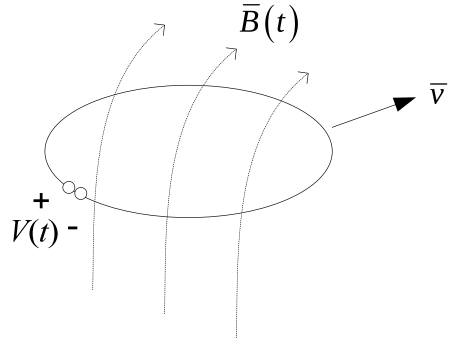
Lecture 4: Faraday's Law and Moving Circuits.

In the previous two lectures, we've been investigating Faraday's law, which is written as

$$\oint_{c(s)} \overline{E} \cdot d\overline{l} = -\frac{d}{dt} \int_{s(c)} \overline{B} \cdot d\overline{s} \qquad (1)$$

As discussed in text Section 9.3, however, there is an alternate form of (1) that is sometimes used when the circuit moves through space. This is especially true if $\overline{B}$ is also a function of time.

Consider the circuit shown below:



This loop is traveling with velocity $\overline{v}$ (relative to the frame defining $\overline{B}$) in a time-varying B field, $\overline{B}(t)$.

The emf measured by a high-impedance voltmeter connected to this circuit is given in (1). Charges in the wire are induced to move under the influence of a force given by the Lorentz force equation

$$\bar{F} = q\left(\bar{E} + \bar{v} \times \bar{B}\right) \quad (2)$$

Motional and Transformer EMF

However, for an observer traveling with the loop, there is no apparent velocity. Instead, this observer would observe a force $\bar{F}'$ due only to an apparent electric field $\bar{E}'$

$$\bar{F}' = q\bar{E}' \quad (3)$$

The two forces $\bar{F}$ and $\bar{F}'$ must be equal, so that from (2) and (3) we find that

$$\bar{E}' = \bar{E} + \bar{v} \times \bar{B} \quad (4)$$

Now, let's substitute (4) back into Faraday's law (1). After considerable simplification we find that

$$\oint_{c(s)} \bar{E}' \cdot d\bar{l} = -\int_{s(c)} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} + \oint_{c(s)} \left(\bar{v} \times \bar{B}\right) \cdot d\bar{l} \quad (5)$$

This is an alternate form to Faraday's law (1). Notice that there is no magnetic flux in this form!

Important points to note in (5):

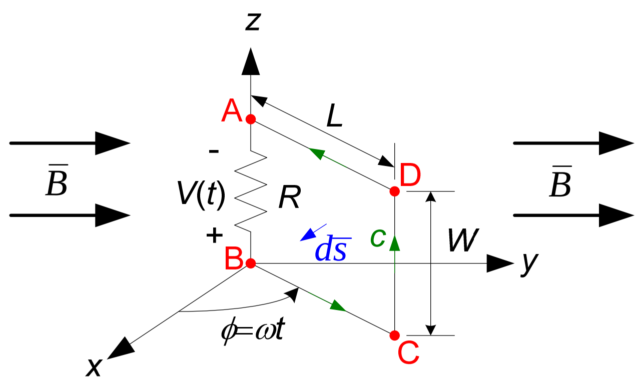
- $\oint_{c(s)} \left(\bar{v} \times \bar{B}\right) \cdot d\bar{l}$ is called the **motional emf**. The circuit defined by contour c is said to "cut" the B field lines, whether or not B is constant or changing with time. (Note that it's assumed here the shape of the circuit does not change with time.)

- $-\int_{s(c)} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$ is called the **transformer emf**. (Notice that the time derivative appears inside the integral.) There is no motion present in this term.
- The same *emf* would be measured by a voltmeter whether it is moving with the circuit as in (5) or stationary as in (1) since

$$\oint_{c(s)} \frac{\bar{F}}{q} \cdot d\bar{l} = \oint_{c(s)} \frac{\bar{F}'}{q} \cdot d\bar{l} = emf$$

Either of these two equations (1) and (5) yield exactly the same *emf*. This property will be illustrated in the two examples shown next. However, one form may be preferred over another depending on the problem.

Example N4.1: Determine the induced resistor voltage V in the rotating loop circuit shown below using the form of Faraday's law given in (5). The loop is immersed in the field $\bar{B} = \hat{a}_y B_o$ [T].



The direction of c was arbitrarily chosen as shown. From (5)

$$emf = -\int_{s(c)} \frac{\partial \overline{B}}{\partial t} \cdot d\overline{s} + \oint_{c(s)} \left(\overline{v} \times \overline{B}\right) \cdot d\overline{l} = \oint_{c(s)} \left(\overline{v} \times \overline{B}\right) \cdot d\overline{l}$$

The first term in the RHS is zero because $\overline{B}$ is not a function of time.

We will consider contributions to $\left(\overline{v} \times \overline{B}\right) \cdot d\overline{l}$ from each of the four line segments that comprise the contour c:

- $\overline{AB}$: $\overline{v} = 0$ such that $\left(\overline{v} \times \overline{B}\right) \cdot d\overline{l} = 0$.
- $\overline{BC}$: $\overline{v} \times \overline{B} \perp d\overline{l}$ such that $\left(\overline{v} \times \overline{B}\right) \cdot d\overline{l} = 0$.
- $\overline{DA}$: $\overline{v} \times \overline{B} \perp d\overline{l}$ such that $\left(\overline{v} \times \overline{B}\right) \cdot d\overline{l} = 0$.
- $\overline{CD}$: $\overline{v} = \hat{a}_\phi L\omega$, therefore

$$\overline{v} \times \overline{B} = \left(\hat{a}_\phi L\omega\right) \times \left(\hat{a}_y B_o\right) = \left(\hat{a}_\phi \times \hat{a}_y\right) L\omega B_o$$

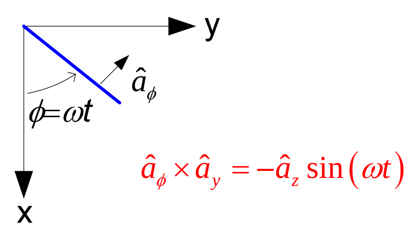
The cross product $\hat{a}_\phi \times \hat{a}_y$ involves unit vectors from two different coordinate systems. A direct calculation approach involves converting $\hat{a}_\phi$ to the Cartesian coordinate system, then performing the cross product. That is,

$$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi \qquad (6)$$

such that

$$\hat{a}_\phi \times \hat{a}_y = -\hat{a}_x \sin(\omega t) \times \hat{a}_y = -\hat{a}_z \sin(\omega t)$$

Another approach to this cross product calculation is more heuristic. The calculation of the cross product of unit vectors is aided by drawing a top-down sketch of the rotating loop:



Consequently,

$$\bar{v} \times \bar{B} = -\hat{a}_z L\omega B_o \sin(\omega t)$$

Therefore,

$$(\bar{v} \times \bar{B}) \cdot d\bar{l} = \left[-\hat{a}_z L\omega B_o \sin(\omega t)\right] \cdot \underbrace{\hat{a}_z dz}_{=d\bar{l}}$$

$$= -L\omega B_o \sin(\omega t)\, dz$$

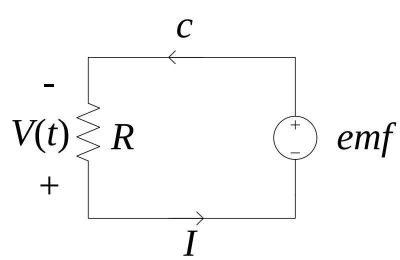
Continuing with the calculation of *emf*, then,

$$emf = \oint_{c(s)} (\bar{v} \times \bar{B}) \cdot d\bar{l} = -\int_0^W L\omega B_o \sin(\omega t)\, dz$$

such that

$$emf = -\omega LWB_o \sin(\omega t) \text{ [V]} \tag{7}$$

Next, draw the equivalent lumped-element circuit:



By KVL:

$$V + emf = 0 \quad \Rightarrow \quad V = -emf = \omega LWB_o \sin(\omega t) \text{ [V]}$$

We can interpret the results of the line integral of $(\bar{v} \times \bar{B}) \cdot d\bar{l}$ for each of the four segments in the context of which segments actually "cut" the $\bar{B}$ field lines. Segments $\overline{BC}$ and $\overline{DA}$ are in motion, but since these segments do not "cut" the $\bar{B}$ field lines, no *emf* is generated by their motion through $\bar{B}$. Their motion is always parallel to the $\bar{B}$ field lines.

Conversely, the vertical section $\overline{CD}$ is "cutting" the $\bar{B}$ field lines as it rotates. Consequently, an *emf* is generated in this segment as it rotates through $\bar{B}$. Note that for $\phi = 0$ and $\phi = 180^{\circ}$, this segment does not cut the $\bar{B}$ field lines so no *emf* is generated near these angles.

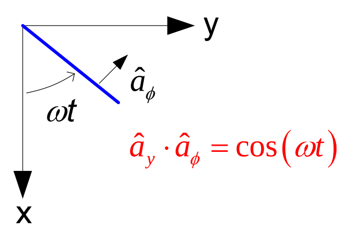
Example N4.2: Repeat the previous example, but use the form of Faraday's law in (1).

From (1):
$$emf = -\frac{d\psi_m}{dt} = -\frac{d}{dt}\oint_s \bar{B}\cdot d\bar{s}$$

As indicated in the figure on page 3, $d\bar{s} = -\hat{a}_\phi drdz$. Consequently,

$$emf = +\frac{d}{dt}\int_0^W\int_0^L\left(\hat{a}_y\cdot\hat{a}_\phi\right)B_o drdz$$

Similar to the previous example problem, this inner product of unit vectors can be evaluated using (6), or the heuristic approach by making a quick sketch:



Therefore,

$$emf = WLB_o\frac{d}{dt}\cos(\omega t) = -WLB_o\omega\sin(\omega t)\ \text{[V]}$$

which is the same answer as in (7) of the previous example, as expected.

As shown in the previous example
$$V = -emf$$

Therefore

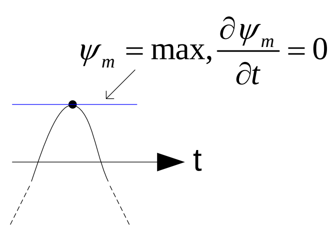
$$V = -emf = WLB_o\omega\sin(\omega t) \quad [\text{V}]$$

Once again, this is the same answer as in the previous example, as expected.

Note that here in Example N4.2 we have solved for V without resorting to motional *emf* and/or transformer *emf*, in contrast to the previous Example N4.1 (where only motional *emf* was present). Instead, we have just applied Faraday's law (1), which is always valid.

Lastly, notice that when the loop is in the *xz* plane, ψ_m is maximum, but $V = 0$! Further, when the loop lies in the *yz* plane, $\psi_m = 0$ but $|V|$ is maximum! Very strange, what is happening?

Recall that the *emf* is not equal to ψ_m in Faraday's law. Rather, it is equal to the negative time-rate-of-change of ψ_m. When the loop is in the *xz* plane:



while when the loop is in the *yz* plane: