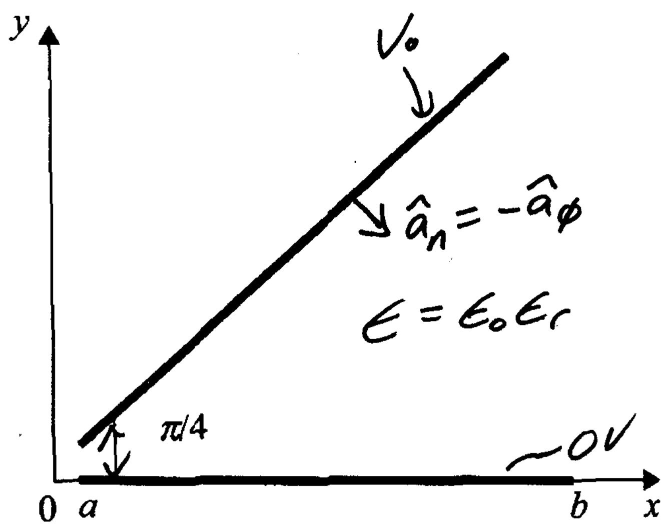
6.52 A capacitor consists of two plates with equal width $(b - a)$, and a length L in the z-direction. The plates are separated by $\phi = \pi/4$, as shown in Figure 6.43. Assume that the plates are separated by a dielectric material ($\varepsilon = \varepsilon_o \varepsilon_r$) and ignore fringing. Determine the capacitance.

- Assume $V(\phi = 0) = 0$ and $V(\phi = \pi/4) = V_0$. As part of the solution process, find expressions for V, $\bar{E}$, and $\bar{D}$ for $0 < \phi < \pi/4$. Hints: Use cylindrical coordinates. Remember dielectric-conductor boundary conditions.

Dielectric w/ no free charge mentioned $\Rightarrow \rho_v = 0$



Use Laplace's Eq'n (6.5) in cylindrical coord., assume $V(\phi=0)=0$ and $V(\phi=\pi/4)=V_0$.

$$\nabla^2 V = \frac{1}{\rho}\underbrace{\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial \rho}\right)}_{\to 0 \text{ No fringing}} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \underbrace{\frac{\partial^2 V}{\partial z^2}}_{\to 0 \text{ No fringing}} = 0$$

$$\frac{1}{\rho^2}\frac{d^2 V}{d\phi^2} = 0 \rightarrow \frac{d^2 V}{d\phi^2} = 0$$

$$\int \frac{d^2 V}{d\phi^2}\,d\phi = \int 0\,d\phi \rightarrow \frac{dV}{d\phi} = A$$

$$\hookrightarrow \int dV = \int A\,d\phi \rightarrow \underline{V = A\phi + B}$$

General sol'n

Apply boundary conditions

$$V(\phi=0) = 0 = A(0) + B \Rightarrow B = 0$$

$$V(\phi=\pi/4) = V_0 = A(\pi/4) + 0 \Rightarrow A = \frac{4V_0}{\pi}$$

$$V(\phi) = \frac{4V_0}{\pi}\phi = 1.27V_0\phi \quad 0 \le \phi \le \pi/4 \text{ between plates}$$

$$(4.76)\ \bar{E} = -\bar{\nabla}V = -\hat{a}_\rho \underbrace{\frac{\partial V}{\partial \rho}}_{0} - \hat{a}_\phi \frac{1}{\rho}\frac{\partial V}{\partial \phi} - \hat{a}_z \underbrace{\frac{\partial V}{\partial z}}_{0}$$

$$\bar{E} = -\hat{a}_\phi \frac{4V_0}{\pi\rho} = -\hat{a}_\phi \frac{1.27V_0}{\rho} \quad 0 < \phi < \pi/4 \text{ between plates}$$

$$\bar{D} = \epsilon\bar{E} = \epsilon_r\epsilon_0\left(-\hat{a}_\phi \frac{4V_0}{\pi\rho}\right)$$

$$\bar{D} = -\hat{a}_\phi \frac{4\epsilon_r\epsilon_0 V_0}{\pi\rho} = -\hat{a}_\phi \frac{1.27\epsilon V_0}{\rho} \quad 0 < \phi < \pi/4 \text{ between plates}$$

Per (5.72), $D_n = \rho_s$, @ $\phi = \pi/4$, $\hat{a}_n = -\hat{a}_\phi$

Therefore, @ $\phi = \pi/4$, $D_n = \rho_s = -\hat{a}_\phi \cdot -\hat{a}_\phi \frac{4\epsilon_r\epsilon_0 V_0}{\pi\rho}$

$$= \frac{4\epsilon_r\epsilon_0 V_0}{\pi\rho}$$

Calculate Q on $\phi = \pi/4$ plate

$$Q = \iint_S \rho_s\, ds = \int_{z=0}^{L}\int_{\rho=a}^{b} \frac{4\epsilon_r\epsilon_0 V_0}{\pi\rho}\, d\rho\, dz$$

$$= \frac{4\epsilon_r\epsilon_0 V_0}{\pi}(z)\Big|_0^L (\ln\rho)\Big|_a^b = \frac{4\epsilon_r\epsilon_0 V_0}{\pi} L\ln(b/a)$$

$$C = \frac{Q}{V} = \frac{4\epsilon_r\epsilon_0 V_0 L\ln(b/a)}{\pi V_0} \Rightarrow C = \frac{4\epsilon_r\epsilon_0 L\ln(b/a)}{\pi}$$