4.14 A four-port network has the scattering matrix shown as follows. (a) Is this network lossless? (b) Is this network reciprocal? (c) What is the return loss at port 1 when all other ports are terminated with matched loads? (d) What is the insertion loss and phase delay between ports 2 and 4 when all other ports are terminated with matched loads? (e) What is the reflection coefficient seen at port 1 if a short circuit is placed at the terminal plane of port 3 and all other ports are terminated with matched loads?

$$[S]=\begin{bmatrix} 0.24\angle 90° & 0.5\angle 45° & 0.6\angle 45° & 0 \\ 0.5\angle 45° & 0 & 0 & 0.2\angle -45° \\ 0.6\angle 45° & 0 & 0 & 0.6\angle -45° \\ 0 & 0.2\angle -45° & 0.6\angle -45° & 0 \end{bmatrix}$$

a) Per (4.51), $[S]^t\,[S]^* = [U]$ for a lossless network.

$$\begin{bmatrix} 0.24\angle 90° & 0.5\angle 45° & 0.6\angle 45° & 0 \\ 0.5\angle 45° & 0 & 0 & 0.2\angle -45° \\ 0.6\angle 45° & 0 & 0 & 0.6\angle -45° \\ 0 & 0.2\angle -45° & 0.6\angle -45° & 0 \end{bmatrix}\begin{bmatrix} 0.24\angle -90° & 0.5\angle -45° & 0.6\angle -45° & 0 \\ 0.5\angle -45° & 0 & 0 & 0.2\angle 45° \\ 0.6\angle -45° & 0 & 0 & 0.6\angle 45° \\ 0 & 0.2\angle 45° & 0.6\angle 45° & 0 \end{bmatrix} = [U]?$$

Check to see if first element of matrix multiplication is equal to 1-

$$(0.24\angle 90°)(0.24\angle {-90°}) + (0.5\angle 45°)(0.5\angle {-45°}) + (0.6\angle 45°)(0.6\angle {-45°}) + 0(0) = 1\ ?$$

$$0.24^2 + 0.5^2 + 0.6^2 = 0.6676 \neq 1 \quad \Rightarrow \quad \boxed{[S] \text{ is NOT a lossless network}}.$$

b) To be reciprocal, $[S]=[S]^t$ (4.48).

$$\begin{bmatrix} 0.24\angle 90° & 0.5\angle 45° & 0.6\angle 45° & 0 \\ 0.5\angle 45° & 0 & 0 & 0.2\angle -45° \\ 0.6\angle 45° & 0 & 0 & 0.6\angle -45° \\ 0 & 0.2\angle -45° & 0.6\angle -45° & 0 \end{bmatrix} = \begin{bmatrix} 0.24\angle 90° & 0.5\angle 45° & 0.6\angle 45° & 0 \\ 0.5\angle 45° & 0 & 0 & 0.2\angle -45° \\ 0.6\angle 45° & 0 & 0 & 0.6\angle -45° \\ 0 & 0.2\angle -45° & 0.6\angle -45° & 0 \end{bmatrix}?$$

$$\Rightarrow \quad \boxed{[S] \text{ is a reciprocal network}}.$$

c) Per (2.38), RL = -20 log |Γ| (dB). When ports 2-4 are matched, $S_{11} = \Gamma_1 = 0.24\angle 90°$ (page 179).

$$\text{RL}_1 = -20\log|S_{11}| = -20\log 0.24 \ \Rightarrow\ \boxed{\text{RL}_1 = 12.396 \text{ dB}}.$$

d) Per (4.40), $[V^-]=[S][V^+]$. The statement that ports 1 & 3 are terminated in matched loads $\Rightarrow V_1^+ = V_3^+ = 0$.

d) cont.

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \\ V_4^- \end{bmatrix} = \begin{bmatrix} 0.24\angle 90^\circ & 0.5\angle 45^\circ & 0.6\angle 45^\circ & 0 \\ 0.5\angle 45^\circ & 0 & 0 & 0.2\angle -45^\circ \\ 0.6\angle 45^\circ & 0 & 0 & 0.6\angle -45^\circ \\ 0 & 0.2\angle -45^\circ & 0.6\angle -45^\circ & 0 \end{bmatrix} \begin{bmatrix} V_1^+ = 0 \\ V_2^+ \\ V_3^+ = 0 \\ V_4^+ \end{bmatrix}$$

Compute $V_2^- = 0.5\angle 45^\circ(0) + 0(V_2^+) + 0(0) + 0.2\angle -45^\circ(V_4^+) = 0.2\angle -45^\circ(V_4^+)$.

The transmission coefficient from port 4 to port 2 is $T_{24} = \dfrac{V_2^-}{V_4^+} = 0.2\angle -45^\circ$.

Use (2.52) IL = -20 log |T| (dB) to get IL_{24} = -20 log 0.2 $\Rightarrow$ $\boxed{\mathbf{IL_{24} = 13.979\ dB}}$.

Put the equation for V_2^- in polar/phasor format

$$|V_2^-|\angle\theta_2^- = (0.2\angle -45^\circ)\,(|V_4^+|\angle\theta_4^+) = 0.2\,|V_4^+|\angle(-45^\circ + \theta_4^+).$$

Equating the phase angles gives $\theta_2^- = -45^\circ + \theta_4^+$ $\Rightarrow$ $\boxed{\textbf{Phase delay}_{24} = -45^\circ}$.

e) The statement that ports 2 & 4 are terminated in matched loads $\Rightarrow$ $V_2^+ = V_4^+ = 0$.

The statement that port 3 is short-circuited implies $\Gamma_3 = \dfrac{V_3^-}{V_3^+} = -1 \;\Rightarrow\; V_3^+ = -V_3^-$.

Now, (4.40) $[V^-] = [S][V^+]$ becomes

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \\ V_4^- \end{bmatrix} = \begin{bmatrix} 0.24\angle 90^\circ & 0.5\angle 45^\circ & 0.6\angle 45^\circ & 0 \\ 0.5\angle 45^\circ & 0 & 0 & 0.2\angle -45^\circ \\ 0.6\angle 45^\circ & 0 & 0 & 0.6\angle -45^\circ \\ 0 & 0.2\angle -45^\circ & 0.6\angle -45^\circ & 0 \end{bmatrix} \begin{bmatrix} V_1^+ \\ 0 \\ -V_3^- \\ 0 \end{bmatrix}$$

Compute $V_1^- = (0.24\angle 90^\circ)V_1^+ + (0.5\angle 45^\circ)(0) + (0.6\angle 45^\circ)(-V_3^-) + 0(0)$. This simplifies to equation (1) $V_1^- = (0.24\angle 90^\circ)V_1^+ - (0.6\angle 45^\circ)V_3^-$.

Compute $V_3^- = (0.6\angle 45^\circ)V_1^+ + 0(0) + 0(-V_3^-) + (0.6\angle 45^\circ)0$.

Next, substitute $V_3^- = (0.6\angle 45^\circ)V_1^+$ into equation (1) to get

$$V_1^- = (0.24\angle 90^\circ)V_1^+ - (0.6\angle 45^\circ)[(0.6\angle 45^\circ)V_1^+] = [(0.24\angle 90^\circ) - (0.6\angle 45^\circ)(0.6\angle 45^\circ)]V_1^+$$

$$V_1^- = (0.12\angle -90^\circ)\,V_1^+$$

By definition, $\Gamma_1 = \dfrac{V_1^-}{V_1^+} = \dfrac{(0.12\angle -90^\circ)\,V_1^+}{V_1^+}$. Solving, we get $\Rightarrow$ $\boxed{\mathbf{\Gamma_1 = 0.12\angle\text{-}90^\circ}}$.