Lecture 5: Displacement Current and Ampère's Law.

One more addition needs to be made to the governing equations of electromagnetics before we are finished. Specifically, we need to clean up a glaring inconsistency.

From Ampère's law in magnetostatics, we learned that

$$\nabla \times \bar{H} = \bar{J} \tag{1}$$

Taking the divergence of this equation gives

$$\nabla \cdot \cancelto{0}{\left(\nabla \times \bar{H} \right)} = \nabla \cdot \bar{J}$$

That is,

$$\nabla \cdot \bar{J} = 0 \tag{2}$$

However, as is shown in Section 5.8 of the text ("Continuity Equation and Relaxation Time), the continuity equation (conservation of charge) requires that

$$\boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t}} \tag{3}$$

We can see that (2) and (3) agree only when there is no time variation or no free charge density. This makes sense since (2) was derived only for magneto**static** fields in Ch. 7.

Ampère's law in (1) is only valid for static fields and, consequently, it violates the conservation of charge principle if we try to directly use it for time varying fields.

Ampère's Law for Dynamic Fields

Well, what is the correct form of Ampère's law for dynamic (time varying) fields?

Enter **James Clerk Maxwell** (ca. 1865) – **The Father of Classical Electromagnetism**. He combined the results of Coulomb's, Ampère's, and Faraday's laws and added a new term to Ampère's law to form the set of fundamental equations of classical EM called **Maxwell's equations**.

It is this addition to Ampère's law that brings it into congruence with the conservation of charge law. Maxwell's contribution was to modify Ampère's law (1) to read

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{4}$$

The second term on the right-hand side is called the **displacement current density**. We will investigate this in more detail shortly.

Consistency with Conservation of Charge

First, however, let's see that adding this term in (4) makes it consistent with conservation of charge in (3). First, we take the divergence of (4)

$$\nabla \cdot \left(\nabla \times \bar{H}\right) = 0 = \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \bar{D}}{\partial t} \tag{5}$$

Next, from Gauss' law

$$\nabla \cdot \bar{D} = \rho$$

we take the time derivative yielding

$$\frac{\partial}{\partial t} \nabla \cdot \bar{D} = \nabla \cdot \frac{\partial \bar{D}}{\partial t} = \frac{\partial \rho}{\partial t} \tag{6}$$

Substituting (6) into (5) gives

$$\nabla \cdot \bar{J} = -\nabla \cdot \frac{\partial \bar{D}}{\partial t} = -\frac{\partial \rho}{\partial t} \tag{7}$$

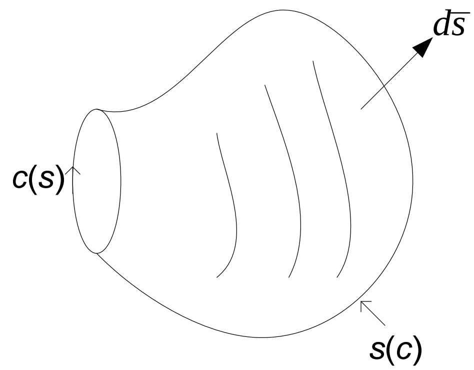
This is the continuity equation (3). Amazing! With this additional displacement current term in (4), Ampère's law is now consistent with conservation of charge.

Of course, this is not a proof that (4) is now the correct form of Ampère's law for dynamic fields, although it is. The correctness of (4) is essentially an experimentally derived proof as with

- Gauss' law,
- Ampère's force law, and
- Faraday's law, discussed earlier in this chapter.

Integral Form of Ampère's Law

The integral form of Ampère's law is obtained by integrating (4) over some arbitrary open surface *s*



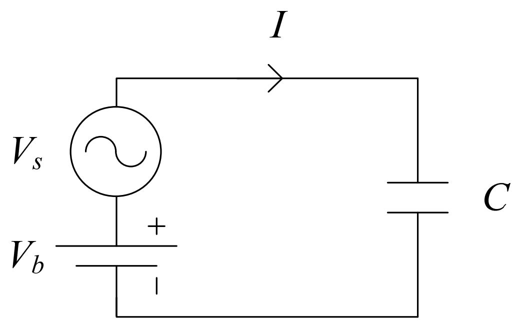
and applying Stokes' theorem to give

$$\oint_{c(s)} \bar{H} \cdot d\bar{l} = \underbrace{\int_{s(c)} \bar{J} \cdot d\bar{s}}_{\text{free current}} + \underbrace{\int_{s(c)} \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}}_{\text{displacement current}} \tag{8}$$

The first term on the right hand side is free current. The second term is this new displacement current.

What is Displacement Current?

So what is displacement current? Consider an example of a simple electrical circuit containing a sinusoidal voltage source and a capacitor:

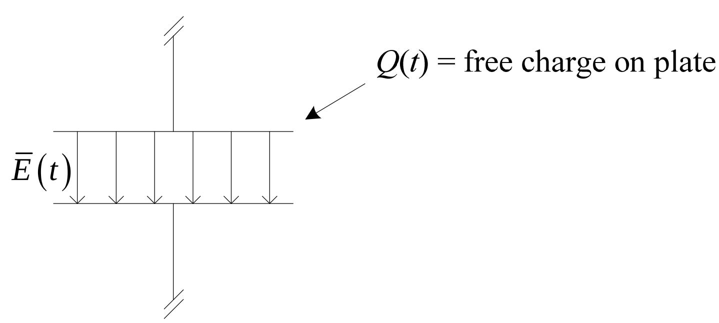


From our experience in the lab, at DC the current $I = 0$. However, if the frequency of the voltage source is not zero, we will observe and measure some nonzero time varying current I.

However, if free charge (i.e., electrons) cannot "jump" the gap between the capacitor plates, then how can there be current in the circuit?

Because there is another type of current that completes the circuit. Between the plates of the capacitor exists the displacement current.

To see this, consider just the capacitor:



You learned previously in EE 381 that under static conditions

$$Q = CV \tag{9}$$

If the voltage source is assumed to vary "slowly enough" with time, then (9) is still valid

$$Q(t) \approx CV(t) \quad (10)$$

In words, (10) reveals to us that as V changes with time, so does the stored charge on the conducting plates of the capacitor.

Now, $V(t)$ as we've seen is not uniquely defined. Nevertheless, if the distance between the plates is sufficiently small (so that $\bar{E}(t)$ is essentially "quasi-static") it may be a very good approximation. Then,

$$V(t) \approx -\int_{\text{plate 1}}^{\text{plate 2}} \bar{E}(t) \cdot d\bar{l}$$

Taking the time derivative of (10) gives

$$\frac{dQ(t)}{dt} = C\frac{dV(t)}{dt} \quad (11)$$

We interpret this equation as **equating two types of current**:

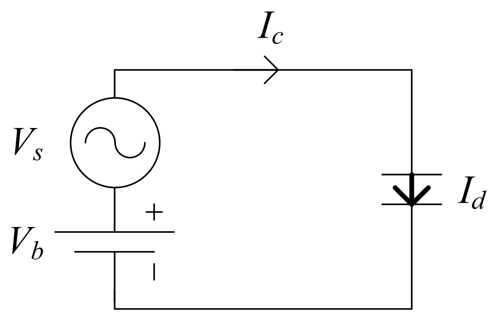
1. $I_c(t) = \dfrac{dQ(t)}{dt} \equiv$ conduction current in the wires
2. $I_d(t) = C\dfrac{dV}{dt} \equiv$ "displacement" current in the capacitor

That is, (11) can be written as

$$I_c(t) = I_d(t) = C\frac{dV(t)}{dt} \quad (12)$$

which should be an extremely familiar equation!

This displacement current has "completed" the electrical circuit:



Importance of Displacement Current

Displacement current is just as important and just as relevant to an electrical circuit as conduction current. As we've just seen in the previous (and very simple) electrical circuit, there would be no (time varying) conduction current in the circuit were not displacement current present in the capacitor.

Furthermore, as we'll see throughout most of the remainder of this course, it is possible for electromagnetic signals to travel through a material – in particular, even in a vacuum. Displacement current is a key reason why such a phenomenon, i.e., electromagnetic radiation, is possible.

Because of this fact, one can argue that most life on Earth is possible because of displacement current! Why? Because the heat from the sun warms our planet through radiative heat transfer (rather than by conduction or convection mechanisms).

This radiation of heat is an electromagnetic wave. Were it not for the displacement current – and, of course, other effects – there would be no heat from the sun warming our planet, and consequently little to no life on Earth as we know it. Thank you Displacement Current!