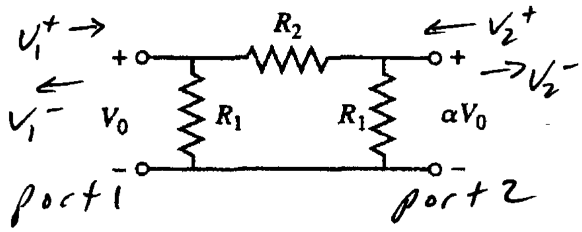
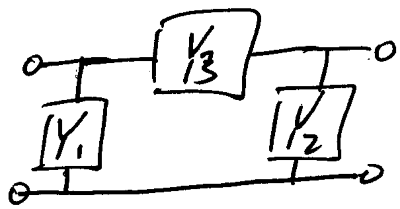
7.7 Consider the π resistive attenuator circuit shown below. If the input and output are matched to Z_0, and the ratio of output voltage to input voltage is α, derive the design equations for R_1 and R_2 for each circuit. If $Z_0 = 50\ \Omega$, compute R_1 and R_2 for 3, 10, and 20 dB attenuators of each type.





- Do π resistive attenuator only.

Per Table 4.1, the ABCD parameters for an admittance π-network are:

$$A = 1 + Y_2/Y_3 = 1 + \frac{1/R_1}{1/R_2} = 1 + R_2/R_1$$

$$B = 1/Y_3 = 1/(1/R_2) = R_2$$

$$C = Y_1 + Y_2 + \frac{Y_1 Y_2}{Y_3} = \frac{1}{R_1} + \frac{1}{R_1} + \frac{1/R_1\,(1/R_1)}{1/R_2} = \frac{2}{R_1} + \frac{R_2}{R_1^2}$$

$$D = 1 + Y_1/Y_3 = 1 + \frac{1/R_1}{1/R_2} = 1 + R_2/R_1$$

For the input to be matched to $Z_0 \Rightarrow S_{11} = 0$.

Per Table 4.2, $S_{11} = \dfrac{A + B/Z_0 - CZ_0 - D}{A + B/Z_0 + CZ_0 + D}$. The only way $S_{11} = 0$ is if the numerator is zero:

$$\left(1 + \cancel{R_2/R_1}\right) + R_2/Z_0 - \left(\frac{2}{R_1} + \frac{R_2}{R_1^2}\right) Z_0 - \left(1 + \cancel{R_2/R_1}\right) = 0$$

$$\frac{R_2}{Z_0} = \left(\frac{2}{R_1} + \frac{R_2}{R_1^2}\right) Z_0 \quad (1)$$

$\times Z_0 R_1^2$

$$R_2 R_1^2 = 2R_1 Z_0^2 + R_2 Z_0^2 \Rightarrow R_2\left(R_1^2 - Z_0^2\right) = 2R_1 Z_0^2$$

$$R_2 = \frac{2R_1 Z_0^2}{R_1^2 - Z_0^2} = \frac{2R_1 Z_0^2}{(R_1 + Z_0)(R_1 - Z_0)} \quad (2)$$

Per (4.41), $S_{12} = \left.\dfrac{V_1^-}{V_2^+}\right|_{V_1^+=0}$ + $S_{21} = \left.\dfrac{V_2^-}{V_1^+}\right|_{V_2^+=0}$

By symmetry, $S_{12} = S_{21} = \frac{\alpha V_0}{V_0} = \alpha$ since

$V_2 = V_2^- = \alpha V_0$ when $V_2^+ = 0$

Per Table 4.2, $S_{21} = \frac{2}{A + B/Z_0 + C Z_0 + D} = \alpha$

$$\frac{2}{\alpha} = (1 + R_2/R_1) + \frac{R_2}{Z_0} + \underbrace{Z_0\left(\frac{2}{R_1} + \frac{R_2}{R_1^2}\right)}_{①} + (1 + R_2/R_1)$$

$$= 2 + \frac{2R_2}{R_1} + \frac{R_2}{Z_0} + \frac{R_2}{Z_0} \quad ①$$

$$\frac{1}{\alpha} = 1 + R_2/R_1 + R_2/Z_0 \xrightarrow{\times R_1 Z_0} \frac{R_1 Z_0}{\alpha} = R_1 Z_0 + R_2 Z_0 + R_2 R_1$$

$$R_1 Z_0\left(\frac{1}{\alpha} - 1\right) = \underbrace{R_2(R_1 + Z_0)}_{②} = \frac{2R_1 Z_0^2}{\cancel{(R_1 + Z_0)}(R_1 - Z_0)} \cancel{(R_1 + Z_0)}$$

$$\frac{1}{\alpha} - 1 = \frac{2Z_0}{R_1 - Z_0} \xrightarrow{\times \alpha(R_1 - Z_0)} (R_1 - Z_0)(1 - \alpha) = 2Z_0\alpha$$

$$R_1(1 - \alpha) = 2Z_0\alpha + Z_0(1 - \alpha) = Z_0(1 + \alpha)$$

$$\underline{\underline{R_1 = Z_0 \frac{1 + \alpha}{1 - \alpha}}}$$

Substitute for R_1 in eq'n ②

$$R_2 = \frac{2\left[Z_0 \frac{1+\alpha}{1-\alpha}\right] Z_0^2}{Z_0^2 \frac{(1+\alpha)^2}{(1-\alpha)^2} - Z_0^2} = \frac{2Z_0(1+\alpha)}{\frac{(1+\alpha)^2}{1-\alpha} - (1-\alpha)}$$

$$= \frac{2Z_0(1+\alpha)(1-\alpha)}{(1+\alpha)^2 - (1-\alpha)^2} = \frac{2Z_0(1-\alpha^2)}{1 + 2\alpha + \alpha^2 - (1 - 2\alpha + \alpha^2)}$$

$$\underline{\underline{R_2 = Z_0 \frac{1-\alpha^2}{2\alpha}}}$$

Per (2.52), $IL = -20 \log|T| = -20 \log \alpha$

$\hookrightarrow \alpha = 10^{-IL/20}$

$\boxed{IL = 3\,dB} \Rightarrow \alpha = 10^{-3/20} = 0.707946$

$$R_1 = 50\,\frac{1+0.70795}{1-0.70795} \Rightarrow \underline{\underline{R_1 = 292.4\,\Omega}}$$

$$R_2 = 50\,\frac{1-0.70795^2}{2(0.70795)} \Rightarrow \underline{\underline{R_2 = 17.6\,\Omega}}$$

$\boxed{IL = 10\,dB} \Rightarrow \alpha = 10^{-10/20} = 0.3162278$

$$R_1 = 50\,\frac{1+0.31623}{1-0.31623} \Rightarrow \underline{\underline{R_1 = 96.25\,\Omega}}$$

$$R_2 = 50\,\frac{1-0.31623^2}{2(0.31623)} \Rightarrow \underline{\underline{R_2 = 71.15\,\Omega}}$$

$\boxed{IL = 20\,dB} \Rightarrow \alpha = 10^{-\frac{20}{20}} = 0.1$

$$R_1 = 50\,\frac{1+0.1}{1-0.1} \Rightarrow \underline{\underline{R_1 = 61.\overline{11}\,\Omega}}$$

$$R_2 = 50\,\frac{1-0.1^2}{2(0.1)} \Rightarrow \underline{\underline{R_2 = 247.5\,\Omega}}$$