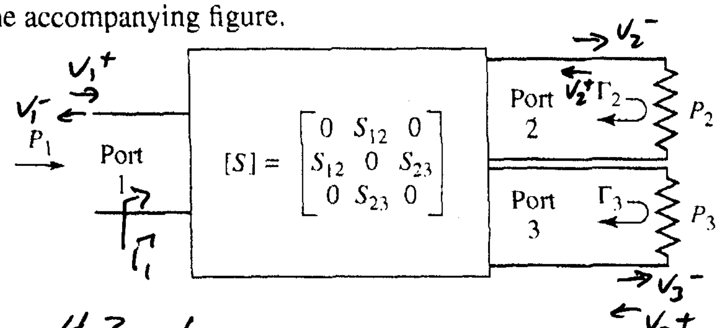
4.28 Use signal flow graphs to find the power ratios P_2/P_1 and P_3/P_1 for the mismatched three-port network shown in the accompanying figure.



Per the section 4.3 discussion on power waves, e.g., (4.59), and assuming each port is connected w/ characteristic impedance Z_0, we can write

$$P_1 = \frac{|V_1^+|^2}{2Z_0} - \frac{|V_1^-|^2}{2Z_0} = \frac{1}{2Z_0}\left[|V_1^+|^2 - |V_1^-|^2\right] = \frac{|V_1^+|^2}{2Z_0}\left(1-|\Gamma_1|^2\right)$$

$$P_2 = \frac{|V_2^-|^2}{2Z_0} - \frac{|V_2^+|^2}{2Z_0} = \frac{|V_2^-|^2}{2Z_0}\left(1-|\Gamma_2|^2\right)$$

$$P_3 = \frac{|V_3^-|^2}{2Z_0} - \frac{|V_3^+|^2}{2Z_0} = \frac{|V_3^-|^2}{2Z_0}\left(1-|\Gamma_3|^2\right)$$

where $\Gamma_1 = \dfrac{V_1^-}{V_1^+}$, $\Gamma_2 = \dfrac{V_2^+}{V_2^-}$, and $\Gamma_3 = \dfrac{V_3^+}{V_3^-}$. For the solution, we will assume Γ_2 + Γ_3 are known.

The power ratios are then:

$$\frac{P_2}{P_1} = \frac{\frac{|V_2^-|^2}{2Z_0}\left(1-|\Gamma_2|^2\right)}{\frac{|V_1^+|^2}{2Z_0}\left(1-|\Gamma_1|^2\right)} = \left|\frac{V_2^-}{V_1^+}\right|^2 \left(\frac{1-|\Gamma_2|^2}{1-|\Gamma_1|^2}\right)$$

and

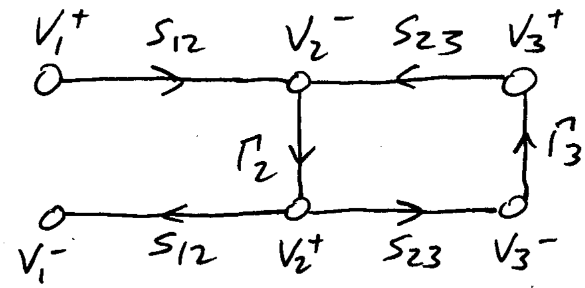
$$\frac{P_3}{P_1} = \frac{\frac{|V_3^-|^2}{2Z_0}\left(1-|\Gamma_3|^2\right)}{\frac{|V_1^+|^2}{2Z_0}\left(1-|\Gamma_1|^2\right)} = \left|\frac{V_3^-}{V_1^+}\right|^2 \frac{\left(1-|\Gamma_3|^2\right)}{\left(1-|\Gamma_1|^2\right)}$$

Using SFG techniques, we need to determine

$$\Gamma_1 = \frac{V_1^-}{V_1^+},\ \frac{V_2^-}{V_1^+},\ \text{and}\ \frac{V_3^-}{V_1^+}.$$

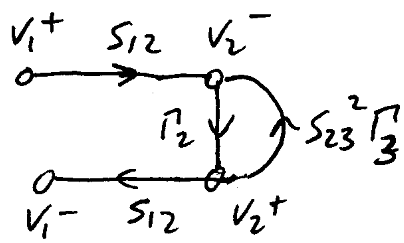
To build the SFG, use $V_2^+ = \Gamma_2 V_2^-$, $V_3^+ = \Gamma_3 V_3^-$, and

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \end{bmatrix} = \begin{bmatrix} 0 & S_{12} & 0 \\ S_{12} & 0 & S_{23} \\ 0 & S_{23} & 0 \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \end{bmatrix} \Rightarrow \begin{aligned} V_1^- &= S_{12} V_2^+ \\ V_2^- &= S_{12} V_1^+ + S_{23} V_3^+ \\ V_3^- &= S_{23} V_2^+ \end{aligned}$$

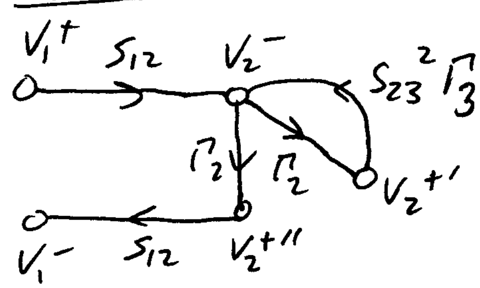


$$\boxed{\text{Find } \Gamma_1 = \frac{V_1^-}{V_1^+} \text{ and } \frac{V_2^-}{V_1^+}}$$

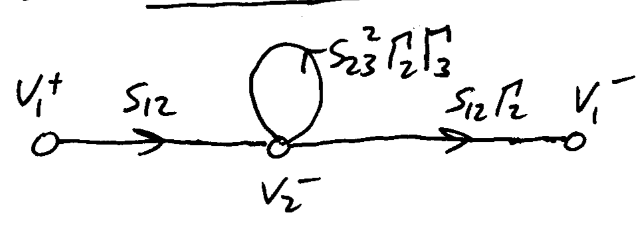
Step 1 Apply series rule to eliminate V_3^- & V_3^+



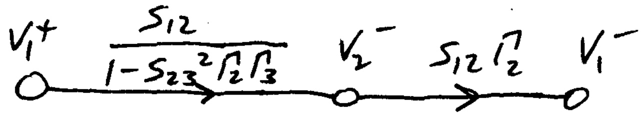
Step 2 Split node V_2^+



Step 3 Use series rule twice to eliminate $V_2^{+\prime}$ & $V_2^{+\prime\prime}$



Step 4 Use self-loop rule



We can now write $V_2^- = V_1^+ \dfrac{S_{12}}{1-S_{23}^2\Gamma_2\Gamma_3}$

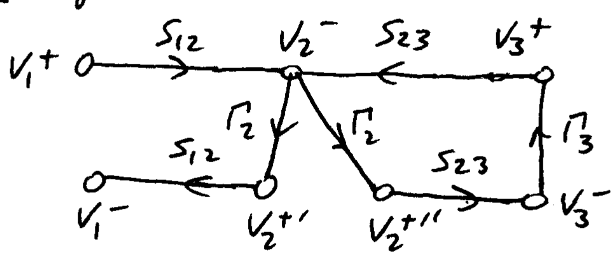
$$\frac{V_2^-}{V_1^+} = \frac{S_{12}}{1-S_{23}^2\Gamma_2\Gamma_3}$$

Using the series rule,

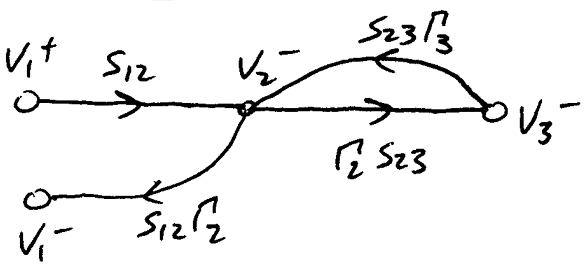
$$V_1^- = \frac{S_{12}^2\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3}V_1^+ \Rightarrow \Gamma_1 = \frac{V_1^-}{V_1^+} = \frac{S_{12}^2\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3}$$

$\boxed{\text{Find } \frac{V_3^-}{V_1^+}}$ Go back to original SFG

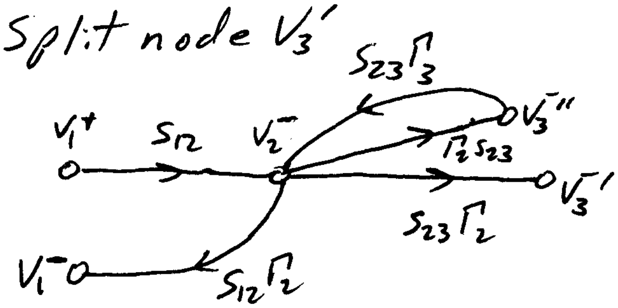
Step 1 Split node V_2^+



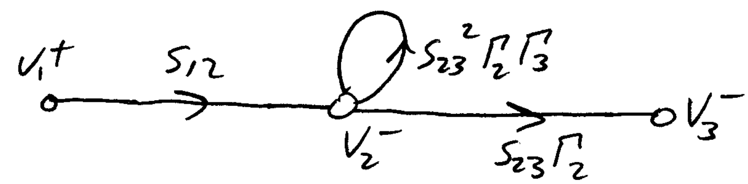
Step 2 Use series rule to eliminate $V_2^{+'}$, $V_2^{+''}$, & V_3^+



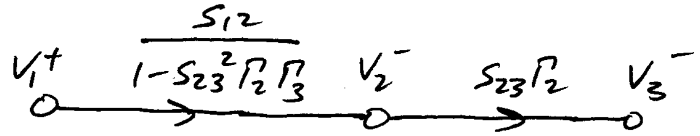
Step 3 Split node V_3'



Step 4 Use series rule to eliminate $V_3^{-''}$ and rename $V_3^{-'}$ to V_3^-



Step 5 Use self-loop rule



Note: $V_2^- = V_1^+ \frac{S_{12}}{1-S_{23}^2\Gamma_2\Gamma_3} \Rightarrow \underline{\frac{V_2^-}{V_1^+} = \frac{S_{12}}{1-S_{23}^2\Gamma_2\Gamma_3}}$ (same)

Using series rule

$$V_3^- = \frac{S_{12}S_{23}\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3} V_1^+ \Rightarrow \underline{\frac{V_3^-}{V_1^+} = \frac{S_{12}S_{23}\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3}}$$

Now, use expressions for $\Gamma_1 = \frac{V_1^-}{V_1^+}$, $\frac{V_2^-}{V_1^+}$, + $\frac{V_3^-}{V_1^+}$ in power ratio expressions

$$\frac{P_2}{P_1} = \left|\frac{S_{12}}{1-S_{23}^2\Gamma_2\Gamma_3}\right|^2 \left(\frac{1-|\Gamma_2|^2}{1-\left|\frac{S_{12}^2\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3}\right|^2}\right) = \left|\frac{V_2^-}{V_1^+}\right|^2 \left(\frac{1-|\Gamma_2|^2}{1-|\Gamma_1|^2}\right)$$

$$\boxed{\frac{P_2}{P_1} = \frac{|S_{12}|^2(1-|\Gamma_2|^2)}{|1-S_{23}^2\Gamma_2\Gamma_3|^2 - |S_{12}^2\Gamma_2|^2}}$$

$$\frac{P_3}{P_1} = \left|\frac{V_3^-}{V_1^+}\right|^2 \left(\frac{1-|\Gamma_3|^2}{1-|\Gamma_1|^2}\right) = \left|\frac{S_{12}S_{23}\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3}\right|^2 \left(\frac{1-|\Gamma_3|^2}{1-\left|\frac{S_{12}^2\Gamma_2}{1-S_{23}^2\Gamma_2\Gamma_3}\right|^2}\right)$$

$$\boxed{\frac{P_3}{P_1} = \frac{|S_{12}|^2|S_{23}|^2|\Gamma_2|^2(1-|\Gamma_3|^2)}{|1-S_{23}^2\Gamma_2\Gamma_3|^2 - |S_{12}^2\Gamma_2|^2}}$$