13.19 In the far field of a particular antenna located at the origin, the magnetic field intensity is

$$\mathbf{H}_s = \frac{j\beta I_o}{4\pi r}(\sin\phi\,\mathbf{a}_\theta + \cos\theta\cos\phi\,\mathbf{a}_\phi)$$

where I_o is the peak value of the input current. Show that the radiation resistance is given by $R_{rad} = 20\,\beta^2$.

- Also, find the corresponding phasor electric field and power radiated.

$$\bar{E}_s = \eta(\bar{H}_s \times \hat{a}_r) = \eta\frac{j\beta I_o}{4\pi r}(\sin\phi\,\hat{a}_\theta + \cos\theta\cos\phi\,\hat{a}_\phi)\times\hat{a}_r$$

$$= \frac{j\eta\beta I_o}{4\pi r}(-\sin\phi\,\hat{a}_\phi + \cos\theta\cos\phi\,\hat{a}_\theta)$$

$$\underline{\underline{\bar{E}_s = \frac{j\eta\beta I_o}{4\pi r}(\cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi)}}$$

From (10.78), the time-average Poynting vector is

$$\bar{P}_{ave} = \tfrac{1}{2}\mathrm{Re}\{\bar{E}_s \times \bar{H}_s^*\}$$

$$= \tfrac{1}{2}\mathrm{Re}\Big\{\frac{j\eta\beta I_o}{4\pi r}(\cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi)\times$$

$$\frac{-j\beta I_o}{4\pi r}(\sin\phi\,\hat{a}_\theta + \cos\theta\cos\phi\,\hat{a}_\phi)\Big\}$$

$$\bar{P}_{ave} = \frac{1}{2}\frac{\eta\beta^2 I_o^2}{(4\pi)^2 r^2}\left[0 + \cos^2\theta\cos^2\phi + \sin^2\phi\right]\hat{a}_r$$

Per (13.40), the power radiated, through a sphere, is

$$P_{rad} = \oiint_S \bar{P}_{ave}\cdot d\bar{S}_r = \frac{\eta\beta^2 I_o^2}{32\pi^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{\cos^2\theta\cos^2\phi + \sin^2\phi}{r^2}\,r^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{\eta\beta^2 I_o^2}{32\pi^2}\underbrace{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}(\cos^2\theta\cos^2\phi + \sin^2\phi)\sin\theta\,d\theta\,d\phi}_{\text{solve numerically} \;\text{“}\; \frac{8\pi}{3}}$$

$$P_{rad} = \frac{\eta \beta^2 I_0^2}{12\pi}$$

From (13.48), $P_{rad} = \frac{1}{2} |I_{in}|^2 R_{rad} = \frac{1}{2} I_0^2 R_{rad}$.
Equating P_{rad} expressions and solving –

$$R_{rad} = \frac{2 P_{rad}}{I_0^2} = \frac{2 \eta \beta^2 I_0^2}{12\pi I_0^2} = \frac{2\eta\beta^2}{12\pi}$$

$$R_{rad} = \frac{2\eta\beta^2}{12\pi}$$ more accurate

using (horrible approximation that $\eta = 120\pi$)

$$R_{rad} = \frac{2(120\pi)\beta^2}{12\pi} = \underline{20\beta^2}$$