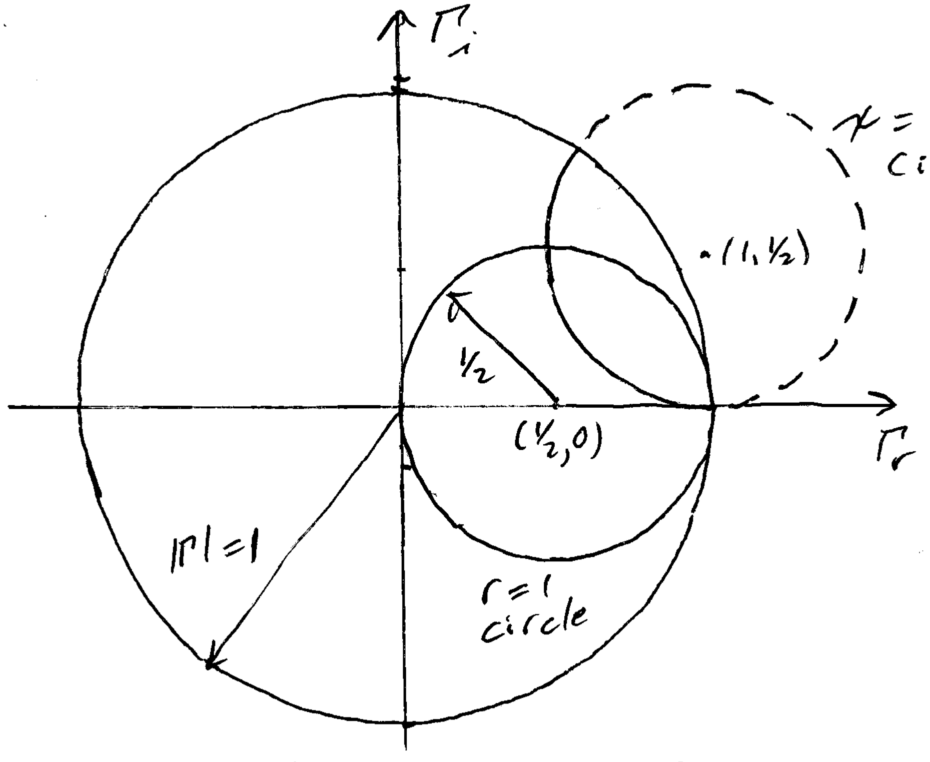
$$\left(\Gamma_r - \frac{1}{2}\right)^2 + \Gamma_i^2 = \left(\frac{1}{2}\right)^2$$

⇒ Circle of radius ½ centered @ (½, 0)

For $x=2$ in (2.56b), we get

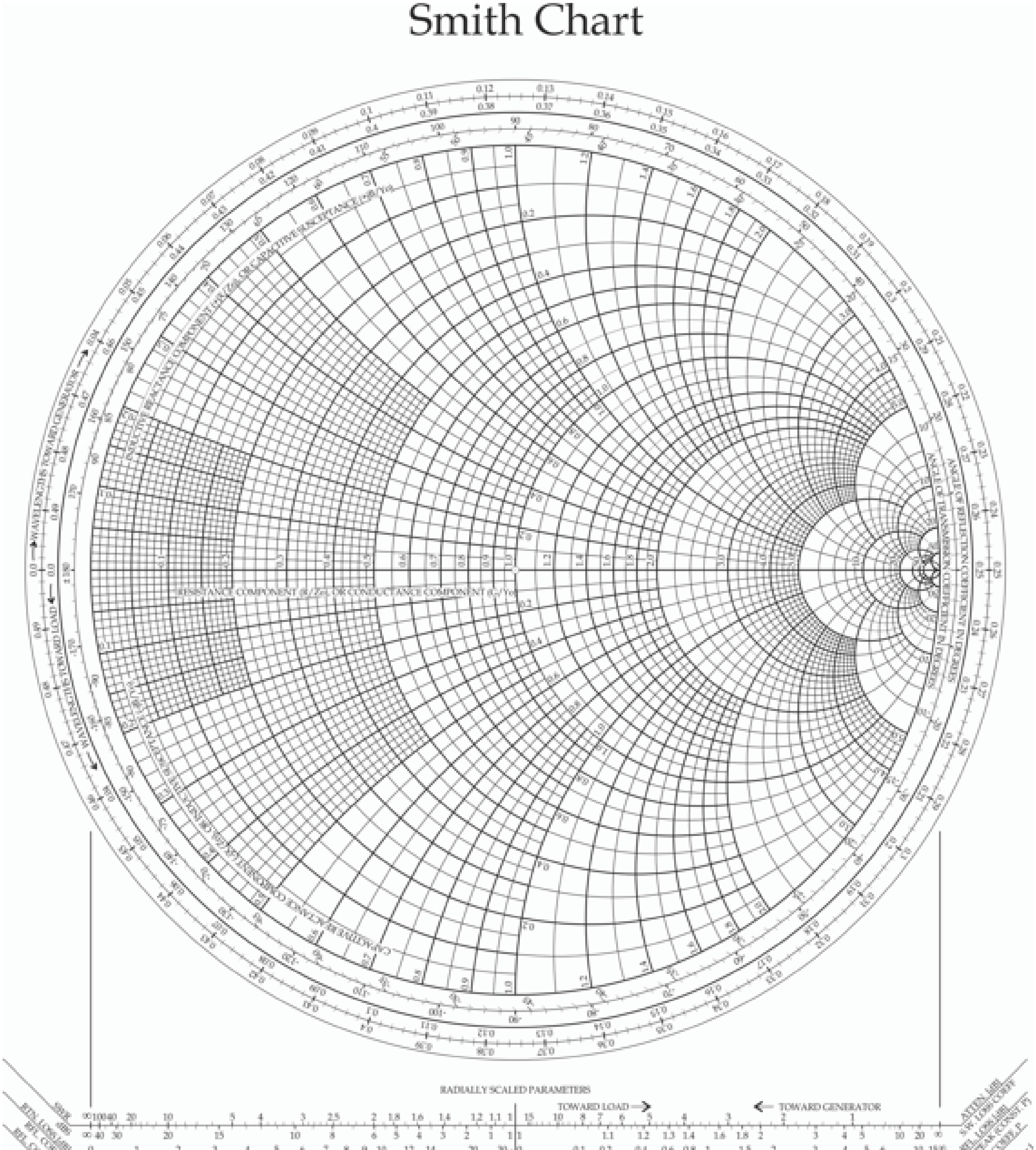
$$\left(\Gamma_r - 1\right)^2 + \left(\Gamma_i - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

⇒ Circle of radius ½ centered @ (1, ½)



If we select a range of r & x values, superimpose them, & impose the condition $|\Gamma| \le 1$, we get the Smith chart. We can then add scales for related quantities, add notes, and give directions for how $\Gamma(z_1) = \Gamma(z_2) e^{j2\beta(z_1 - z_2)}$ are related, i.e., 'TWD generator' or 'TWD load.'

2.4 cont.


Smith Chart
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN

2.4 cont.

① To plot or read reflection coefficients (Γ) on a Smith Chart, use the polar convention that is, plot/read magnitude (scale at bottom of chart ranging linearly from 0 (center of Smith chart) to 1 (outer edge of Smith chart) and angle (circular scale on outer edge of Smith Chart). Note that positive angles are above the horizontal axis and negative angles are below.

② To plot/read normalized ($z = Z/R_0$) impedances on a Smith chart, locate the appropriate real "r"-circle(s) (centered on horizontal axis) and find intersection w/ or trace to arc of the imaginary "x"-circle (centered on $\Gamma_r = 1$ on right side of Smith chart). Note that positive values of x are above the horizontal axis, while negative values are below.

③ Given that either $|\Gamma|$, Γ, or z are plotted on the Smith Chart. The VSWR (voltage standing wave ratio) or SWR can be found using the Smith chart, scale at bottom of the chart that ranges from 1 (center of Smith Chart) to ∞. To find the VSWR measure the distance from the center of the Smith chart to the $|\Gamma|$ circle or Γ/z point.

Example-

1) Plot reflection coefficient Γ = 0.707∠-45° for a 50 Ω transmission line

- Use straight edge to draw radial line from center of Smith chart through the -45° mark on "ANGLE OF REFLECTION COEFFCIENT IN DEGREES" scale (inner ring surrounding Smith chart).
- Use "REFL. COEFF. V or I" scale at bottom right of chart to set compass to $|\Gamma| = 0.707$, and draw arc, centered on Smith chart, through -45° radial line.
- The intersection of radial line & arc marks Γ = 0.707∠-45° on Smith chart.

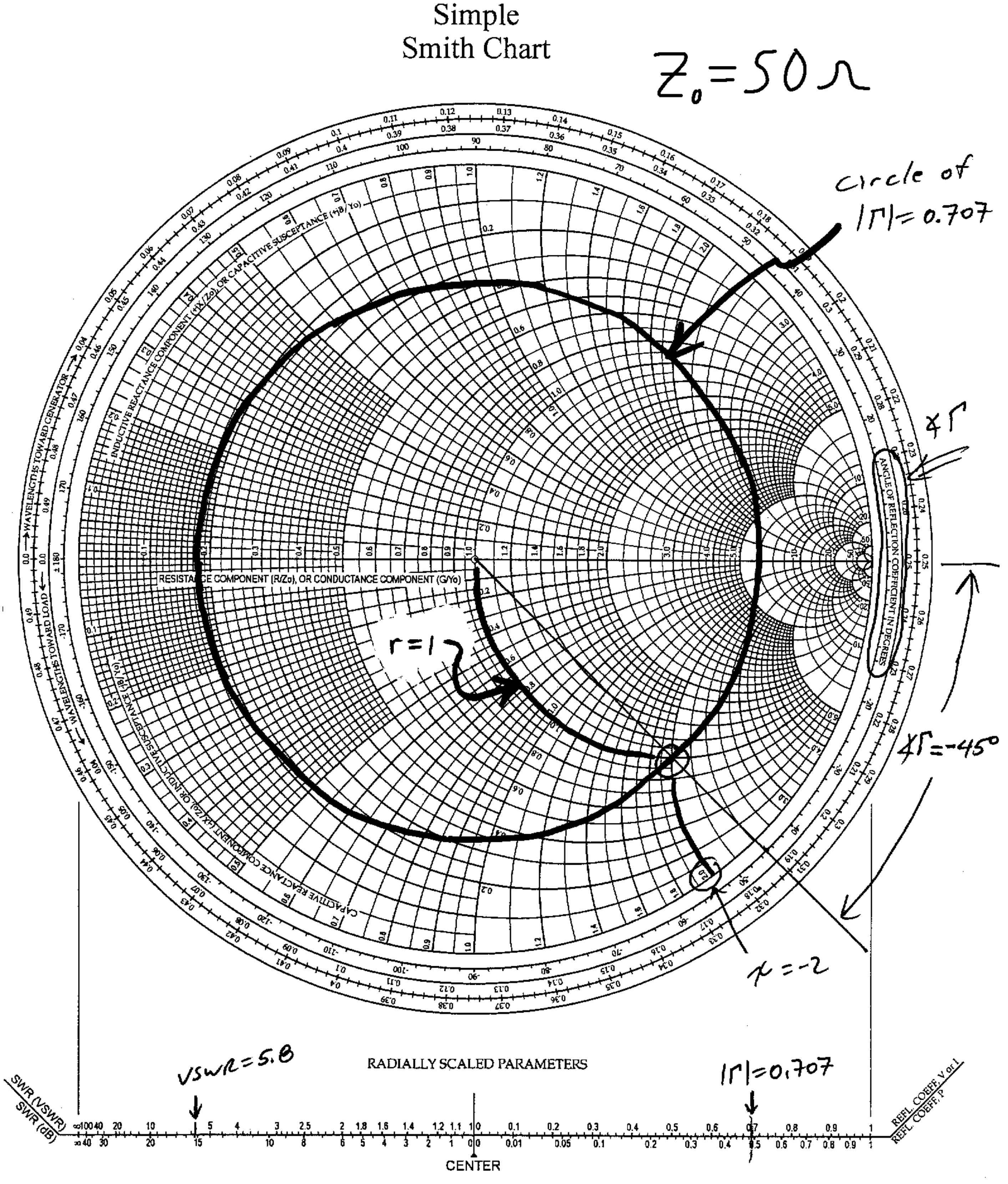
2) Read normalized impedance *z* corresponding to Γ = 0.707∠-45°

- On Smith chart, at Γ = 0.707∠-45° point, locate and read/interpolate value of appropriate "*r*" circle (family of circles centered on horizontal axis and with values shown on horizontal axis) as ***r* = 1**.
- On Smith chart, at Γ = 0.707∠-45° point, locate and read/interpolate value of appropriate "*x*" arc (reactance values shown on inside of outer ring of Smith chart; values above horizontal axis are positive/inductive while those below are negative/capacitive) as ***x* = -2**.
- Put together to get normalized impedance **$z = 1 - j2$ Ω/Ω.**
- Find impedance corresponding to Γ = 0.707∠-45° by multiplying *z* w/ characteristic impedance to get $Z = Z_0 z = 50(1 - j2) \Rightarrow$ **$Z = 50 - j100$ Ω**.

3) Read standing wave ratio SWR (VSWR) corresponding to Γ = 0.707∠-45°

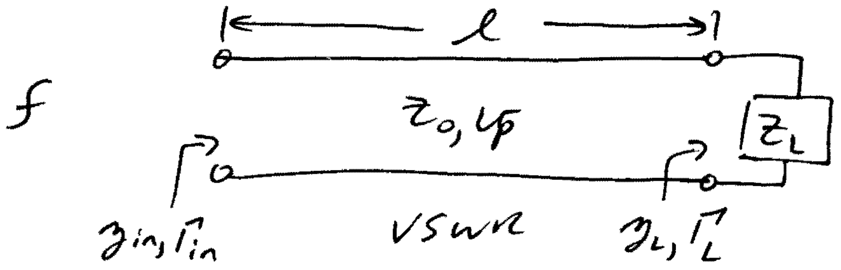
- Use "REFL. COEFF. V or I" scale at bottom right to set your compass to $|\Gamma| = 0.707$.
- Draw 0.707 arc, centered on Smith chart scales, through SWR (VSWR) scale on bottom left.
- Read standing wave ratio to be **VSWR = 5.8 .**

ex. cont.


Simple
Smith Chart
$Z_0 = 50\,\Omega$
circle of $|\Gamma| = 0.707$
$\angle\Gamma$
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
$\angle\Gamma = -45°$
r=1
x = -2
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RADIALLY SCALED PARAMETERS
VSWR = 5.8
$|\Gamma| = 0.707$
SWR (VSWR)
SWR (dB)
REFL. COEFF. V or I
REFL. COEFF. P
CENTER

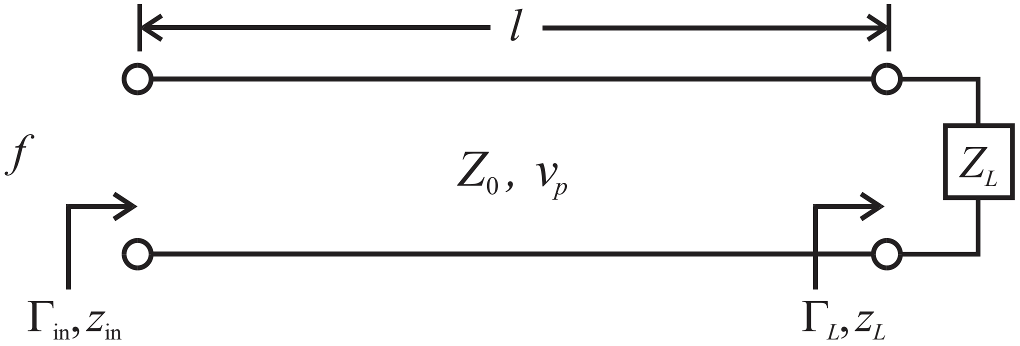
2.4 cont.

Relating load end parameters to input end using Smith Chart.



1) Plot z_L or Γ_L on Smith chart. If necessary, draw a radial line from center of Smith chart thru z_L/Γ_L point to outer rings.
2) Read Γ_L or z_L off Smith chart.
3) Find SWR (VSWR) by using ruler or compass to measure distance from center of Smith chart to z_L/Γ_L point. Mark this distance on SWR (VSWR) Scale @ bottom and read off SWR.
4) Calculate length of TL in wavelengths, i.e., $\ell/\lambda = \ell f/u$. Subtract out $n\,1/2$ to get a value between 0 and 0.5.
5) Starting @ z_L/Γ_L point, use compass to trace arc of length ℓ/λ in the "wavelengths toward generator" direction (easiest to use that scale and add ℓ/λ to value read where radial thru z_L/Γ_L)
6) Read off values of z_{in} and Γ_{in} where arc terminates (easiest to draw radial line from center of Smith Chart through z_{in}/Γ_{in} point).

Example- ee481_581_Smith_chart_example_2.docx



For the lossless transmission line circuit shown: $f = 100$ MHz, $v_p = 3\times10^8$ m/s, $l = 3.3$ m, $Z_0 = 50\,\Omega$, and $Z_L = 75 + j\,50\,\Omega$.

1) Normalize and plot load impedance

- Normalize $z_L = Z_L / Z_0 = (75 + j\,50) / 50 \Rightarrow$ **$\underline{z_L = 1.5 + j1\ \Omega/\Omega}$**.
- Plot z_L on Smith chart by finding intersection of $r = 1.5$ circle with $x = 1$ arc.

2) Find load reflection coefficient and VSWR

- Set compass to distance between center of Smith chart and z_L. Use compass to mark the "REFL. COEFF. V or I" scale at bottom right of Smith chart to determine $\underline{|\Gamma_L| = 0.42}$.
- Use compass to draw $|\Gamma| = 0.42$ arc, centered on Smith chart scales, through SWR (VSWR) scale on bottom left. Read **<u>VSWR = 2.4</u>**.
- Use straight-edge to draw radial line from center of Smith chart through z_L and outer rings of Smith chart. Use "ANGLE OF REFLECTION COEFFCIENT IN DEGREES" scale to read $\underline{\angle \Gamma_L = 41.8°}$.
- Put magnitude and angle together to get **$\underline{\Gamma_L = 0.42\angle 41.8°}$**. For comparison, the analytic result is $\Gamma_L = 0.4152\angle 41.63°$.

3) Find input reflection coefficient

- Calculate $l/\lambda = l f / v_p = 3.3\,(100\times10^6) / 3\times10^8 = 1.1$. Subtract $2(0.5) = 1$ (i.e., remove integer multiples of $n\lambda/2$) to get $\Rightarrow \underline{l/\lambda = 0.1}$.
- On the Smith chart, the radial line through z_L reads 0.192 on the "WAVELENGTHS TOWARD GENERATOR" scale. Add $0.192 + l/\lambda$ to get 0.292 and draw a radial line from the center of the Smith chart through this point on the scale.

- Draw an arc, centered on Smith chart, from z_L through radial line at 0.292. The intersection of the arc and radial line is the Γ_{in} / z_{in} point. Use the "ANGLE OF REFLECTION COEFFCIENT IN DEGREES" scale to read $\underline{\angle \Gamma_{in} = -30.2°}$ and note $\underline{|\Gamma_{\text{in}}| = |\Gamma_L| = 0.42}$.
- Put magnitude and angle together to get $\underline{\boldsymbol{\Gamma_{\text{in}} = 0.42\angle\text{-}30.2°}}$.

4) Find input impedance

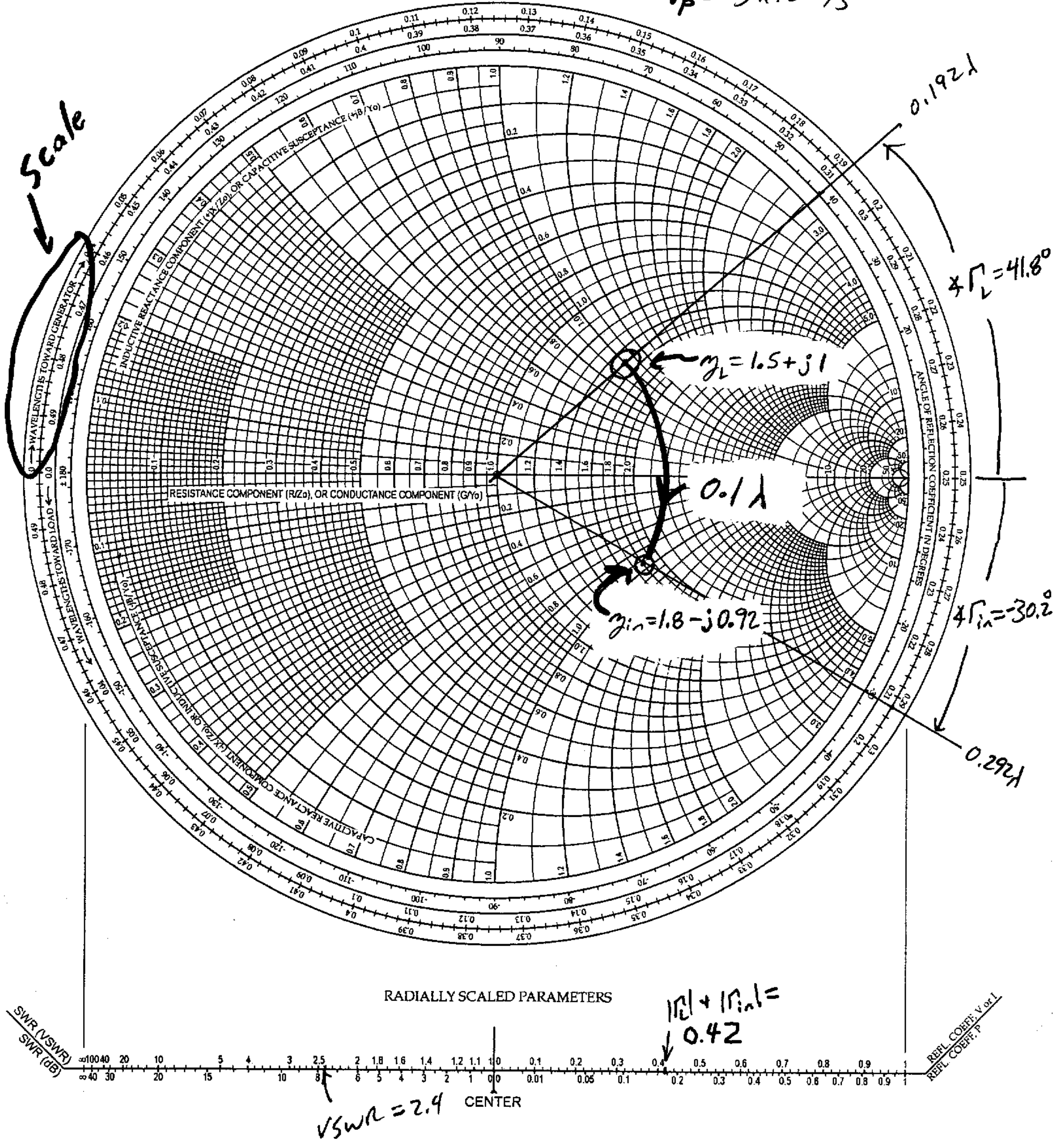
- At $\Gamma_{\text{in}} = 0.42\angle\text{-}30.2°$ point, locate and read/interpolate value of appropriate "r" circle as $\underline{r_{\text{in}} = 1.8}$.
- At $\Gamma_{\text{in}} = 0.42\angle\text{-}30.2°$ point, locate and read/interpolate value of appropriate "x" arc as $\underline{x_{\text{in}} = -0.92}$.
- Put together to get normalized input impedance $\underline{\boldsymbol{z_{\text{in}} = 1.8 - j0.92\ \Omega/\Omega}}$.
- Find input impedance by multiplying z_{in} w/ characteristic impedance to get $Z_{\text{in}} = Z_0 z_{\text{in}} = 50(1.8 - j0.92) \quad \Rightarrow \underline{\boldsymbol{Z_{\text{in}} = 90 - j46\ \Omega}}$.

Simple Smith Chart

$Z_0 = 50\Omega$

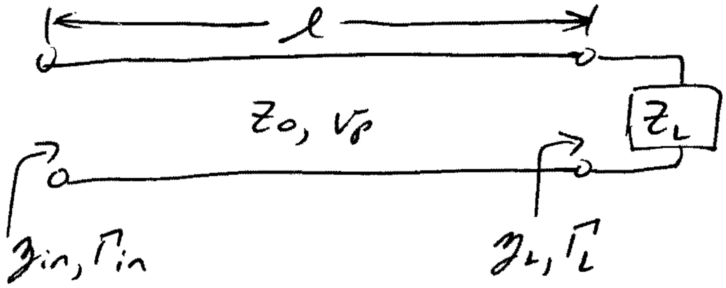
$f = 100\ MHz$

$V_p = 3\times10^8\ m/s$



2.4 cont.

Relating input end parameters to load end using Smith chart.



1) Plot z_{in} or Γ_{in} on Smith chart. If necessary draw a radial line through z_{in}/Γ_{in}.

2) Read Γ_{in} or z_{in} off Smith chart.

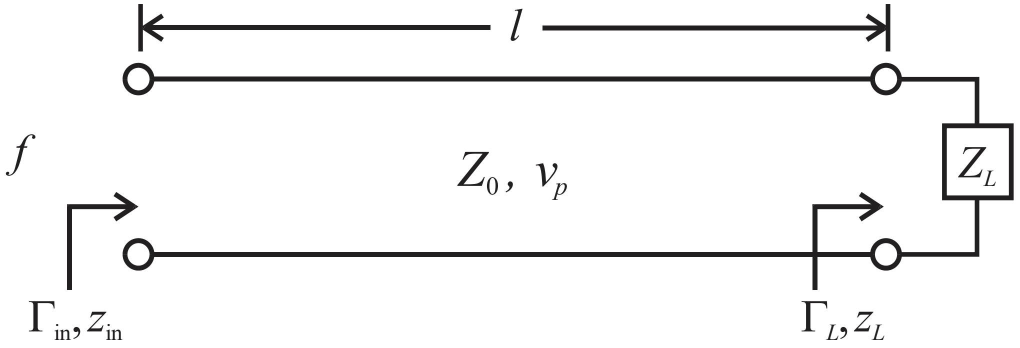
3) Read SWR (VSWR) off Smith chart.

4) Calculate length of TL in wavelengths, i.e., ℓ/λ. Subtract out $n\lambda/2$ to make $0 \le \ell/\lambda \le 0.5$.

5) Trace arc of radius $|\Gamma_{in}|$ and length ℓ/λ starting @ z_{in}/Γ_{in} point in the "WAVELENGTHS TOWARD LOAD" direction to arrive @ z_L/Γ_L point. Hint: Add ℓ/λ to value read off "WAVELENGTHS TOWARD LOAD" scale where radial line through z_{in}/Γ_{in} point. Draw radial line from center of Smith chart through this value.

6) Read $z_L = r_L + jx_L$ and $\Gamma_L = |\Gamma_L| \underline{\angle \Gamma_L}$ from the Smith Chart

Example- ee481_581_Smith_chart_example_3.docx



For the lossless transmission line circuit shown: $f = 500$ MHz, $v_p = 2\times10^8$ m/s, $l = 1.242$ m, $Z_0 = 75\ \Omega$, and $\Gamma_{\text{in}} = 0.8\angle{-117.5°}$.

1) Plot input reflection coefficient and find VSWR

- Use straight edge to draw radial line from center of Smith chart through the -117.5° mark on the "ANGLE OF REFLECTION COEFFCIENT IN DEGREES" scale.
- Use "REFL. COEFF. V or I" scale at bottom right to set compass to $|\Gamma| = 0.8$, and draw arc, centered on Smith chart, through -117.5° radial line.
- The intersection of radial line & arc marks $\underline{\Gamma_{\text{in}} = 0.8\angle{-117.5°}}$.
- Use compass to draw $|\Gamma| = 0.8$ arc, centered on Smith chart scales, through SWR (VSWR) scale on bottom left. Read **VSWR = 9** .

2) Find input impedance

- At $\Gamma_{\text{in}} = 0.8\angle{-117.5°}$ point, locate and read/interpolate value of appropriate "r" circle as $\underline{r_{\text{in}} = 0.15}$.
- At $\Gamma_{\text{in}} = 0.8\angle{-117.5°}$ point, locate and read/interpolate value of appropriate "x" arc as $\underline{x_{\text{in}} = -0.60}$.
- Put together to get normalized input impedance $\underline{\mathbf{z_{\text{in}} = 0.15 - j0.60\ \Omega/\Omega}}$.
- Find input impedance by multiplying z_{in} w/ characteristic impedance to get $Z_{\text{in}} = Z_0\, z_{\text{in}} = 75(0.15 - j0.60) \ \Rightarrow \underline{\mathbf{Z_{\text{in}} = 11.25 - j45\ \Omega}}$.

3) Find load reflection coefficient

- Calculate $l/\lambda = lf/v_p = 1.242(500 \times 10^6)/2 \times 10^8 = 3.105$. Subtract $6(0.5) = 3$ (i.e., remove integer multiples of $n\lambda/2$) to get $\Rightarrow \underline{l/\lambda = 0.105}$.
- Leave compass set to $|\Gamma| = 0.8$ and draw circle centered on Smith chart.
- Using radial line for $\angle\Gamma_{\text{in}} = -117.5°$, read 0.087 on the "WAVELENGTHS TOWARD LOAD" scale. Add $0.087 + l/\lambda$ to get 0.192 and draw a radial line from the center of the Smith chart through this point on the scale.
- Use "ANGLE OF REFLECTION COEFFCIENT IN DEGREES" scale to read $\underline{\angle \Gamma_L = -41.6°}$.
- Put magnitude and angle together to get $\underline{\boldsymbol{\Gamma_L = 0.8\angle\text{-}41.6°}}$.

4) Find load impedance

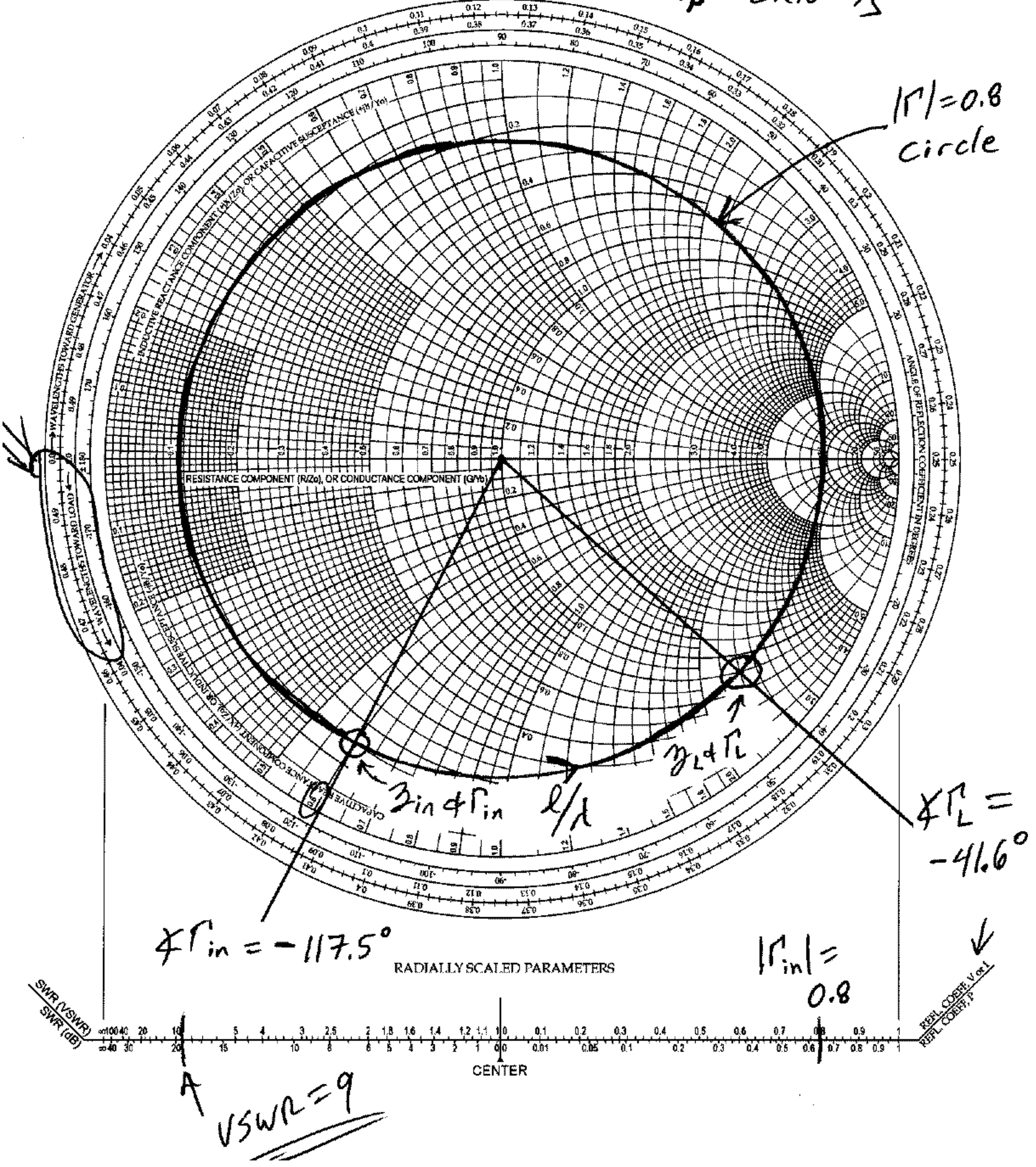
- At $\Gamma_L = 0.8\angle\text{-}41.6°$ point, locate and read/interpolate value of appropriate "*r*" circle as $\underline{r_L = 0.8}$.
- At $\Gamma_L = 0.8\angle\text{-}41.6°$ point, locate and read/interpolate value of appropriate "*x*" arc as $\underline{x_L = -2.4}$.
- Put together to get <u>normalized</u> load impedance $\underline{\boldsymbol{z_L = 0.8 - j2.4\ \Omega/\Omega}}$.
- Find load impedance by multiplying z_L w/ characteristic impedance to get $Z_L = Z_0 z_L = 75(0.8 - j2.4) \Rightarrow \underline{\boldsymbol{Z_L = 60 - j180\ \Omega}}$.

Simple Smith Chart

$l = 1.242\,m$

$Z_0 = 75\Omega$

$v_p = 2\times10^8\,m/s$



2.4 cont.

On the Smith chart, draw a circle of radius $|\Gamma|$ for your particular TL circuit.

1) Where the $|\Gamma|$ circle intersects the real axis to the right of the center,

$$\boxed{z_{max} = r_{max} = \frac{Z_{max}}{Z_0} = \frac{1+|\Gamma|}{1-|\Gamma|} = SWR.}$$

Note, this is the location(s) where V_{max} & I_{min} occur.

2) Where the $|\Gamma|$ circle intersects the real axis to the left of the center,

$$\boxed{z_{min} = r_{min} = \frac{Z_{min}}{Z_0} = \frac{1-|\Gamma|}{1+|\Gamma|} = \frac{1}{SWR}.}$$

Note, this is the location(s) where V_{min} & I_{max} occur.

What about admittances, $Y = \frac{1}{Z}$ (S), and the Smith Chart?

Normalized admittance $\equiv y = \frac{1}{z} = \frac{Z_0}{Z} = Z_0 Y = \frac{Y}{Y_0}$

Note: $y = |y| \angle\theta_y = \frac{1}{|z| \angle\theta_z} \Rightarrow \theta_y = -\theta_z \Rightarrow \underline{\underline{\text{"}-\text{"} = \pm 180^0}}$

2.4 cont.

Therefore, for any z on the Smith chart, we can find the corresponding y by moving ±180° ($\lambda/4$) around the circle of constant $|\Gamma|$!

Alternatively, there are combined impedance and admittance Smith Charts where the r circles and x arcs have been rotated ±180° to yield g circles and b arcs!

$$\left[z = r + jx \ \ \& \ \ y = \frac{1}{z} = g + jb \right]$$

This almost necessitates the use of color-coded lines to distinguish the normalized impedance r circles & x arcs from the normalized admittance g circles & b arcs.

Analytically, the 'g' & 'b' circles are found using

$$y = \frac{1}{z} = \frac{1-\Gamma(z)}{1+\Gamma(z)} = \frac{1-(\Gamma_r + j\Gamma_i)}{1+(\Gamma_r + j\Gamma_i)} = g + jb$$

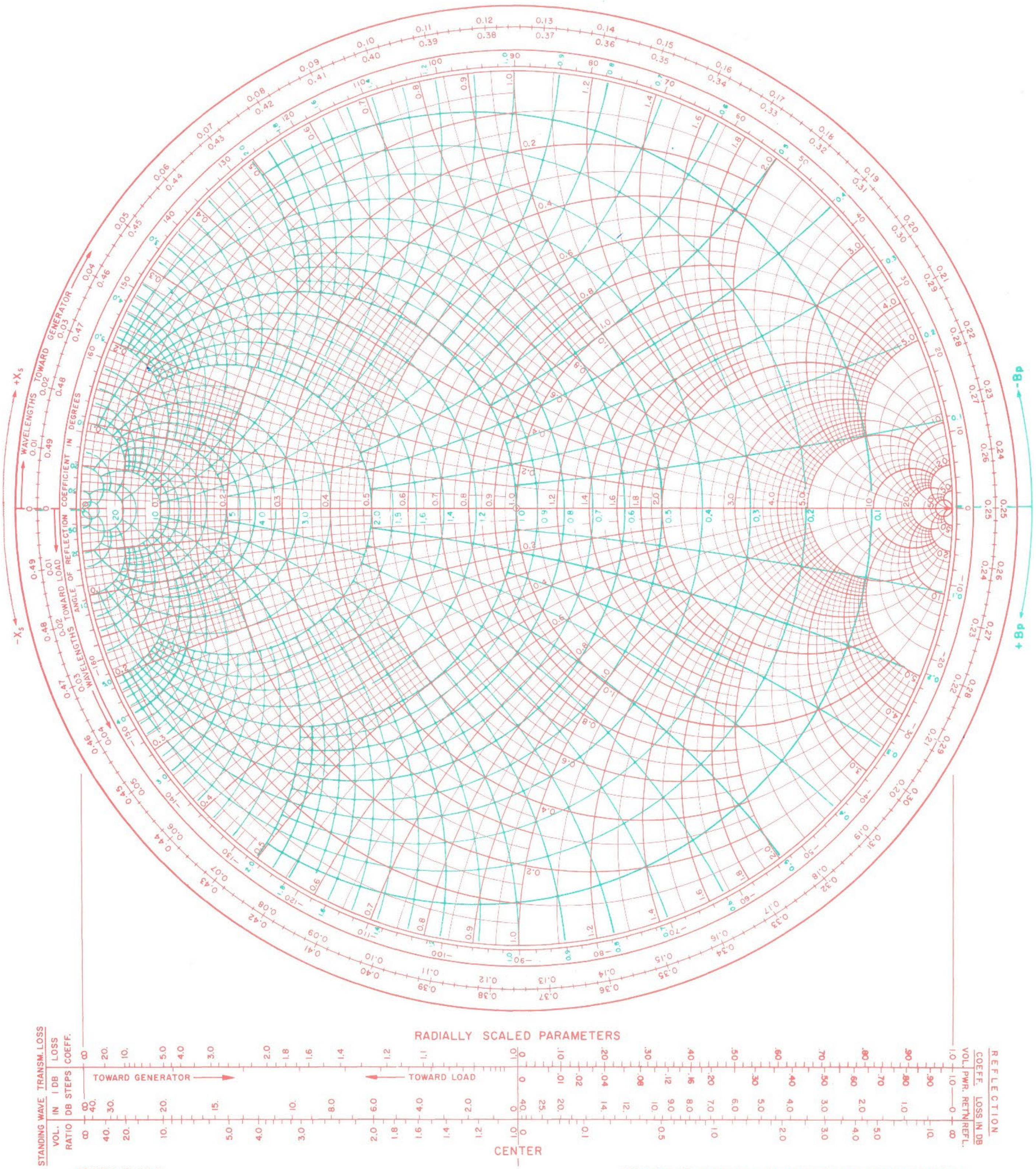
to get:

$$\left(\Gamma_r + \frac{g}{1+g}\right)^2 + \Gamma_i^{\,2} = \left(\frac{1}{1+g}\right)^2 \quad \text{'g circles'}$$

$$\left(\Gamma_r + 1\right)^2 + \left(\Gamma_i + \frac{1}{b}\right)^2 = \left(\frac{1}{b}\right)^2 \quad \text{'b circles'}$$

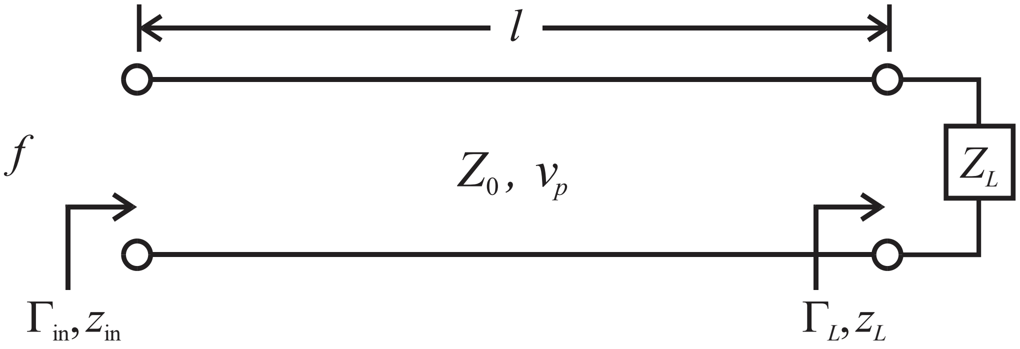
NAME	TITLE	DWG. NO.
SMITH CHART FORM ZY-01-N	ANALOG INSTRUMENTS COMPANY, NEW PROVIDENCE, N.J. 07974	DATE

NORMALIZED IMPEDANCE AND ADMITTANCE COORDINATES



PRINTED IN U.S.A.

Example- ee481_581_Smith_chart_example_4.docx



For the lossless transmission line circuit above, the frequency, length, and phase velocity will be left unspecified while $Z_0 = 75\ \Omega$ and $Z_L = 56.25 - j\,75\ \Omega$.

1) Normalize and plot load impedance

- Normalize $z_L = Z_L / Z_0 = (56.25 - j\,75) / 75 \Rightarrow$ **$z_L = 0.75 - j\,1\ \Omega/\Omega$**.
- Plot z_L on Smith charts by finding the intersection of the $r = 0.75$ circle with the $x = -1$ arc.

2) Find load reflection coefficient, RL, and VSWR (method 1)

- Set compass to distance between center of Smith charts and z_L. Use "REFL. COEFF. V or I" scale at bottom right to determine $|\Gamma_L| = 0.5$ or 0.51.
- Use straight-edge to draw radial line from center of Smith chart through z_L and outer rings of Smith charts. Use "ANGLE OF REFLECTION COEFFCIENT IN DEGREES" scale to read $\angle\,\Gamma_L = -74°$.
- Put magnitude and angle together to get **$\Gamma_L = 0.5\angle{-74°}$** or **$0.51\angle{-74°}$**. For comparison, the analytic result is $\Gamma_L = 0.5114\angle{-74.29°}$.
- Use compass to draw $|\Gamma| = 0.5$ arc, centered on Smith chart scales, through SWR (VSWR) scale on bottom left. Read **VSWR = 3.1** .
- Use compass to draw $|\Gamma| = 0.5$ arc, centered on Smith chart scales, through RETN LOSS scale on bottom right. Read **RL = 6 dB** or **5.8 dB.**

3) Find VSWR (method 2)

- Draw a circle, centered on Smith charts, through z_L.
- Read value of normalized resistance r where the $|\Gamma| = 0.5$ circle crosses the horizontal/real axis to the right of the origin to get **r_{max} = VSWR = 3.1**.

4) Find load admittance

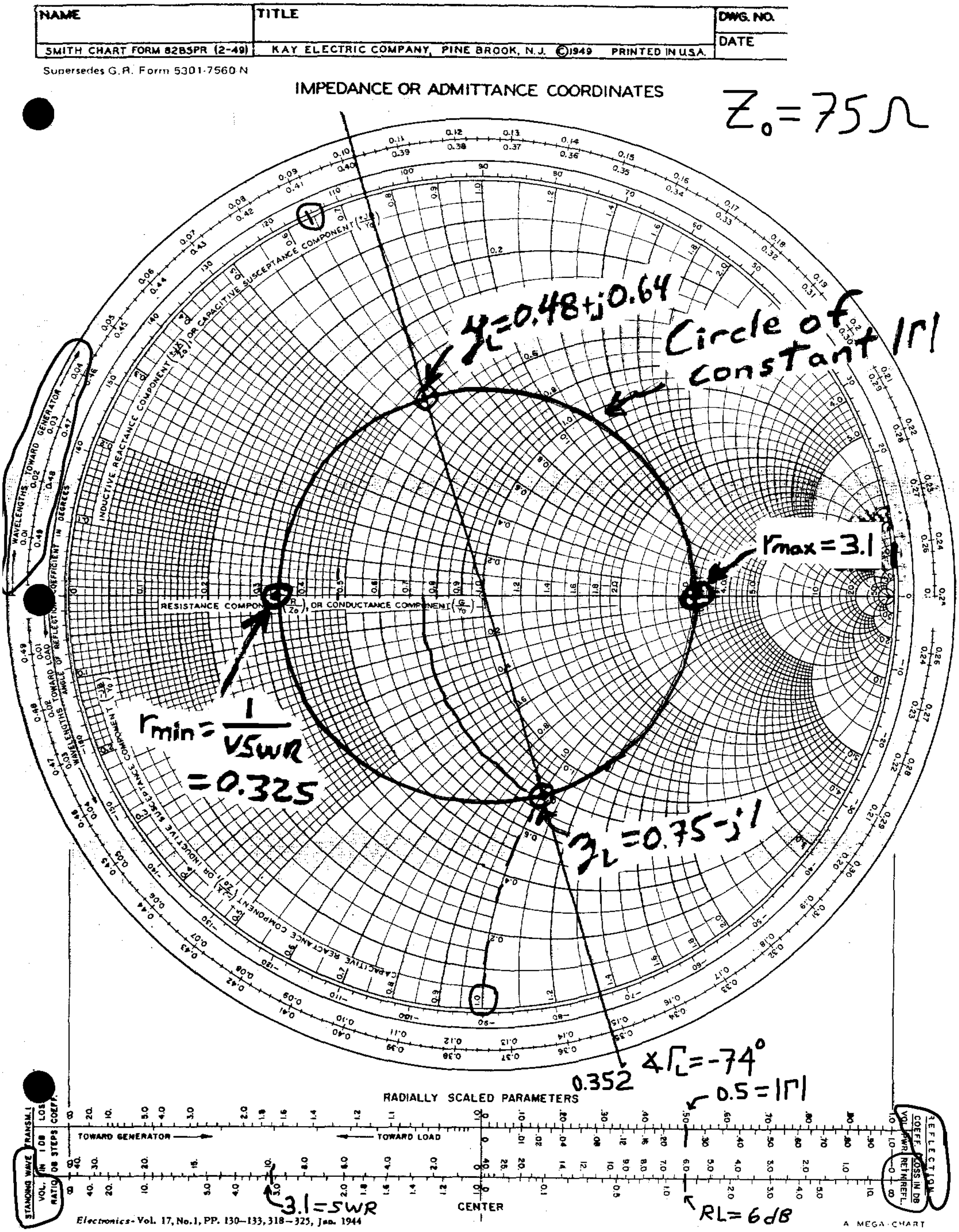
- Use straight-edge to draw line from edge-to-edge of Smith charts through center of Smith charts and z_L point.
- Where the line intersects the $|\Gamma| = 0.5$ circle on the side opposite to z_L, locate and read/interpolate value of appropriate "*g*" circle as $\underline{g_L = 0.48}$.
- Where the line intersects the $|\Gamma| = 0.5$ circle on the side opposite to z_L, locate and read/interpolate value of appropriate "*b*" arc as $\underline{b_L = 0.64}$.
- Put together to get <u>normalized</u> load admittance $\underline{\boldsymbol{y_L = 0.48 + j0.64}\ \textbf{S/S}}$.
- Find load admittance by dividing y_L by characteristic impedance Z_0 to get $Y_L = y_L / Z_0 = (0.48 + j\,0.64)/75 \Rightarrow \underline{\boldsymbol{Y_L = 0.0064 + j0.0083}\ \textbf{S} = \boldsymbol{6.4 + j8.3}\ \textbf{mS}}$.

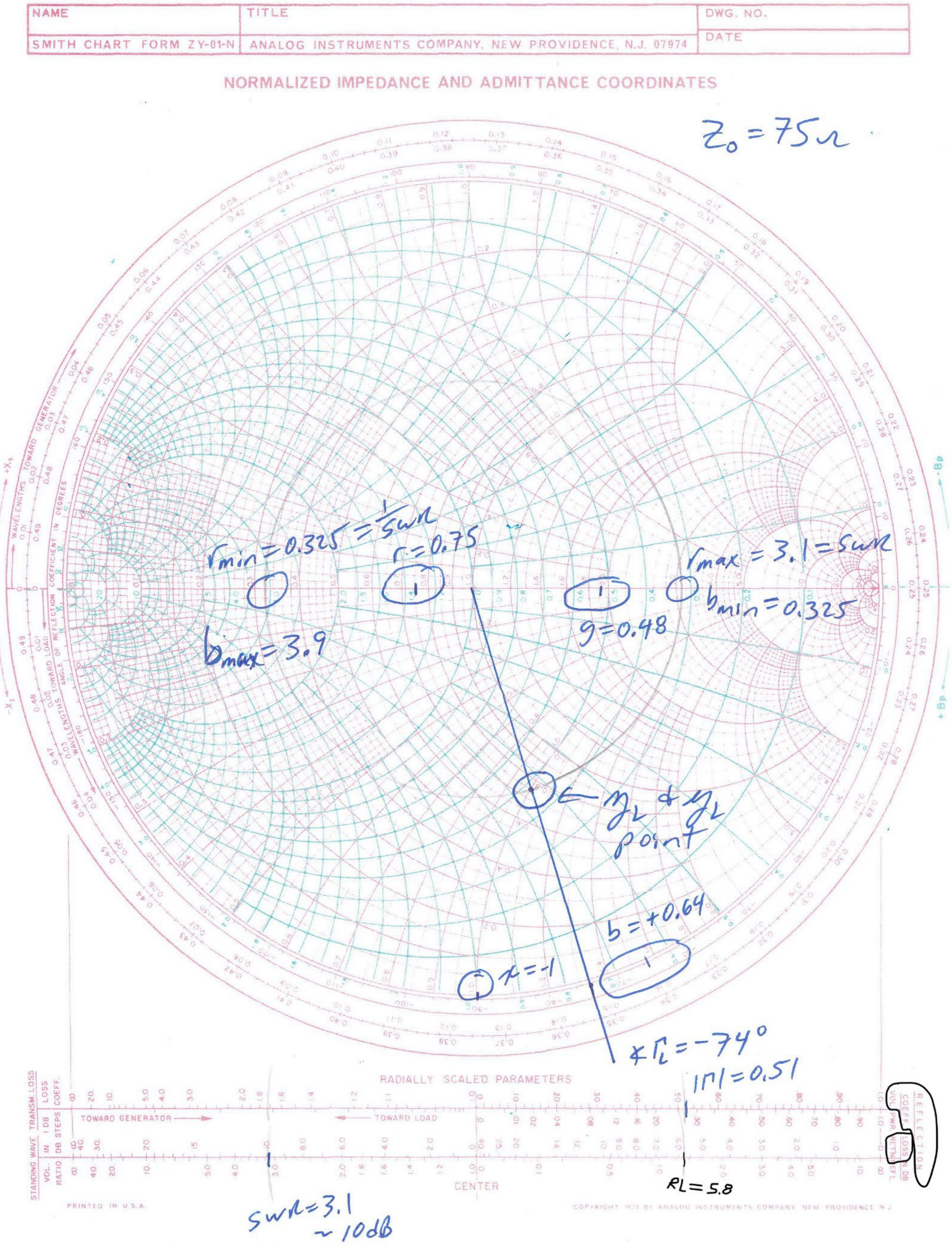
5) Find/locate voltage and impedance maxima

- Impedance maxima occur where the $|\Gamma| = 0.5$ circle crosses the real axis to the right of origin. Read/interpolate "*r*" circles to get $\underline{\boldsymbol{r_{\max} = 3.1}}$.
- The maximum impedance along the transmission line is found by multiplying $r_{\max}$ w/ Z_0 to get $Z_{\max} = Z_0\, z_{\max} = 75(3.1) \Rightarrow \underline{\boldsymbol{Z_{\max} = 232.5\ \Omega}}$.
- Voltage maxima along the transmission line occur at $r_{\max}$. Starting where the radial line through z_L crosses the "WAVELENGTHS TOWARD GENERATOR" scale at 0.352, move <u>toward the generator</u> to the real axis to **right** of origin ($r_{\max}$ location) where the scale reads 0.25. The total distance is $(0.5\text{-}0.352)\,\lambda + 0.25\lambda = 0.398\lambda$.
- As everything repeats at $\lambda/2$ intervals on lossless TLs, the voltage maxima locations in <u>distance from the load</u> are $\underline{\boldsymbol{l_{\max} = 0.398\lambda + n\lambda/2}}$.

5) Find/locate voltage and impedance minima

- Impedance minima occurs where the $|\Gamma| = 0.5$ circle crosses the real axis to the left of origin. Read/interpolate "*r*" circles to get $\underline{\boldsymbol{r_{\min} = 0.325}}$.
- The minimum impedance along the transmission line is found by multiplying $r_{\min}$ w/ Z_0 to get $Z_{\min} = Z_0\, z_{\min} = 75(0.325) \Rightarrow \underline{\boldsymbol{Z_{\min} = 24.375\ \Omega}}$.
- Voltage minima along the TL occur at $r_{\min}$. Starting where the radial line through z_L crosses "WAVELENGTHS TOWARD GENERATOR" scale at 0.352, move <u>toward the generator</u> to the real axis **left** of origin ($r_{\min}$ location) where the scale reads 0.5. The total distance is $(0.5\text{-}0.352)\,\lambda = 0.148\lambda$.
- As everything repeats at $\lambda/2$ intervals, the voltage minima locations in <u>distance from the load</u> are $\underline{\boldsymbol{l_{\min} = 0.148\lambda + n\lambda/2}}$.


NAME
TITLE
DWG. NO.
DATE
SMITH CHART FORM 82BSPR (2-49)
KAY ELECTRIC COMPANY, PINE BROOK, N.J. ©1949 PRINTED IN U.S.A.
Supersedes G.R. Form 5301-7560-N
IMPEDANCE OR ADMITTANCE COORDINATES
Z₀ = 75Ω
y_L = 0.48 + j0.64
Circle of constant |Γ|
r_max = 3.1
r_min = 1/VSWR = 0.325
z_L = 0.75 − j1
0.352
∠Γ_L = −74°
0.5 = |Γ|
3.1 = SWR
RL = 6dB
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
RESISTANCE COMPONENT (R/Z₀), OR CONDUCTANCE COMPONENT (G/Y₀)
INDUCTIVE REACTANCE COMPONENT (+jX/Z₀), OR CAPACITIVE SUSCEPTANCE COMPONENT (+jB/Y₀)
CAPACITIVE REACTANCE COMPONENT (−jX/Z₀), OR INDUCTIVE SUSCEPTANCE COMPONENT (−jB/Y₀)
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
STANDING WAVE VOL. RATIO
IN DB
TRANSM. LOSS COEFF.
LOSS IN DB STEPS
REFLECTION COEFF. VOL.
REFLECTION COEFF. PWR.
RET'N LOSS IN DB
REFL. LOSS IN DB
CENTER
Electronics - Vol. 17, No. 1, PP. 130–133, 318–325, Jan. 1944
A MEGA-CHART


NAME
TITLE
DWG. NO.
SMITH CHART FORM ZY-01-N
ANALOG INSTRUMENTS COMPANY, NEW PROVIDENCE, N.J. 07974
DATE
NORMALIZED IMPEDANCE AND ADMITTANCE COORDINATES
Z₀ = 75Ω
rmin = 0.325 = 1/SWR
r = 0.75
rmax = 3.1 = SWR
bmin = 0.325
bmax = 3.9
g = 0.48
zL & yL point
b = +0.64
x = -1
∢ΓL = -74°
|Γ| = 0.51
RL = 5.8
SWR = 3.1
~ 10dB
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
STANDING WAVE VOL. RATIO
IN DB
TRANSM. LOSS 1 DB STEPS
LOSS COEFF.
REFLECTION COEFF. VOL. PWR.
RETN. LOSS IN DB
CENTER
PRINTED IN U.S.A.
COPYRIGHT 1970 BY ANALOG INSTRUMENTS COMPANY, NEW PROVIDENCE, N.J.
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
+Xs
-Xs
-Bp
+Bp

2.4 cont.

Another common use of Smith charts is in determining unknown loads. Any test instrument will need some sort of connecting TL (see below). Therefore, we are measuring input impedance or reflection coefficient! To get load impedance &/or reflection coefficient, we need either the electrical length (ℓ/λ) of the TL or some other way of translating to the load from the input.

Method 1 (Use Smith chart and impedance analyzer or vector network analyzer)

Known: Z_0, v_p, f or λ, physical length ℓ_{phys} of TL, and either $Z_{in,meas}$ or $\Gamma_{in,meas} = S_{11,meas}$

1) Attach a known load to end of TL, usually a short or open circuit, and get $Z_{in,known}$ or $\Gamma_{in,known}$ from test instrument.
2) Plot either $z_{L,known}$ or $\Gamma_{L,known}$ on Smith chart.
3) Plot either $z_{in,known}$ or $\Gamma_{in,known}$ on Smith chart.
4) Move on arc from $z_{in,known}/\Gamma_{in,known}$ point to $z_{L,known}/\Gamma_{L,known}$ point. Draw radial lines from center to edge of Smith chart through both points. Move 'WAVELENGTHS TOWARD LOAD' direction!

2.4 cont.

5) Using 'WAVELENGTHS TOWARD LOAD' scale find $\ell_{TL,known}/\lambda$ (i.e., distance from input to load). However, $\ell = \ell_{TL,known} + n\,\lambda/2$. Use ℓ_{phys} & λ information to find correct value for ℓ.

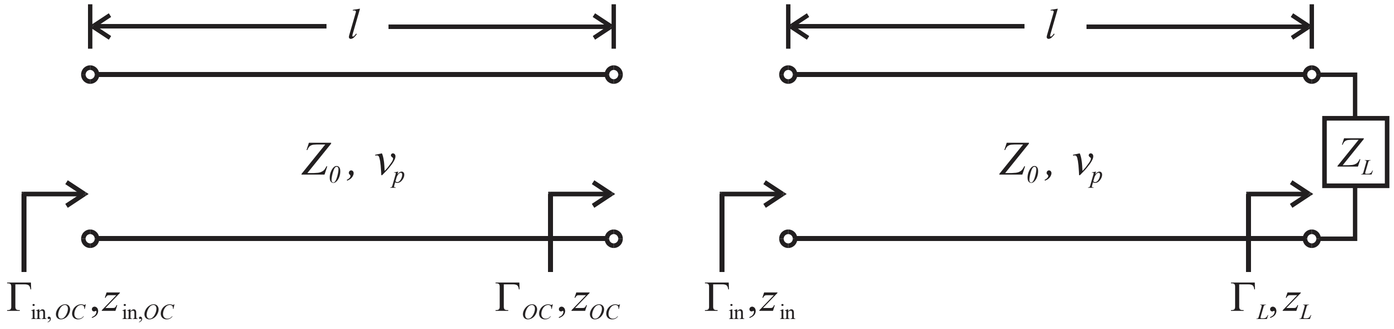
6) Next, with unknown load connected to end of TL, measure either $Z_{in,unknown}$ or $\Gamma_{in,unknown}$ with test instrument.

7) Plot $z_{in,unknown}$ or $\Gamma_{in,unknown}$ on Smith Chart.

8) Move ℓ/λ "WAVELENGTHS TOWARD LOAD" on arc/circle passing through $z_{in,unknown}$/ $\Gamma_{in,unknown}$ point to arrive at $z_{L,unknown}$/ $\Gamma_{L,unknown}$ point. Note: It is usually easist to draw radial lines for start and stop points before drawing arc/circle.

9) Read $z_{L,unknown}$ + $\Gamma_{L,unknown}$ off Smith Chart. Multiply $z_{L,unknown}$ by Z_0 to get $Z_{L,unknown}$.

Example- ee481_581_Smith_chart_example_5.docx



For the lossless transmission line (TL) circuits above: $f = 1$ GHz, $v_p = 3 \times 10^8$ m/s, $Z_0 = 50\ \Omega$, and the TL has length $l_{\text{tape}} = 63.6$ cm as measured by a tape measure. The wavelength is calculated to be $\lambda = u/f = 3 \times 10^8 / 1 \times 10^9 = 30$ cm.

Open Circuit Termination (known load)

For an open circuit, we know $z_{OC} = Z_{OC}/ Z_0 \to \infty$ and $\Gamma_{OC} = 1$. For the left hand circuit above, an input impedance of $Z_{\text{in},OC} = -j50\ \Omega$ is measured.

1) Normalize and plot open circuit termination input impedance

- Normalize $z_{\text{in},OC} = Z_{\text{in},OC} / Z_0 = (-j\,50) / 50 \Rightarrow$ **$z_{\text{in},OC} = -j1\ \Omega/\Omega$**.
- Plot $z_{\text{in},OC}$ on Smith chart by finding the intersection of the $r = 0$ circle (outer edge) with the $x = -1$ arc.

2) Find length of transmission line

- Use straight-edge to draw radial line from center of Smith chart through $z_{\text{in},OC}$ and outer rings of Smith chart. Where the radial line crosses the "WAVELENGTHS TOWARD LOAD" scale, read off 0.125.
- The $z_{OC} \to \infty$ point, on the right edge of the Smith chart, reads 0.25 on the "WAVELENGTHS TOWARD LOAD" scale. The distance toward the load from $z_{\text{in},OC}$ is then $l = (0.25\text{-}0.125)\,\lambda + n\lambda/2 = \mathbf{0.125\lambda} + n\lambda/2$.
- Using $\lambda = 30$ cm, the transmission line length must be $l = 3.75 + n15$ cm. When $n = 4$, $l = 3.75 + (4)15 \Rightarrow$ **$l = 63.75$ cm $= 2.125\lambda$**, quite close to $l_{\text{tape}} = 63.6$ cm.

Unknown Load Termination

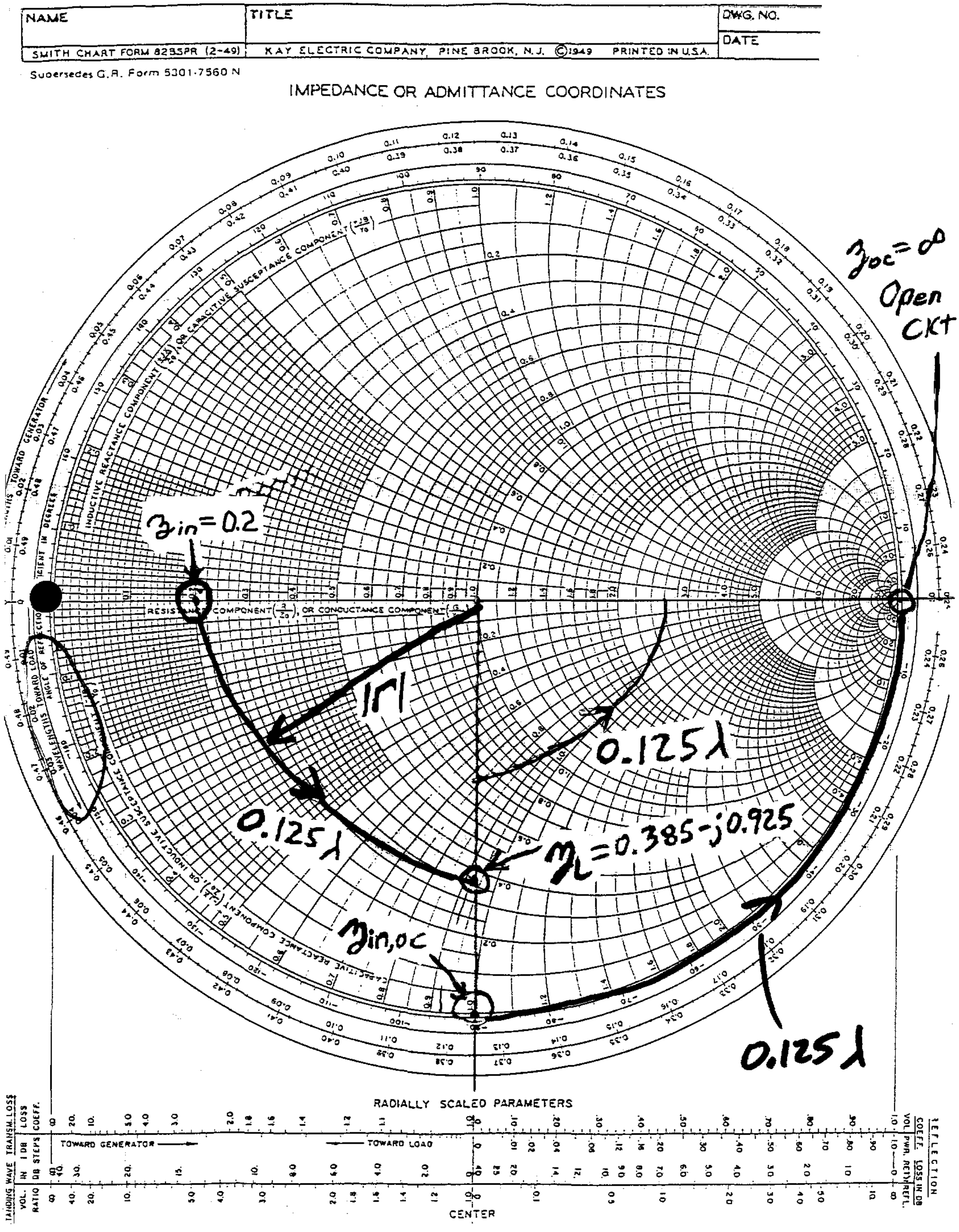
For the righthand circuit w/ unknown load, an input impedance of $Z_{\text{in}} = 10\ \Omega$ is measured.

1) Normalize and plot TL input impedance for unknown load

- Normalize $z_{\text{in}} = Z_{\text{in}} / Z_0 = 10 / 50 \Rightarrow$ **$\underline{z_{\text{in}} = 0.2\ \Omega/\Omega}$**.
- Plot z_{in} on Smith chart by finding the intersection of the $r = 0.2$ circle with the $x = 0$ arc/line (i.e., horizontal/real axis).

2) Find unknown load impedance

- Note that the horizontal axis of the Smith chart passes through z_{in} where the "WAVELENGTHS TOWARD LOAD" scale reads 0.
- Draw a radial line from the center of the Smith chart through 0.125 (i.e., the TL length w/out the extra 2λ) on the "WAVELENGTHS TOWARD LOAD" scale.
- Draw an arc, centered on Smith chart, from z_{in} to the radial line at 0.125 on "WAVELENGTHS TOWARD LOAD" scale.
- Read/interpolate value of normalized load resistance at intersection of arc and radial line as $\underline{r_L = 0.385}$.
- Read/interpolate value of normalized load reactance at intersection of arc and radial line as $\underline{x_L = -0.925}$.
- Put together to get <u>normalized</u> load impedance **$\underline{z_L = 0.385 - j0.925\ \Omega/\Omega}$**.
- Find load impedance by multiplying z_L by characteristic impedance Z_0 to get $Z_L = z_L Z_0 = (0.385 - j0.925)\ 50 \Rightarrow$ **$\underline{Z_L = 19.25 - j\ 46.25\ \Omega}$**.


NAME
TITLE
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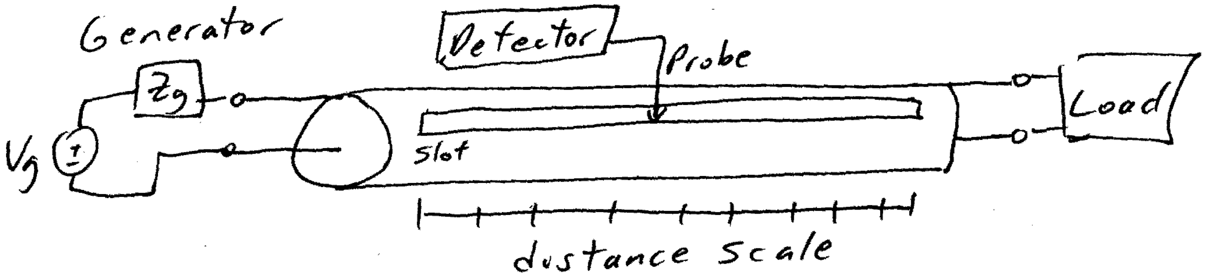
SMITH CHART FORM 82BSPR (2-49)
KAY ELECTRIC COMPANY, PINE BROOK, N.J. ©1949 PRINTED IN U.S.A.
Supersedes G.R. Form 5301-7560 N
IMPEDANCE OR ADMITTANCE COORDINATES
z_oc = ∞
Open CKt
z_in = 0.2
|Γ|
0.125λ
0.125λ
z_L = 0.385-j0.925
z_in,oc
0.125λ
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
CENTER

2.4 cont.

Method 2 (Slotted Line and Smith Chart)

"Slotted Line"? A section of TL with a thin slot cut to allow a probe to either measure voltage or electric field magnitude. With the advent of low cost VNAs, they're not widely used anymore.



Known: Z_o and f

1) Attach unknown load to end of slotted line. Move probe back and forth to measure either V_{max} + V_{min} or $|E_{max}|$ & $|E_{min}|$.

2) Calculate $SWR = \frac{V_{max}}{V_{min}} = \frac{|E_{max}|}{|E_{min}|}$. Use SWR scale at bottom of Smith Chart to set compass and draw circle of constant SWR (& $|\Gamma|$). $\Rightarrow z_L$ is somewhere on this circle!

3) Replace load with a short circuit ($z_{sc} = 0$) and move probe to a voltage minimum along the slotted line closest to the load. Record this location $l_{sc,min1}$. Moving toward the generator record the locations of voltage minima $l_{sc,min2}, \ldots$

2.4 cont.

4) From TL theory, we know that $\ell_{sc,min1}$, $\ell_{sc,min2}$, ... are $\lambda/2$ apart. Use this information to determine λ. For accuracy, it is best to take an average of multiple measurements.

5) Return probe to $\ell_{sc,min1}$ location, disconnect the short circuit, and re-attach the unknown load. Now, move the probe toward the generator to the first new voltage minima and record the location $\ell_{unknown,min}$.

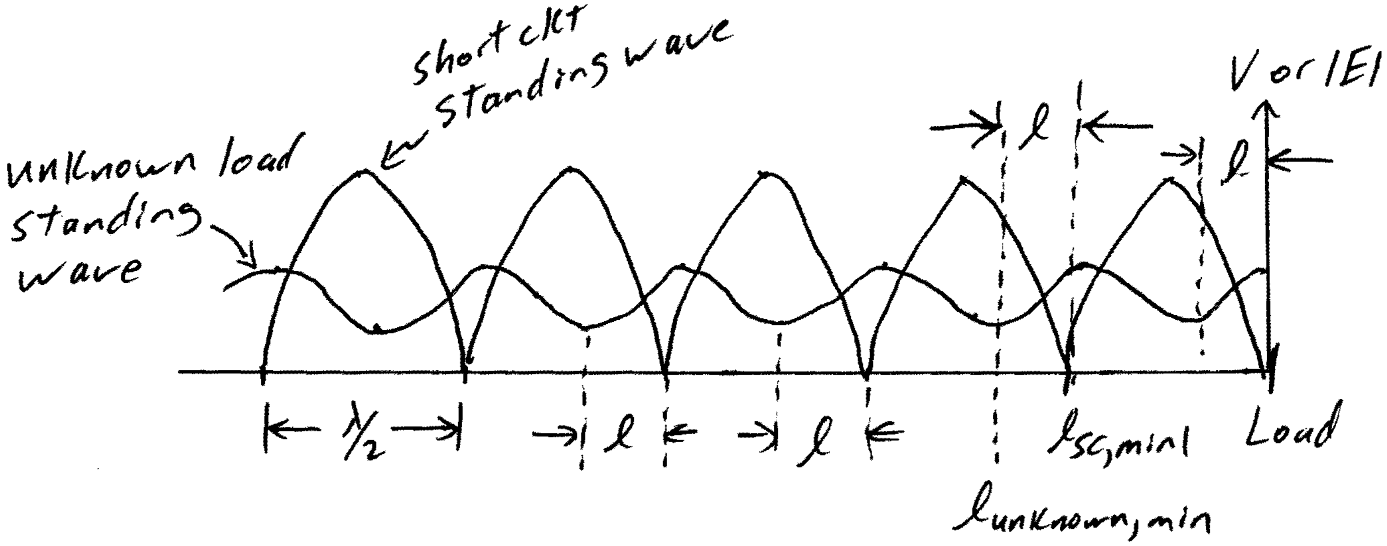
6) Calculate $\ell = |\ell_{sc,min1} - \ell_{unknown,min}|$ and ℓ/λ. This is the distance from the voltage minimum location (i.e., r_{min} on $|\Gamma|$ circle) to the unknown load z_L.

7) On Smith Chart, draw radial line at/through r_{min} and draw a radial line at ℓ/λ in the 'WAVELENGTHS TOWARD LOAD' direction from the first radial line.

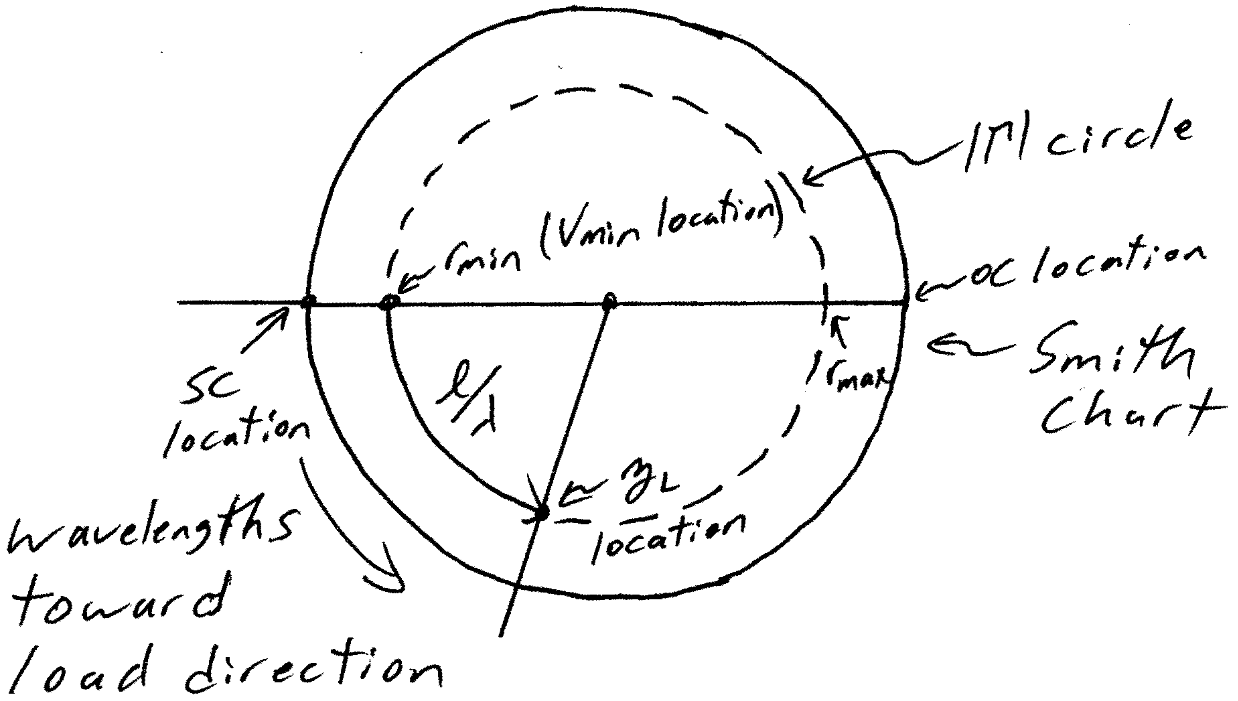
8) Read off r_L + x_L where the second radial line intersects the $|\Gamma|$ circle.

9) $z_L = r_L + j x_L$. Calculate $\underline{Z_L = Z_0 z_L}$.

2.4 cont.



Note how the spacing l repeats along the slotted line, same as the voltage minima repeating every $\lambda/2$.



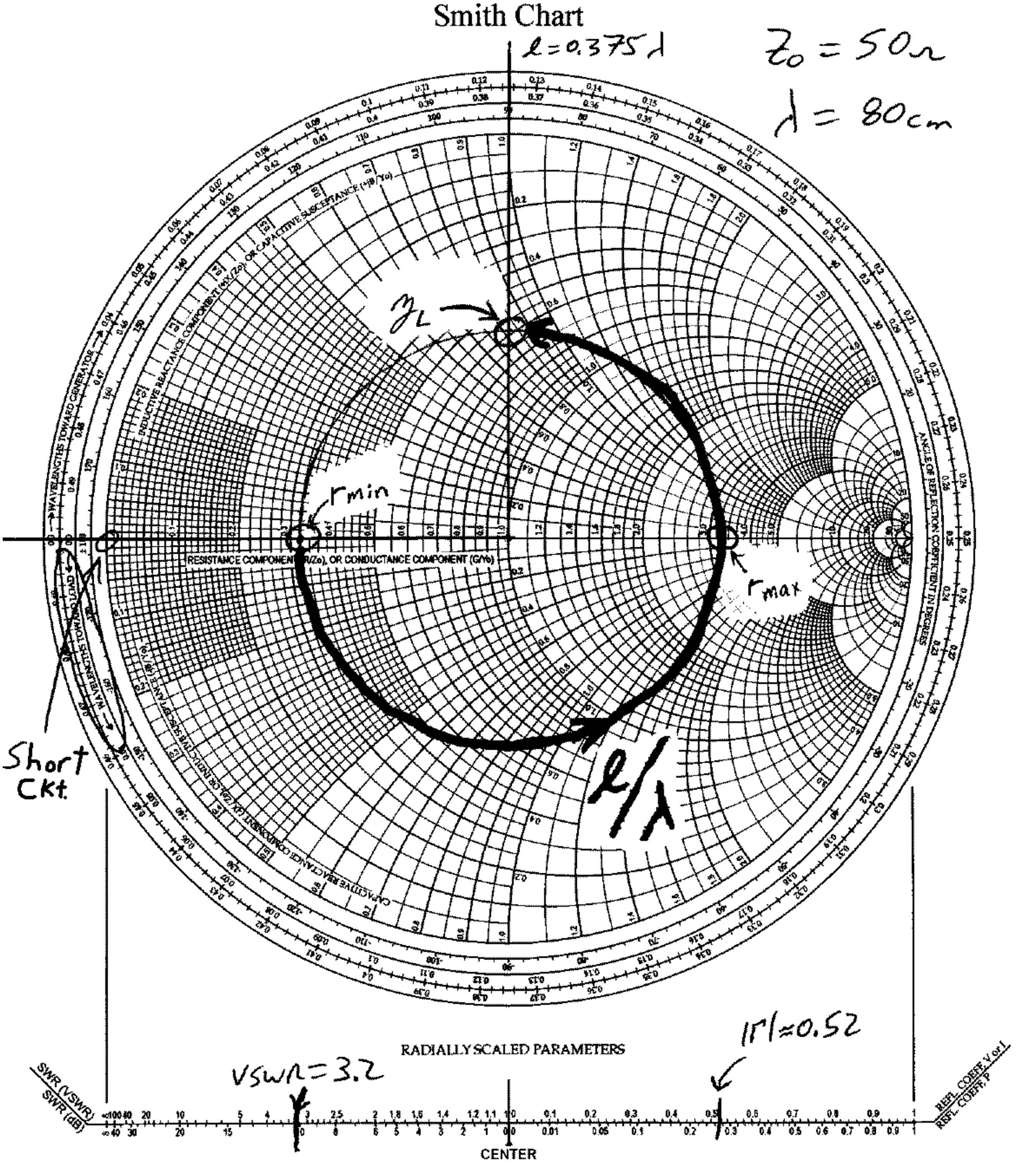
Example- ee481_581_Smith_chart_slotted_line.docx

We are using an air-dielectric 50 Ω slotted line to determine an unknown load.

1) Attach **unknown load** to end of slotted line. Moving probe back-n-forth, measure the voltage maxima $V_{\max}$ = -5 dBmV (multiple locations) and the voltage minima $V_{\min}$ = -15 dBmV (multiple locations).
2) Find **VSWR** and **magnitude of reflection coefficient** $|\Gamma|$ along slotted line.
   - Using $V_{\max} = -5$ dBmV $= 20\log_{10}(V_{\max}/1\text{ mV})$, we calculate the maximum voltage magnitude $V_{\max} = 10^{-5/20}$ (1 mV) ⇒ $\underline{\mathbf{V_{max} = 0.562\ mV}}$.
   - Using $V_{\min} = -15$ dBmV $= 20\log_{10}(V_{\max}/1\text{ mV})$, we calculate the minimum voltage magnitude $V_{\min} = 10^{-15/20}$ (1 mV) ⇒ $\underline{\mathbf{V_{min} = 0.178\ mV}}$.
   - By definition, the VSWR $= V_{\max} / V_{\min} = 10^{-5/20} / 10^{-15/20}$ ⇒ **VSWR = 3.162**.
   - Set compass using “SWR (VSWR)” scale at bottom left of Smith chart.
   - Use compass to mark “REFL. COEFF., V OR I” scale on bottom right of Smith chart. Read $\underline{\mathbf{|\Gamma| = 0.52}}$.
   - Using compass, draw a circle of $|\Gamma| = 0.52$ on Smith chart. We know that z_L is somewhere on this circle. Also, voltage minima $V_{\min}$ for the unknown load occur at the $r_{\min}$ point on the circle where it crosses the horizontal axis to left of origin. Read $r_{\min} = 0.31$ Ω/Ω.
3) Attach **short circuit** to end of slotted line. Measure adjacent voltage minima at location $l_1 = 90$ cm & $l_2 = 50$ cm along the slotted line.
4) Adjacent voltage minima are separated by half a wavelength.
   - Calculate $\lambda/2 = 90 - 50 = 40$ cm ⇒ $\underline{\mathbf{\lambda = 80\ cm}}$.
   - Bonus: frequency $f = c/\lambda = 3\times10^8/0.8 = 375\times10^6$ Hz ⇒ $\underline{\mathbf{f = 375\ MHz}}$.
5) Return probe to voltage minima to location $l_{sc,\min1} = l_1 = 90$ cm along the slotted line. This is the minimum closest to the load as my ruler measurements get larger toward the load. Re-attaching the unknown load, move **toward the generator** and measure voltage minima $V_{\min} = -15$ dBmV at $l_{\text{unknown},\min} = 60$ cm.
6) Calculate the distance toward the load from the $V_{\min}/r_{\min}$ point.
   - $l = |60 - 90| = 30$ cm, and $l/\lambda = 30/80$ ⇒ $\underline{\mathbf{l/\lambda = 0.375}}$.
7) The horizontal axis of the Smith chart goes through $r_{\min}$ and the short circuit points to the left of the origin. Draw a radial line from the center of the Smith chart through **0.375** on the “WAVELENGTHS TOWARD LOAD” scale.

8) Where the radial line at **0.375** on the "WAVELENGTHS TOWARD LOAD" scale intersects the |Γ| = 0.52 circle, read/interpolate values of normalized load resistance and reactance as $\underline{r_L = 0.58}$ and $\underline{x_L = 0.82}$.

9) Put $\underline{r_L = 0.58}$ and $\underline{x_L = 0.82}$ together to get
   - <u>Normalized</u> load impedance is $\underline{\mathbf{z_L = 0.58 + j0.82\ \Omega/\Omega}}$.
   - Load impedance is $Z_L = z_L Z_0 = (0.58 + j0.82)\,50 \Rightarrow \underline{\mathbf{Z_L = 29 + j41\ \Omega}}$.



2.7 Lossy Transmission Lines

What if our TL is long enough that losses can NOT be neglected or is operated at a high enough frequency that losses are appreciable?

Earlier (Section 2.1), we found:

Most accurate

Prop. constant ≡ $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$ $(1/m)$

$$= j\omega\sqrt{LC}\sqrt{1-j\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)-\frac{RG}{\omega^2 LC}}$$

$$= \alpha + j\beta = \text{atten constant} + j\,\text{phase constant}$$

Characteristic impedance ≡ $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ (Ω)

For low-loss TLs, we can make some approximations.

Low-loss? $R \ll \omega L$ and $G \ll \omega C$.

$$\gamma \simeq j\omega\sqrt{LC}\sqrt{1-j\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)} \qquad \sqrt{1+x} \simeq 1+x/2$$

$$\simeq j\omega\sqrt{LC}\left[1-j\,\tfrac{1}{2}\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)\right]$$

$$\simeq \tfrac{1}{2}\,\omega\sqrt{LC}\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)+j\omega\sqrt{LC}$$

$$\gamma \simeq \underbrace{\tfrac{1}{2}\left(\frac{R}{Z_0}+GZ_0\right)}_{\alpha}+j\underbrace{\omega\sqrt{LC}}_{\beta} \qquad \left(\frac{1}{m}\right)$$

&

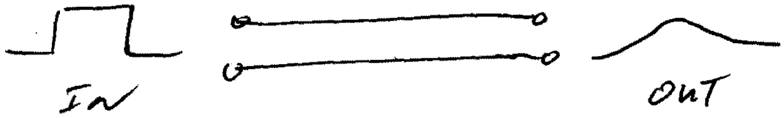
$$Z_0 \simeq \sqrt{L/C} \quad (\Omega)$$

Essentially, this is what TL manufacturers are assuming when they give $Z_0 = 50\Omega$ and $\alpha = 1\,\text{dB/km}$.

2.7 cont.

Distortionless TLs

In general, both α and β are dependent on the frequency and β is NOT linearly dependent for lossy TLs. Since phase velocity, $v_p = \omega/\beta$, this implies that signal consisting of more than a single frequency will suffer from dispersion (distortion) as some frequencies will travel at different speeds than others. and some will be attenuated more than others.



(Oliver Heaviside)

This led some clever engineer/scientist to develop the 'distortionless' TL where we require $\frac{R}{L} = \frac{G}{C}$.

Then, $\gamma = j\omega\sqrt{LC}\sqrt{1 - j2\frac{R}{\omega L} - \frac{R^2}{\omega^2 L^2}}$ Assume $R \ll \omega L$ & $\sqrt{1+x} = 1 + \frac{1}{2}x$

$$\simeq j\omega\sqrt{LC}\left(1 - j\frac{R}{\omega L}\right)$$

$$\simeq R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} = \underset{\overset{\uparrow}{\alpha}}{\frac{R}{Z_0}} + \underset{\overset{\uparrow}{\beta}}{j\omega\sqrt{LC}} \quad (m^{-1})$$

$$Z_0 = \sqrt{L/C}$$

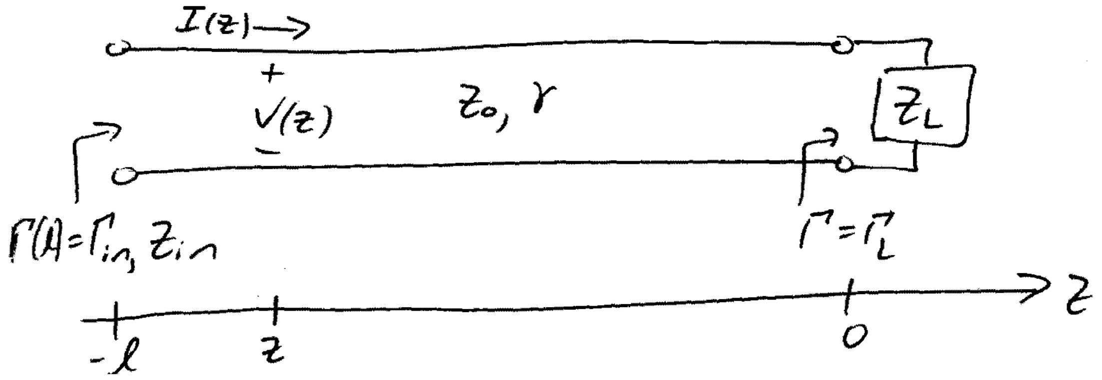
Now, $\alpha = \frac{R}{Z_0}$ is frequency independent!

$\beta = \omega\sqrt{LC} \Rightarrow v_p = \omega/\beta = \frac{1}{\sqrt{LC}}$ is also!

$\Rightarrow$ Not common these days w/ fiber optics.

2.7 cont.

Terminated Lossy TL



$$\Gamma = \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} \quad \text{where } Z_o = |Z_o| e^{j\theta_{Z_o}}$$

Γ @ $z=0$

$$V(z) = V_o^+\left[e^{-\gamma z} + \Gamma e^{\gamma z}\right] = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

$$I(z) = \frac{V_o^+}{Z_o}\left[e^{-\gamma z} - \Gamma e^{\gamma z}\right] = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z}$$

$$\Gamma(\ell) = \Gamma_{in} = \Gamma e^{-2\gamma\ell} = \Gamma_L e^{-2\alpha\ell} e^{-j2\beta\ell}$$

Note: $|\Gamma(\ell)| = |\Gamma_L| e^{-2\alpha\ell}$ gets smaller!

$$Z_{in}(\ell) = Z_o \frac{1+\Gamma_{in}}{1-\Gamma_{in}} = Z_o\left[\frac{Z_L + Z_o \tanh(\gamma\ell)}{Z_o + Z_L \tanh(\gamma\ell)}\right]$$

$$P_{in} = \tfrac{1}{2}\,\mathrm{Re}\left\{V(-\ell)\, I^*(-\ell)\right\}$$ ← actually easiest

$$P_{in} = \frac{|V_o^+|^2}{2|Z_o|}\left[\cos\theta_{Z_o}\, e^{2\alpha\ell}\left(1 - \overset{|\Gamma_{in}|^2}{|\Gamma(\ell)|^2}\right) - \sin\theta_{Z_o}\, 2\,\mathrm{Im}\left(\Gamma e^{-j2\beta\ell}\right)\right]$$

For low-loss TLs, θ_{Z_o} is small so $Z_o \simeq |Z_o|$

$$P_{in} \cong \frac{|V_o^+|^2}{2Z_o} e^{2\alpha\ell}\left[1 - \underset{\rightarrow |\Gamma| e^{-4\alpha\ell}}{|\Gamma(\ell)|^2}\right] \quad (2.92)$$

2.7 cont.

actually easiest

$$P_L = \tfrac{1}{2}\,\mathrm{Re}\{V(0)I^*(0)\} = \frac{|V_0^+|^2}{2|Z_0|}\left[\cos\theta_{Z_0}\left(1-\overset{|\Gamma_L|^2}{|\Gamma|^2}\right) - \sin\theta_{Z_0}\,2\,\mathrm{Im}(\Gamma)\right]$$

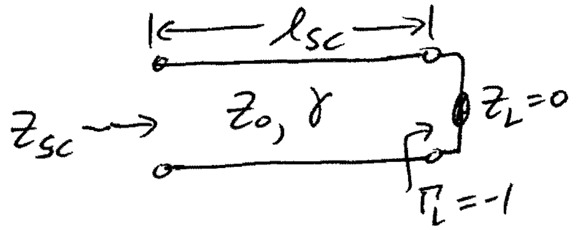
Again, for a low-loss TL, θ_{Z_0} is small and we get

$$P_L = \frac{|V_0^+|^2}{2Z_0}\left(1-|\Gamma|^2\right) \qquad (2.93)$$

Note that $P_{in} > P_L$. The difference is the power lost in the TL. For a low-loss TL,

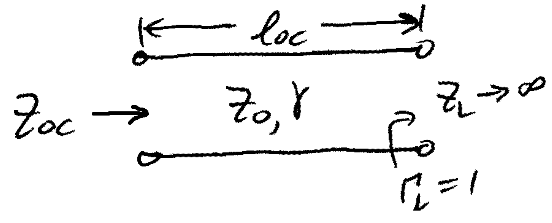
$$(2.94) \quad P_{loss} = P_{in} - P_L = \frac{|V_0^+|^2}{2Z_0}\left[\underbrace{\left(e^{2\alpha l}-1\right)}_{\text{Incident}} + \underbrace{|\Gamma|^2\left(1-e^{-2\alpha l}\right)}_{\text{Reflected}}\right]$$

What happens to open & short circuit stubs?



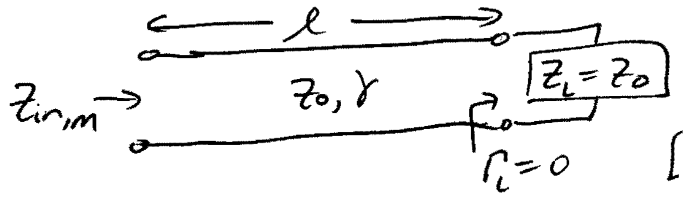
$Z_{sc} = Z_0 \tanh(\gamma l_{sc})$

← will have a real component



$$Z_{oc} = Z_0 \coth(\gamma l_{oc}) = \frac{Z_0}{\tanh(\gamma l_{oc})}$$

Matched load



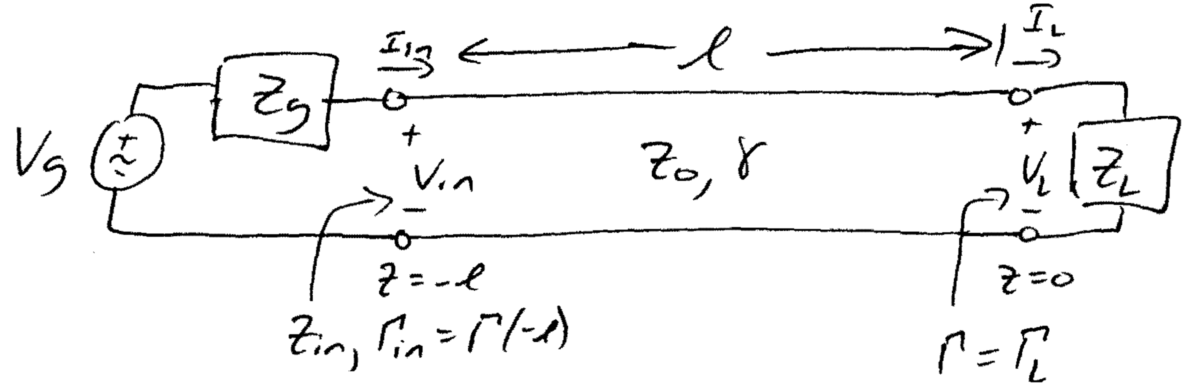
$Z_{in,m} = Z_0$ ← complex

length independent

[For lossy TLs, $Z_{in} \to Z_0$ as $l \to \infty$]

2.7 cont.

Lossy TL circuit



By circuit theory,

$$V_{in} = V_g\left(\frac{Z_{in}}{Z_g + Z_{in}}\right)$$

$$I_{in} = \frac{V_g}{Z_g + Z_{in}} \Rightarrow P_{in} = 1/2 \,\mathrm{Re}\{V_{in} I_{in}^*\}$$

Compare this $V_{in} = V(-\ell)$ value to the phasor voltage expression when $z = -\ell$

$$V_{in} = V_g\left(\frac{Z_{in}}{Z_g + Z_{in}}\right) = V_0^+\left(e^{\gamma\ell} + \Gamma_L e^{-\gamma\ell}\right) = V(-\ell)$$

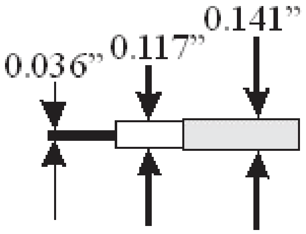
$$\underline{V_0^+ = \frac{V_{in}}{e^{\gamma\ell} + \Gamma_L e^{-\gamma\ell}} = V_g\left(\frac{Z_{in}}{Z_g + Z_{in}}\right)\frac{1}{e^{\gamma\ell} + \Gamma_L e^{-\gamma\ell}}}$$

Note: For a known TL and load as well as generator, we can easily find V_0^+. With V_0^+, we can easily evaluate $V(z)$ and $I(z)$ as well as $P(z)$ for all $-\ell \le z \le 0$.

EE 481/581 Lossy RG 402 Transmission Line Example

For this example we will be considering a lossy TL circuit built using 10 cm of RG 402 Type 0.141 semi-rigid coaxial cable with a solid copper shield/outer conductor, a silver-plated copper-clad steel inner conductor, and PTFE (Teflon) insulation. The physical dimensions of the coax are shown in the figure. At 10 GHz, it is specified to have a phase velocity of 69.5% of light and an attenuation of 147.64 dB/100 m. We will assume it is a low-loss TL.

$f := 10 \cdot 10^{9}$ Hz $\qquad \omega := 2 \cdot \pi \cdot f$



$\varepsilon 0 := 8.8541878 \cdot 10^{-12}$ F/m

$\mu 0 := 4 \cdot \pi \cdot 10^{-7}$ H/m

$c := 2.99792458 \cdot 10^{8}$ m/s

Calculate/define lossy transmission line parameters

$\alpha dB := 1.4764$ dB/m $\qquad \alpha := \frac{\alpha dB}{20 \cdot \log(e)} \qquad \boxed{\alpha = 0.16998}$ Np/m, atten constant

Find relative permittivity of the PTFE by using phase velocity relation that $v_p = c/\text{sqrt}(\varepsilon_r)$. For Teflon, we expect a number close to 2.1.

$\varepsilon r\_PTFE := \frac{1}{0.695^{2}} \qquad \boxed{\varepsilon r\_PTFE = 2.07029}$

$vp := 0.695 \cdot c \qquad \boxed{vp = 2.08356 \times 10^{8}}$ m/s

$\beta := \frac{\omega}{vp} \qquad \boxed{\beta = 301.56043}$ rad/m, phase constant

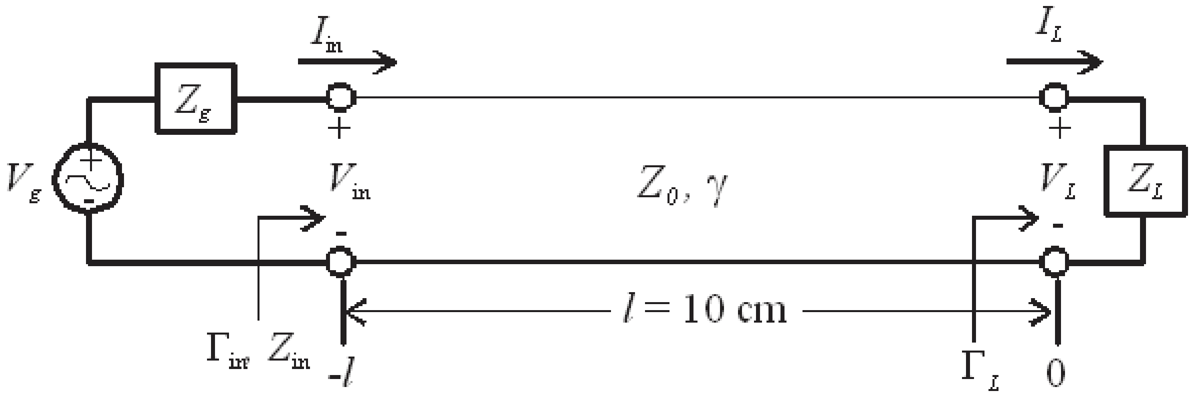
$\gamma := \alpha + j \cdot \beta \qquad \boxed{\gamma = 0.16998 + 301.56043i}$ 1/m, propag. constant

$\lambda := \frac{vp}{f} \qquad \boxed{\lambda = 0.02084}$ m, wavelength

$Z0 := 50 \qquad \boxed{Z0 = 50}$ Ω, characteristic impedance

Lossy transmission line circuit

$Zg := 50 - j \cdot 10 \quad \Omega \qquad ZL := 60 + j \cdot 50 \quad \Omega \qquad l := 0.1 \quad m \qquad Vg := 100 \quad V$



$$\Gamma L := \frac{ZL - Z0}{ZL + Z0}$$

$$\Gamma L = 0.2466 + 0.3425i \qquad \boxed{|\Gamma L| = 0.422} \qquad \boxed{\arg(\Gamma L) \cdot \frac{180}{\pi} = 54.2461} \quad \text{deg}$$

$$\Gamma in := \Gamma L \cdot e^{-2 \cdot \gamma \cdot l} \qquad Zin := Z0 \cdot \frac{(1 + \Gamma in)}{(1 - \Gamma in)}$$

$$\Gamma in = -0.3865 - 0.1302i \qquad \boxed{|\Gamma in| = 0.4079} \qquad \boxed{\arg(\Gamma in) \cdot \frac{180}{\pi} = -161.3819} \quad \text{deg}$$

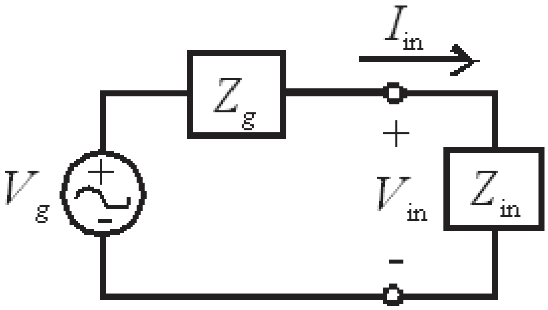
$$\boxed{Zin = 21.491 - 6.7144i} \quad \Omega \qquad \boxed{|Zin| = 22.5155} \quad \Omega \qquad \boxed{\arg(Zin) \cdot \frac{180}{\pi} = -17.35} \quad \text{deg}$$

OR

$$Zin := Z0 \cdot \frac{(ZL + Z0 \cdot \tanh(\gamma \cdot l))}{Z0 + ZL \cdot \tanh(\gamma \cdot l)} \qquad \boxed{Zin = 21.491 - 6.7144i} \quad \Omega$$

SAME!

Now, we can draw the equivalent circuit seen by the generator.



Use simple circuit theory to calculate the input voltage, current, and power.

$\mathrm{Iin} := \frac{\mathrm{Vg}}{\mathrm{Zg} + \mathrm{Zin}}$ $\boxed{|\mathrm{Iin}| = 1.362}$ A $\boxed{\arg(\mathrm{Iin})\cdot\frac{180}{\pi} = 13.1592}$ deg

$\mathrm{Vin} := \frac{\mathrm{Vg}\cdot\mathrm{Zin}}{\mathrm{Zg} + \mathrm{Zin}}$ $\boxed{|\mathrm{Vin}| = 30.6671}$ V $\boxed{\arg(\mathrm{Vin})\cdot\frac{180}{\pi} = -4.191}$ deg

$\mathrm{V0p} := \frac{\mathrm{Vin}}{e^{\gamma\cdot 1} + \Gamma L\cdot e^{-\gamma\cdot 1}}$ $\boxed{|\mathrm{V0p}| = 48.0771}$ V $\boxed{\arg(\mathrm{V0p})\cdot\frac{180}{\pi} = 79.9798}$ deg

$\mathrm{Pin} := 0.5\cdot\mathrm{Re}\left(\mathrm{Vin}\cdot\overline{\mathrm{Iin}}\right)$ $\boxed{\mathrm{Pin} = 19.93476}$ W **Power into transmission line.**

$\mathrm{Pin2} := \frac{(|\mathrm{V0p}|)^2}{2\cdot\mathrm{Z0}}\cdot e^{2\cdot\alpha\cdot 1}\cdot\left[1 - (|\Gamma\mathrm{in}|)^2\right]$ $\boxed{\mathrm{Pin2} = 19.93476}$ W **Same!**

Next, we'll find how much power makes it to the load and how much is lost.

$\mathrm{VL} := \mathrm{V0p}\cdot\left(e^{-\gamma\cdot 0} + \Gamma L\cdot e^{\gamma\cdot 0}\right)$ $\boxed{|\mathrm{VL}| = 62.152}$ V $\boxed{\arg(\mathrm{VL})\cdot\frac{180}{\pi} = 95.341}$ deg

$\mathrm{IL} := \frac{\mathrm{VL}}{\mathrm{ZL}}$ $\boxed{|\mathrm{IL}| = 0.7958}$ A $\boxed{\arg(\mathrm{IL})\cdot\frac{180}{\pi} = 55.536}$ deg

$\mathrm{PL} := 0.5\cdot\mathrm{Re}\left(\mathrm{VL}\cdot\overline{\mathrm{IL}}\right)$ $\boxed{\mathrm{PL} = 18.9979}$ W **Power delivered to load.**

$\mathrm{PL2} := \frac{(|\mathrm{V0p}|)^2}{2\cdot\mathrm{Z0}}\cdot\left[1 - (|\Gamma L|)^2\right]$ $\boxed{\mathrm{PL2} = 18.9979}$ W **Same!**

$\mathrm{Ploss} := \mathrm{Pin} - \mathrm{PL}$ $\boxed{\mathrm{Ploss} = 0.93686}$ W **Power lost in transmission line.**

$\mathrm{Ploss\_fwd} := \frac{(|\mathrm{V0p}|)^2}{2\cdot\mathrm{Z0}}\cdot\left(e^{2\cdot\alpha\cdot 1} - 1\right)$ $\boxed{\mathrm{Ploss\_fwd} = 0.79928}$ W

$\mathrm{Ploss\_bwd} := \frac{(|\mathrm{V0p}|)^2}{2\cdot\mathrm{Z0}}\cdot\left[(|\Gamma L|)^2\cdot\left(1 - e^{-2\cdot\alpha\cdot 1}\right)\right]$ $\boxed{\mathrm{Ploss\_bwd} = 0.13758}$ W

$\mathrm{Ploss2} := \mathrm{Ploss\_fwd} + \mathrm{Ploss\_bwd}$ $\boxed{\mathrm{Ploss2} = 0.93686}$ W, Same!

$$n := 0 .. 1000 \qquad z_n := \frac{n}{1000}\cdot -l \qquad z\lambda_n := \frac{z_n}{\lambda}$$

$$V_n := V0p\cdot\left(e^{-\gamma\cdot z_n} + \Gamma L\cdot e^{\gamma\cdot z_n}\right) \qquad I_n := \frac{V0p}{Z0}\cdot\left(e^{-\gamma\cdot z_n} - \Gamma L\cdot e^{\gamma\cdot z_n}\right) \qquad P_n := 0.5\cdot \mathrm{Re}\left(V_n\cdot\overline{I_n}\right)$$