10.46 An antenna is located at the origin of a spherical coordinate system. The fields produced by the antenna in free space are

$$\mathbf{E} = \frac{E_o}{r}\sin\theta \sin\omega(t - r/c)\mathbf{a}_\theta = \frac{E_o}{r}\sin\theta\cos\left[\omega t - \frac{\omega}{c}r - \pi/2\right]\hat{a}_\theta$$

$$\mathbf{H} = \frac{E_o}{\eta r}\sin\theta \sin\omega(t - r/c)\mathbf{a}_\phi = \frac{E_o}{\eta r}\sin\theta\cos\left[\omega t - \frac{\omega}{c}r - \pi/2\right]\hat{a}_\phi$$

where $c = \frac{1}{\sqrt{\mu_o\varepsilon_o}}$ and $\eta = \sqrt{\frac{\mu_o}{\mu_o}}$. Determine the time-average power radiated by the antenna.

- First, find the phasor electric $\overline{E}_s$ and magnetic $\overline{H}_s$ fields. Then, find the time-average Poyting vector.

$$\boxed{\overline{E}_s = \hat{a}_\theta \frac{E_o}{r}\sin\theta\, e^{-j\frac{\omega}{c}r}\, \underbrace{e^{-j\pi/2}}_{-j}}$$

$$\boxed{\overline{H}_s = \hat{a}_\phi \frac{E_o}{\eta r}\sin\theta\, e^{-j\frac{\omega}{c}r}\, \underbrace{e^{-j\pi/2}}_{-j}}$$

Find time-average Poynting vector per (10.78)

$$\overline{P}_{ave} = \frac{1}{2}\mathrm{Re}\{\overline{E}_s \times \overline{H}_s^*\} = \frac{1}{2}\mathrm{Re}\left[\hat{a}_\theta \frac{E_o}{r}\sin\theta\, e^{-j\frac{\omega}{c}r}(-j) \times \hat{a}_\phi \frac{E_o}{\eta r}\sin\theta\, e^{+j\frac{\omega}{c}r}(+j)\right]$$

$$= \frac{1}{2}\mathrm{Re}\left\{\hat{a}_r \frac{E_o^2}{\eta r^2}\sin^2\theta\, e^0\, \underbrace{(-j^2)}_{1}\right\} = \hat{a}_r \frac{E_o^2}{2\eta r^2}\sin^2\theta$$

Per (10.80), the time-average power radiated can be found by integrating over the area of a sphere of radius r around the antenna.

$$P_{ave} = \iint_S \overline{P}_{ave}\cdot d\overline{S} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \hat{a}_r \frac{E_o^2}{2\eta r^2}\sin^2\theta \cdot \hat{a}_r r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{E_o^2}{2\eta}\int_{\theta=0}^{\pi}\sin^3\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi = \frac{E_o^2}{2\eta}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]\Bigg|_{\theta=0}^{\pi} \phi\Bigg|_{\phi=0}^{2\pi}$$

$$= \frac{E_o^2}{2\eta}\left[(+1+1) + \left(-\frac{1}{3} - \frac{1}{3}\right)\right](2\pi) \Rightarrow \underline{\underline{P_{ave} = \frac{4\pi}{3}\frac{E_o^2}{\eta}}}$$