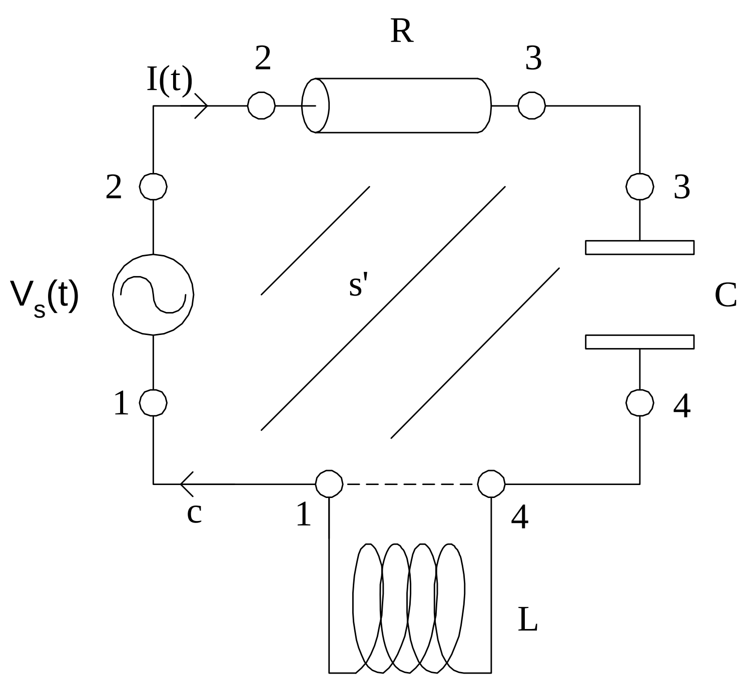
Lecture 8: Maxwell's Equations and Electrical Circuits.

Electrical circuit analysis is usually presented as a theory unto itself. However, the basis of electrical circuit analysis actually comes from electromagnetics, i.e., Maxwell's equations.

It is important to recognize this since electrical circuit theory is really only an approximation and under the "right" conditions, it can **fail**.

We will illustrate how electrical circuit analysis is derived from Maxwell's equations by considering the following physical circuit (i.e., an electrical circuit that has physical dimensions):



First, we will apply Faraday's law to the contour *c*

$$\oint_{c(s)} \bar{E}(t)\cdot d\bar{l} = -\frac{d}{dt}\int_{s(c)} \bar{B}(t)\cdot d\bar{s} \tag{1}$$

Between any two adjacent terminals *a* and *b*, we will define voltage as

$$V_{ba}(t) = -\int_a^b \bar{E}(t)\cdot d\bar{l} \tag{2}$$

To make the LHS of (1) fit (2), we will move the minus sign in (1) so that

$$-\oint_{c(s)} \bar{E}(t)\cdot d\bar{l} = \frac{d}{dt}\int_{s(c)} \bar{B}(t)\cdot d\bar{s} \tag{3}$$

As we apply (3) to the physical circuit in the figure, we will choose to ignore the effects of the leads and the wires connecting the elements. (We can come back later and add these effects in, if we wish.)

The integral on the LHS of (3) will be broken up into four subsections as

$$\underbrace{-\int_1^2 \bar{E}\cdot d\bar{l}}_{=V_{21}}\underbrace{-\int_2^3 \bar{E}\cdot d\bar{l}}_{=V_{32}}\underbrace{-\int_3^4 \bar{E}\cdot d\bar{l}}_{=V_{43}}\underbrace{-\int_4^1 \bar{E}\cdot d\bar{l}}_{=V_{14}} = \frac{d}{dt}\int_{s(c)} \bar{B}\cdot d\bar{s} \tag{4}$$

We will consider separately each of the five terms in (4).

Source Voltage

V_{21} – This is the **source voltage** (or emf):

$$-\int_1^2 E(t) \cdot d\bar{l} \equiv V_s(t) \qquad (5)$$

Resistor Voltage

V_{32} – This is the **resistor voltage**:

$$-\int_2^3 \bar{E}(t) \cdot d\bar{l} = V_{32}(t)$$

You considered resistors supporting direct current previously in EE 381. For sinusoidal steady state, we will assume here that the resistors are electrically small. That is, their dimensions are much smaller than a wavelength ($\lambda = c / f$).

Inside a conductive material, by Ohm's law $\bar{J}(t) = \sigma \bar{E}(t)$. Therefore,

$$V_{32}(t) = -\int_2^3 \bar{E}(t) \cdot d\bar{l} = -\int_2^3 \frac{\bar{J}(t)}{\sigma} \cdot d\bar{l}$$

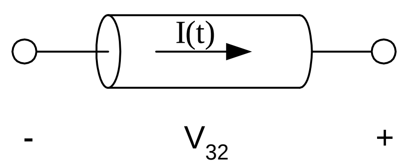
Assuming the frequency f is small enough so that $\bar{J}$ is nearly uniform over the cross section (which is not true at high f), then

$$V_{32}(t) = -\int_2^3 \frac{I(t)}{\sigma A} dl = -I(t) R \tag{6}$$

where

$$R = \int_2^3 \frac{1}{\sigma A} dl$$

This is the typical formula for compute the DC resistance of a conductor that has a uniform current density over its cross section. Why the minus sign in (6)? Because of the assumed polarity:



Capacitor Voltage

V_{43} – This is the **capacitor voltage**:

$$-\int_3^4 \bar{E}(t) \cdot d\bar{l} = V_{43}(t)$$

We considered the capacitor in detail earlier in Lecture 5 in connection with displacement current. Under the quasi-static assumption $Q(t) \approx CV(t)$ then

$$\underbrace{\frac{dQ(t)}{dt}}_{=I(t)} = C\frac{dV(t)}{dt}$$

Forming the antiderivative of this expression gives

$$V(t) = \frac{1}{C}\int_{-\infty}^{t} I(t')\,dt' = V_{34}$$

In terms of V_{43}, we can write this result as

$$\boxed{V_{43}(t) = -V_{34}(t) = -\frac{1}{C}\int_{-\infty}^{t} I(t')\,dt'} \qquad (7)$$

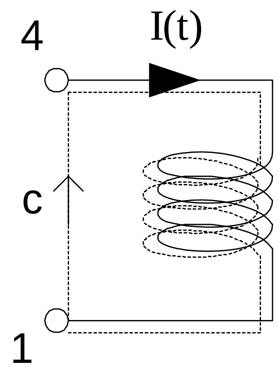
We see here once again that the **basic circuit operation of the capacitor arises because of displacement current**, as we discussed earlier in Lecture 5.

Inductor Voltage

V_{14} – This is the **inductor voltage**:

$$-\int_{4}^{1} \bar{E}(t)\cdot d\bar{l} = V_{14}(t)$$

Consider the contour c



And apply Faraday's law to this contour

$$\oint_{c(s)} \bar{E} \cdot d\bar{l} = -\frac{d}{dt} \int_{s(c)} \bar{B}(t) \cdot d\bar{s}$$

We can separate the line integral into two parts

$$\int_1^4 \bar{E} \cdot d\bar{l} + \underbrace{\int_{\text{coil}} \bar{E} \cdot d\bar{l}}_{=0} = -\frac{d}{dt} \int_{\text{coil}} \bar{B}(t) \cdot d\bar{s} \tag{8}$$

The second term on the LHS equals zero if there is no resistance in the wires of the inductor. Otherwise, there would be an R_L term similar to V_{32} earlier.

For the RHS of (8)

$$\int_{\text{coil}} \bar{B}(t) \cdot d\bar{s} = \lambda(t) = N\psi_m(t) \tag{9}$$

where $\lambda(t)$ is the flux linkage through the surface formed by the coil of wire. Substituting (9) into (8) gives

$$-\int_1^4 \bar{E}(t) \cdot d\bar{l} = \frac{d\lambda(t)}{dt} \tag{10}$$

For magnetostatic fields, you saw in EE 381 that

$$\lambda = LI \tag{11}$$

We will assume here that the frequency is small enough (i.e., is "quasi-static") that

$$\begin{aligned} \frac{d\lambda(t)}{dt} &= \frac{d}{dt}(LI) = L\frac{dI(t)}{dt} + I(t)\underbrace{\frac{dL}{dt}}_{=0} \\ &= L\frac{dI(t)}{dt} \end{aligned} \tag{12}$$

Substituting (12) into (10) gives

$$V_{41} = L\frac{dI(t)}{dt}$$

or

$$\boxed{V_{14} = -V_{41} = -L\frac{dI(t)}{dt}} \tag{13}$$

We see that the **basic operation of the inductor arises directly from Faraday's law**!

Collecting the Results

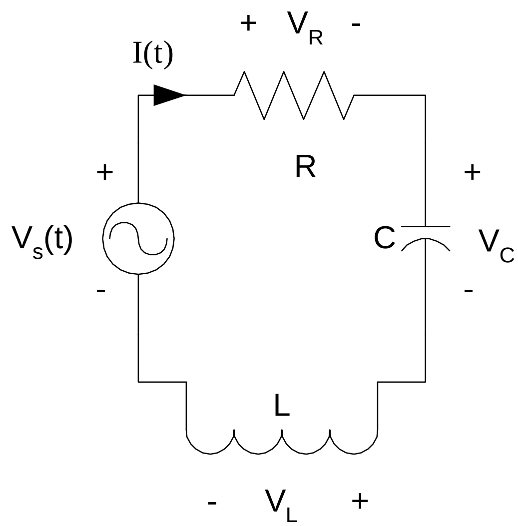
Now, we'll pull all of this together. Substituting (13), (7), (6), and (5) into (4) gives

$$\boxed{V_s(t) - I(t)R - \frac{1}{C}\int_{-\infty}^{t} I(t')\,dt' - L\frac{dI(t)}{dt} = \frac{d}{dt}\int_{s'} \overline{B}\cdot d\overline{s}} \tag{14}$$

This result was derived directly from Maxwell's equations!

In (14), the surface s' is the open surface bounded by c, minus the surface bounded by the inductor. The effect of the magnetic flux through the inductor surface is accounted for in the inductor term in (14).

Next, let's apply KVL to the lumped element representation for this physical circuit:



$$V_s(t) = RI(t) + \frac{1}{C}\int_{-\infty}^{t} I(t')\,dt' + L\frac{dI(t)}{dt} \qquad (15)$$

This result was derived directly from circuit theory.

Comparing (14) and (15) we see that if the RHS of (14) is negligible, then these two equations are equal!

We have derived the terminal (V-I) characteristics for the lumped element circuit from Maxwell's equations. This is a **BIG** accomplishment.

In effect, every electrical circuit is a little electromagnetic "test bed:"

- Resistor: behavior governed by Ohm's law,
- Capacitor: behavior governed by "Maxwell's law,"
- Inductor: behavior governed by Faraday's law.

"Missing" Term in KVL?

What is the extra term $\frac{d}{dt}\int_{s'} \overline{B} \cdot d\overline{s}$ in (14)? It is the (negative) emf contribution from the magnetic flux linking the surface s' of the entire circuit (i.e., the entire surface s minus the inductor):

- At low f, we can often safely ignore this term.
- At high f and overall physical circuit dimensions approaching tens of centimeters or larger, this term may not be negligible.