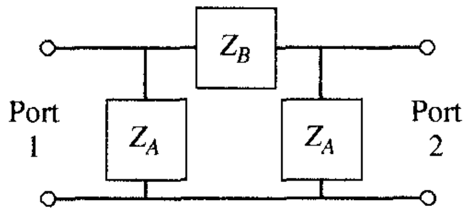
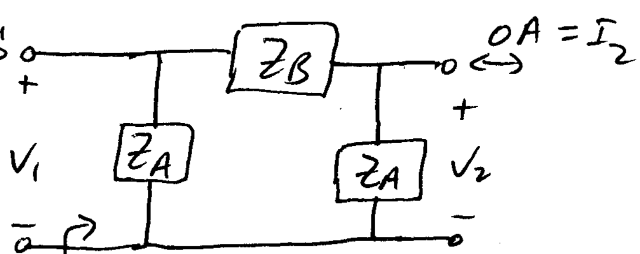
4.7 Derive the [Z] and [Y] matrices for the two-port networks shown in the figure below.



(a)

Per (4.28), $Z_{ij} = \left.\dfrac{V_i}{I_j}\right|_{I_k=0 \text{ for } k \neq j}$

$\boxed{Z_{11}}$



Find I_1 in terms of V_1.

$$Z_{eq} = Z_A \,//\, (Z_A + Z_B) = \frac{Z_A(Z_A+Z_B)}{Z_A + (Z_A+Z_B)} = \frac{Z_A(Z_A+Z_B)}{2Z_A+Z_B}$$

$$I_1 = V_1/Z_{eq} = V_1 \frac{2Z_A+Z_B}{Z_A(Z_A+Z_B)}$$

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{V_1}{V_1 \dfrac{2Z_A+Z_B}{Z_A(Z_A+Z_B)}} = \underline{\frac{Z_A(Z_A+Z_B)}{2Z_A+Z_B}} = Z_{22} \text{ (symmetry)}$$

$\boxed{Z_{21}}$ same circuit. Find V_2 in terms of I_1.

$$V_1 = I_1 Z_{eq} = I_1 \frac{Z_A(Z_A+Z_B)}{2Z_A+Z_B}$$

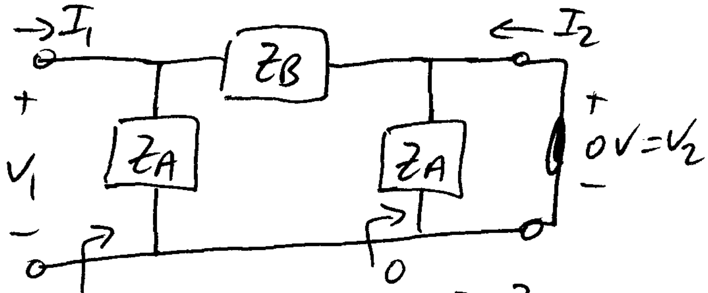
$$V_2 = V_1 \frac{Z_A}{Z_A+Z_B} = I_1 \frac{Z_A^2}{2Z_A+Z_B} \cancelto{1}{\frac{Z_A+Z_B}{Z_A+Z_B}}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{I_1 \dfrac{Z_A^2}{2Z_A+Z_B}}{I_1} = \frac{Z_A^2}{2Z_A+Z_B} = Z_{12} \text{ (symmetry)}$$

$$\boxed{[Z] = \begin{bmatrix} \dfrac{Z_A(Z_A+Z_B)}{2Z_A+Z_B} & \dfrac{Z_A^2}{2Z_A+Z_B} \\ \dfrac{Z_A^2}{2Z_A+Z_B} & \dfrac{Z_A(Z_A+Z_B)}{2Z_A+Z_B} \end{bmatrix}}$$

Per (4.29), $Y_{ij} = \left.\dfrac{I_i}{V_j}\right|_{V_k = 0 \text{ for } k \neq j}$

$\boxed{Y_{11}}$



Find V_1 in terms of I_1.

$$Z_{eq} = Z_A \parallel Z_B = \frac{Z_A Z_B}{Z_A + Z_B}$$

$$V_1 = I_1 Z_{eq} = I_1 \frac{Z_A Z_B}{Z_A + Z_B}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = \frac{I_1}{I_1 \frac{Z_A Z_B}{Z_A + Z_B}} = \underline{\frac{Z_A + Z_B}{Z_A Z_B}} = Y_{22} \text{ (symmetry)}$$

$\boxed{Y_{21}}$ Same circuit. Find I_2 in terms of V_1

$$I_2 = \frac{-V_2}{Z_B}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = \frac{-V_1/Z_B}{V_1} = \frac{-1}{Z_B} = Y_{12} \text{ (symmetry)}$$

$$\boxed{[Y] = \begin{bmatrix} \dfrac{Z_A + Z_B}{Z_A Z_B} & \dfrac{-1}{Z_B} \\[2ex] \dfrac{-1}{Z_B} & \dfrac{Z_A + Z_B}{Z_A Z_B} \end{bmatrix}}$$