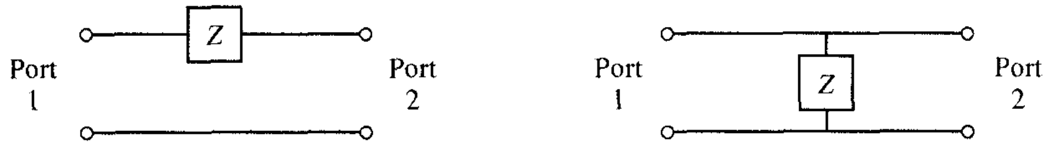
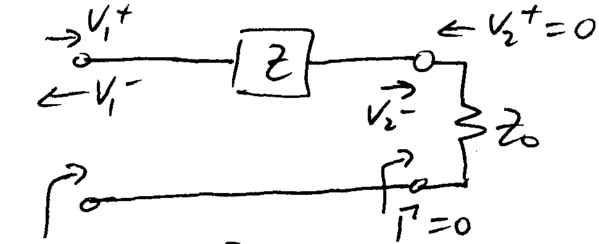
4.11 Find the scattering parameters for the series and shunt loads shown below. Show that $S_{12} = 1 - S_{11}$ for the series case, and that $S_{12} = 1 + S_{11}$ for the shunt case. Assume a characteristic impedance Z_0.



Per (4.41), $S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0 \text{ for } k \neq j}$

$\boxed{\text{Series}}$ $\boxed{S_{11}}$



$Z_{in} = Z + Z_0$

$$S_{11} = \Gamma_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{(Z + Z_0) - Z_0}{(Z + Z_0) + Z_0} = \frac{Z}{2Z_0 + Z} = S_{22} \text{ (Symmetry)}$$

$\boxed{S_{21}}$ Same circuit, Find V_2^- (voltage across Z_0) in terms of V_1^+

$$V_2 = V_2^- = V_1\left(\frac{Z_0}{Z + Z_0}\right) = (V_1^+ + \overset{\Gamma V_1^+ = \Gamma_{11} V_1^+}{V_1^-})\frac{Z_0}{Z + Z_0} = V_1^+(1 + \overset{Z/(2Z_0+Z)}{\Gamma_{11}})\frac{Z_0}{Z + Z_0}$$

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{V_1^+\left(1 + \frac{Z}{2Z_0 + Z}\right)\frac{Z_0}{Z + Z_0}}{V_1^+} = \left(1 + \frac{Z}{2Z_0 + Z}\right)\frac{Z_0}{Z + Z_0}$$

$$= \frac{Z_0}{Z + Z_0} + \frac{Z Z_0}{(2Z_0 + Z)(Z + Z_0)} = \frac{Z_0(2Z_0 + Z) + Z Z_0}{(2Z_0 + Z)(Z + Z_0)}$$

$$= \frac{2Z_0(Z + Z_0)}{(2Z_0 + Z)(Z + Z_0)} = \frac{2Z_0}{2Z_0 + Z} = S_{12} \text{ (symmetry)}$$

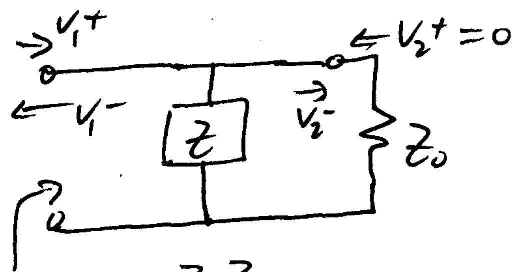
$$\boxed{\text{series} \quad [S] = \begin{bmatrix} \frac{Z}{2Z_0 + Z} & \frac{2Z_0}{2Z_0 + Z} \\ \frac{2Z_0}{2Z_0 + Z} & \frac{Z}{2Z_0 + Z} \end{bmatrix}}$$

$$S_{12} \overset{?}{=} 1 - S_{11}$$

$$\frac{2Z_0}{2Z_0+Z} \overset{?}{=} 1 - \frac{Z}{2Z_0+Z} = \frac{(2Z_0+Z)-Z}{2Z_0+Z} = \frac{2Z_0}{2Z_0+Z}$$

$$\underline{\frac{2Z_0}{2Z_0+Z} = \frac{2Z_0}{2Z_0+Z}} \quad \therefore \text{ Yes!}$$

$\boxed{\text{Shunt}}$ $\boxed{S_{11}}$



$$Z_{in} = \frac{Z Z_0}{Z+Z_0}$$

$$S_{11} = \Gamma_{11} = \frac{Z_{in}-Z_0}{Z_{in}+Z_0} = \frac{\frac{ZZ_0}{Z+Z_0} - Z_0}{\frac{ZZ_0}{Z+Z_0} + Z_0}\left(\frac{Z+Z_0}{Z+Z_0}\right) = \frac{ZZ_0 - Z_0(Z+Z_0)}{ZZ_0 + Z_0(Z+Z_0)}$$

$$= \frac{-Z_0^2}{2ZZ_0+Z_0^2} = \underline{\frac{-Z_0}{2Z+Z_0}} = S_{22} \text{ (symmetry)}$$

$\boxed{S_{21}}$ Same Circuit. Find V_2^- (voltage across Z_0) in terms of V_1^+

$$V_2 = V_2^- = V_1 = V_1^+ + V_1^- = V_1^+(1+\Gamma_{11})$$

$$= V_1^+\left(1 + \frac{-Z_0}{2Z+Z_0}\right)$$

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{V_1^+\left(1+\frac{-Z_0}{2Z+Z_0}\right)}{V_1^+} = \frac{(2Z+Z_0)-Z_0}{2Z+Z_0}$$

$$= \underline{\frac{2Z}{2Z+Z_0}} = S_{12} \text{ (symmetry)}$$

Shunt

$$\boxed{[S] = \begin{bmatrix} \dfrac{-Z_0}{2Z+Z_0} & \dfrac{2Z}{2Z+Z_0} \\ \dfrac{2Z}{2Z+Z_0} & \dfrac{-Z_0}{2Z+Z_0} \end{bmatrix}}$$

$$S_{12} \stackrel{?}{=} 1 + S_{11}$$

$$\frac{2Z}{2Z+Z_0} \stackrel{?}{=} 1 + \frac{-Z_0}{2Z+Z_0} = \frac{(2Z+Z_0)-Z_0}{2Z+Z_0}$$

$$\underline{\frac{2Z}{2Z+Z_0} = \frac{2Z}{2Z+Z_0}} \quad \therefore \text{ Yes!}$$