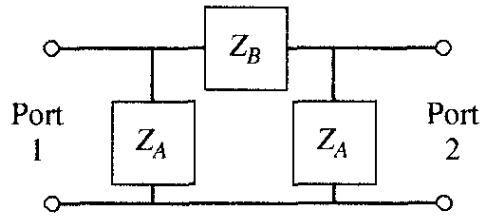
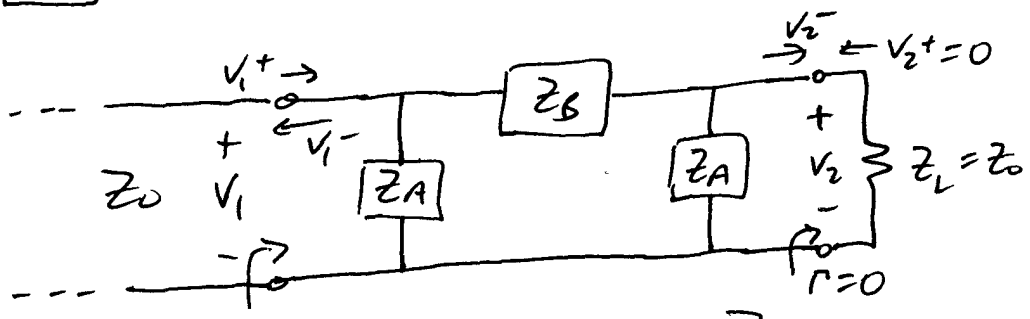
Find [*S*] for the pi-network of problem 4.7a.



(a)

Per (4.41), $S_{ij} = \left.\dfrac{V_i^-}{V_j^+}\right|_{V_k^+ = 0 \text{ for all } k \neq j}$

S_{11} ⇒ Match port 2 w/ $Z_L = Z_0$ ⇒ $V_2^+ = 0$



$$Z_{in} = Z_A \,//\, [Z_B + Z_A \,//\, Z_0]$$

$$= Z_A \,//\, \left[Z_B + \frac{Z_A Z_0}{Z_A + Z_0}\right] = Z_A \,//\, \left(\frac{Z_B(Z_A+Z_0) + Z_A Z_0}{Z_A + Z_0}\right)$$

$$= \frac{Z_A\left(\dfrac{Z_A Z_B + Z_B Z_0 + Z_A Z_0}{Z_A + Z_0}\right)}{Z_A + \left(\dfrac{Z_A Z_B + Z_B Z_0 + Z_A Z_0}{Z_A + Z_0}\right)} = \frac{Z_A^2 Z_B + Z_A Z_B Z_0 + Z_A^2 Z_0}{Z_A(Z_A+Z_0) + Z_A Z_B + Z_B Z_0 + Z_A Z_0}$$

$$= \frac{Z_A^2 Z_B + Z_A Z_B Z_0 + Z_A^2 Z_0}{Z_A^2 + Z_A Z_B + Z_B Z_0 + 2 Z_A Z_0}$$

$$S_{11} = \Gamma_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{\left(\dfrac{Z_A^2 Z_B + Z_A Z_B Z_0 + Z_A^2 Z_0}{Z_A^2 + Z_A Z_B + Z_B Z_0 + 2 Z_A Z_0}\right) - Z_0}{\left(\dfrac{Z_A^2 Z_B + Z_A Z_B Z_0 + Z_A^2 Z_0}{Z_A^2 + Z_A Z_B + Z_B Z_0 + 2 Z_A Z_0}\right) + Z_0}$$

$$= \frac{(Z_A^2 Z_B + Z_A Z_B Z_0 + Z_A^2 Z_0) - Z_0(Z_A^2 + Z_A Z_B + Z_B Z_0 + 2 Z_A Z_0)}{(Z_A^2 Z_B + Z_A Z_B Z_0 + Z_A^2 Z_0) + Z_0(Z_A^2 + Z_A Z_B + Z_B Z_0 + 2 Z_A Z_0)}$$

$$S_{11} = \frac{z_A^2 z_B - z_B z_0^2 - 2 z_A z_0^2}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + z_B z_0^2 + 2 z_A z_0^2} = S_{22} \quad \text{(Symmetry)}$$

S_{21} Same circuit. Find $V_2^- = V_2$ (voltage across z_0) in terms of V_1^+.

By voltage division $V_2 = V_2^- = V_1 \left(\frac{z_A /\!/ z_0}{z_B + z_A /\!/ z_0} \right) = V_1 \left(\frac{\frac{z_A z_0}{z_A + z_0}}{z_B + \frac{z_A z_0}{z_A + z_0}} \right)$

$$V_2^- = V_1^+ (1 + S_{11}) \left(\frac{z_A z_0}{z_B (z_A + z_0) + z_A z_0} \right)$$

$$S_{21} = \left. \frac{V_2^-}{V_1^+} \right|_{V_1^+ = 0} = \left[1 + \frac{z_A^2 z_B - z_B z_0^2 - 2 z_A z_0^2}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + z_B z_0^2 + 2 z_A z_0^2} \right] \left(\frac{z_A z_0}{z_A z_B + z_A z_0 + z_B z_0} \right)$$

$$= \left[\frac{\left(z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + z_B z_0^2 + 2 z_A z_0^2 \right) + \left(z_A^2 z_B - z_B z_0^2 - 2 z_A z_0^2 \right)}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + z_B z_0^2 + 2 z_A z_0^2} \right] \left(\frac{z_A z_0}{z_A z_B + z_A z_0 + z_B z_0} \right)$$

$$S_{21} = \left[\frac{2 z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + 2 z_A z_0^2 + z_B z_0^2} \right] \left(\frac{z_A z_0}{z_A z_B + z_A z_0 + z_B z_0} \right) = S_{12} \quad \text{(Symmetry)}$$

$$[S] = \begin{bmatrix} \frac{z_A^2 z_B - z_B z_0^2 - 2 z_A z_0^2}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + z_B z_0^2 + 2 z_A z_0^2} & \left[\frac{2 z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + 2 z_A z_0^2 + z_B z_0^2} \right] \left(\frac{z_A z_0}{z_A z_B + z_A z_0 + z_B z_0} \right) \\ \left[\frac{2 z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + 2 z_A z_0^2 + z_B z_0^2} \right] \left(\frac{z_A z_0}{z_A z_B + z_A z_0 + z_B z_0} \right) & \frac{z_A^2 z_B - z_B z_0^2 - 2 z_A z_0^2}{z_A^2 z_B + 2 z_A z_B z_0 + 2 z_A^2 z_0 + z_B z_0^2 + 2 z_A z_0^2} \end{bmatrix}$$