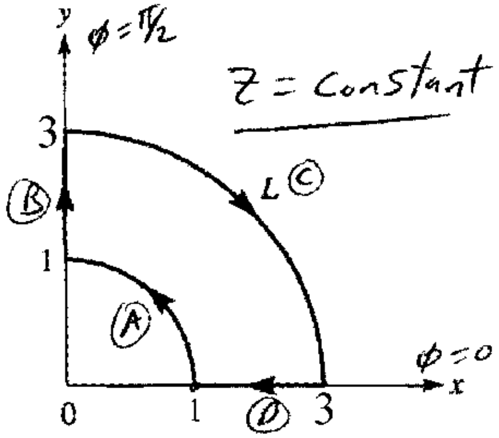
3.45 Let $\mathbf{A} = \rho \sin\phi\, \mathbf{a}_\rho + \rho^2 \mathbf{a}_\phi$ **(ants/m)**; evaluate $\oint_L \mathbf{A}\cdot d\mathbf{l}$ if L is the contour of Fig 3.31.

- modified so L bounds the area $1 \le \rho \le 3$, $0 \le \phi \le \pi/2$. Verify Stoke's theorem by evaluating circulation using both sides of Stoke's theorem.



Verify Stoke's Theorem

$$\oint_L \bar{A}\cdot d\bar{\ell} = \int_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s}$$

work in cylindrical coordinates

$$\oint_L \bar{A}\cdot d\bar{\ell} = \int_A \bar{A}\cdot d\bar{\ell} + \int_B \bar{A}\cdot d\bar{\ell} + \int_C \bar{A}\cdot d\bar{\ell} + \int_D \bar{A}\cdot d\bar{\ell}$$

$$\bar{A}\cdot d\bar{\ell} = (\rho\sin\phi\,\hat{a}_\rho + \rho^2\hat{a}_\phi)\cdot(d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z)$$
$$= \rho\sin\phi\, d\rho + \rho^3 d\phi + 0 \text{ (ants)}$$

$$\int_{A} \bar{A}\cdot d\bar{\ell} = \int_{\phi=0}^{\pi/2} (\rho\sin\phi\, \cancelto{0}{d\rho} + 1^3 d\phi) = \phi\Big|_0^{\pi/2} = \underline{\pi/2}$$
$\rho = 1,\ \phi = 0 \text{ to } \pi/2$

$$\int_{B} \bar{A}\cdot d\bar{\ell} = \int_{\rho=1}^{3} (\rho\underbrace{\sin\tfrac{\pi}{2}}_{1} d\rho + \rho^3 \cancelto{0}{d\phi}) = \left(\frac{\rho^2}{2}\right)\Big|_{\rho=1}^{3} = \left(\frac{3^2}{2} - \frac{1^2}{2}\right) = \underline{4}$$
$\phi = \pi/2,\ \rho = 1 \text{ to } 3$

$$\int_{C} \bar{A}\cdot d\bar{\ell} = \int_{\phi=\pi/2}^{0} (\rho\sin\phi\, \cancelto{0}{d\rho} + 3^3 d\phi) = 27\phi\Big|_{\phi=\pi/2}^{0} = 27(0 - \pi/2) = \underline{-13.5\pi}$$
$\rho = 3,\ \phi = \pi/2 \text{ to } 0$

$$\int_{D} \bar{A}\cdot d\bar{\ell} = \int_{\rho=3}^{1} (\rho\underbrace{\sin\phi}_{0} d\rho + \rho^3 \cancelto{0}{d\phi}) = \underline{0}$$
$\phi = 0,\ \rho = 3 \text{ to } 1$

$$\oint_L \bar{A}\cdot d\bar{\ell} = \pi/2 + 4 - 13.5\pi + 0 \Rightarrow \underline{\underline{\oint_L \bar{A}\cdot d\bar{\ell} = -36.841 \text{ (ants)}}}$$

Next, compute curl of $\bar{A}$ ($\bar{\nabla} \times \bar{A}$) and the RHS of Stoke's Theorem

$$\bar{\nabla} \times \bar{A} = \left[\frac{1}{\rho}\cancelto{0}{\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z}\right]\hat{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \cancelto{0}{\frac{\partial A_z}{\partial \rho}}\right]\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\hat{a}_z$$

$$= -\cancelto{0}{\frac{\partial(\rho^2)}{\partial z}}\hat{a}_\rho + \cancelto{0}{\frac{\partial(\rho \sin\phi)}{\partial z}}\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho^3)}{\partial \rho} - \frac{\partial(\rho \sin\phi)}{\partial \phi}\right]\hat{a}_z$$

$$= \frac{1}{\rho}\left[\rho^2 - \rho\cos\phi\right]\hat{a}_z$$

$$\underline{\bar{\nabla} \times \bar{A} = (3\rho - \cos\phi)\hat{a}_z \text{ (ants/m}^2\text{)}}$$

For $\iint_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{s}$ by RHR $d\bar{s} = -d\bar{s}_z = -\hat{a}_z \rho \, d\rho \, d\phi$

$$\iint_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{s} = \iint_S (3\rho - \cos\phi)\hat{a}_z \cdot -\hat{a}_z \rho \, d\rho \, d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\rho=1}^{3} (\rho\cos\phi - 3\rho^2)\, d\rho \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \left(\frac{\rho^2\cos\phi}{2} - \rho^3\right)\Bigg|_{\rho=1}^{3} d\phi$$

$$= \int_{\phi=0}^{\pi/2} \left(\frac{8\cos\phi}{2} - 26\right) d\phi$$

$$= \left(\frac{8\sin\phi}{2} - 26\phi\right)\Bigg|_{\phi=0}^{\pi/2} = \left(\frac{8(1)}{2} - \frac{26\pi}{2}\right) - (0 - 0)$$

$$\underline{\underline{\iint_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{s} = -36.8407 \text{ (ants)} = \oint \bar{A} \cdot d\bar{l}}}$$ $\therefore$ Same!