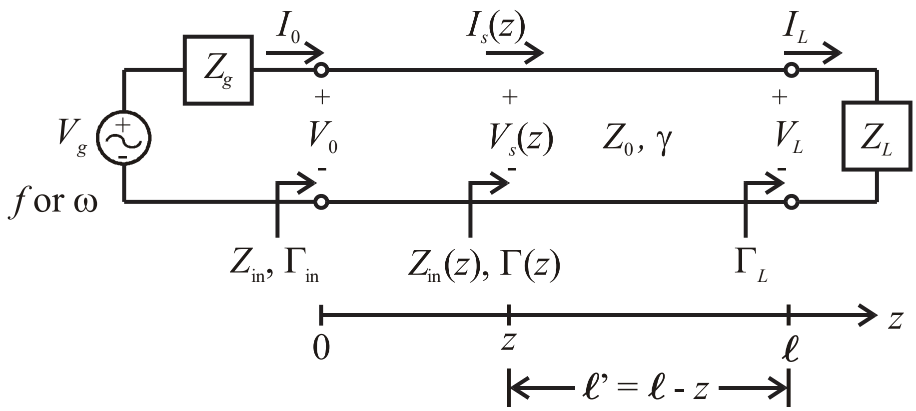
Lossy Transmission Lines (TLs)



Propagation constant (1/m): $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$

Attenuation constant (np/m): $\alpha = \mathrm{Re}(\gamma)$

Phase constant (rad/m): $\beta = \mathrm{Im}(\gamma) = \dfrac{\omega}{u} = \dfrac{2\pi}{\lambda}$

Phase velocity (m/s): $u = f\lambda = \dfrac{\omega}{\beta}$ where f is frequency (Hz) and ω is radian frequency (rad/s).

Wavelength (m): $\lambda = \dfrac{u}{f} = \dfrac{2\pi}{\beta}$

Characteristic impedance (Ω): $Z_0 = \dfrac{V_0^+}{I_0^+} = \dfrac{-V_0^-}{I_0^-} = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}} = \dfrac{R + j\omega L}{\gamma} = \dfrac{\gamma}{G + j\omega C} = R_0 + jX_0 = |Z_0| \angle \theta_{Z0}$

Phasor voltage: $V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} = V_0^+ e^{-\gamma z}[1 + \Gamma(z)]$ for $0 \le z \le \ell$.

Phasor current: $I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \dfrac{V_0^+}{Z_0} e^{-\gamma z} - \dfrac{V_0^-}{Z_0} e^{+\gamma z} = \dfrac{V_0^+}{Z_0} e^{-\gamma z}[1 - \Gamma(z)]$ for $0 \le z \le \ell$.

where $V_0^+ = |V_0^+| \angle \theta_V^+$, $V_0^- = |V_0^-| \angle \theta_V^-$, $I_0^+ = |I_0^+| \angle \theta_I^+$, and $I_0^- = |I_0^+| \angle \theta_I^+$ are complex coefficients representing the amplitudes of the forward and backward traveling voltages and currents.

Forward and backward traveling voltages (V) and currents (A):

Note that, at the input (z = 0), $V_S(0) = V_0 = V_0^+ + V_0^-$ and $I_S(0) = I_0 = I_0^+ + I_0^- = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}$

$$V_0^+ = 0.5(V_0 + I_0 Z_0) = \frac{V_0}{1 + \Gamma(0)} = 0.5(V_L + I_L Z_0)\, e^{+\gamma\ell} \quad \text{and} \quad I_0^+ = \frac{V_0^+}{Z_0}$$

$$V_0^- = 0.5(V_0 - I_0 Z_0) = V_0 - V_0^+ = 0.5(V_L - I_L Z_0)\, e^{-\gamma\ell} = V_0^+ \Gamma(0) \quad \text{and} \quad I_0^- = \frac{-V_0^-}{Z_0}.$$

Time-domain voltage: $V(z,t) = |V_0^+|\, e^{-\alpha z} \cos(\omega t - \beta z + \theta_V^+) + |V_0^-|\, e^{\alpha z} \cos(\omega t + \beta z + \theta_V^-)$ for $0 \le z \le \ell$.

Time-domain current:

$$\begin{aligned} I(z,t) &= |I_0^+|\, e^{-\alpha z} \cos(\omega t - \beta z + \theta_I^+) + |I_0^-|\, e^{\alpha z} \cos(\omega t + \beta z + \theta_I^-) \\ &= \frac{|V_0^+|}{|Z_0|} e^{-\alpha z} \cos(\omega t - \beta z + \theta_V^+ - \theta_{Z0}) - \frac{|V_0^-|}{|Z_0|} e^{\alpha z} \cos(\omega t + \beta z + \theta_V^- - \theta_{Z0}) \end{aligned} \quad 0 \le z \le \ell.$$

Reflection coefficient (unitless): $\Gamma(z) = \dfrac{V_{\text{ref}}(z)}{V_{\text{inc}}(z)} = \dfrac{-I_{\text{ref}}(z)}{I_{\text{inc}}(z)} = \dfrac{Z_{\text{in}}(z) - Z_0}{Z_{\text{in}}(z) + Z_0} = \Gamma_L\, e^{-2\gamma(l-z)} = \Gamma_{\text{in}}\, e^{2\gamma z}$ for $0 \le z \le \ell$.

of particular interest are: **load reflection coefficient** $\Gamma_L = \dfrac{V_0^-}{V_0^+} e^{+2\gamma\ell} = \Gamma(0)\, e^{+2\gamma\ell} = \Gamma_{\text{in}}\, e^{+2\gamma\ell} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

and

input reflection coefficient $\Gamma_{\text{in}} = \Gamma(0) = \dfrac{V_0^-}{V_0^+} = \dfrac{Z_{\text{in}}(0) - Z_0}{Z_{\text{in}}(0) + Z_0} = \dfrac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \Gamma_L\, e^{-2\gamma\ell}$.

Note: $0 \le |\Gamma(z)| \le 1$ for passive loads and transmission lines.

Input impedance (Ω): $Z_{\text{in}}(z) = \dfrac{V_s(z)}{I_s(z)} = Z_0\left[\dfrac{1+\Gamma(z)}{1-\Gamma(z)}\right] = Z_0\left[\dfrac{Z_L + Z_0 \tanh\left[\gamma(\ell - z)\right]}{Z_0 + Z_L \tanh\left[\gamma(\ell - z)\right]}\right]$ for $0 \le z \le \ell$.

at input $z = 0$: $Z_{\text{in}}(0) = Z_{\text{in}} = \dfrac{V_0}{I_0} = Z_0\left[\dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right] = Z_0\left[\dfrac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}\right] = Z_0\left[\dfrac{1+\Gamma(0)}{1-\Gamma(0)}\right] = Z_0\left[\dfrac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}\right]$

at load $z = \ell$: $Z_{\text{in}}(l) = Z_L = \dfrac{V_L}{I_L} = Z_0\left[\dfrac{1+\Gamma_L}{1-\Gamma_L}\right]$

Standing wave ratio* (unitless): $SWR = S = \dfrac{V_{\max}}{V_{\min}} = \dfrac{I_{\max}}{I_{\min}} = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|}$ and $1 \le S < \infty$ for passive loads.

[* Defined at **load** as a measure of mismatch. Really only meaningful for lossless/low loss &/or short TLs.]

Power (W):

$$P_{\text{ave}}(z) = 0.5\,\text{Re}\left\{V_s(z)\, I_s^*(z)\right\} \qquad 0 \le z \le \ell$$

$$= 0.5\frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha z}\cos(\theta_{Z0}) - 0.5\frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha z}\cos(\theta_{Z0})\left|\Gamma(z)\right|^2 - \frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha z}\sin(\theta_{Z0})\,\text{Im}\left(\Gamma(z)\right)$$

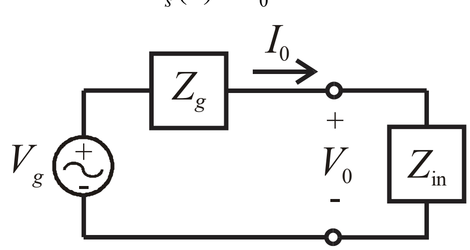
forward traveling power backward traveling power cross coupled power (usually very small)

Input Power (W): $P_{\text{ave}}(0) = P_{\text{in}} = 0.5\,\text{Re}\left\{V_0\, I_0^*\right\} = 0.5\dfrac{|V_0^+|^2}{|Z_0|}\cos(\theta_{Z0})\left[1 - |\Gamma_{\text{in}}|^2\right] - \dfrac{|V_0^+|^2}{|Z_0|}\sin(\theta_{Z0})\,\text{Im}(\Gamma_{\text{in}})$

Load Power (W):

$$P_{\text{ave}}(\ell) = P_L = 0.5\,\text{Re}\left\{V_L\, I_L^*\right\} = 0.5\frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha\ell}\cos(\theta_{Z0})\left[1 - |\Gamma_L|^2\right] - \frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha\ell}\sin(\theta_{Z0})\,\text{Im}\left(\Gamma_L\right)$$

The input phasor voltage $V_s(0) = V_0$ and current $I_s(0) = I_0$ can be found using the equivalent circuit-



where, by circuit theory, $V_0 = V_g\left(\dfrac{Z_{\text{in}}}{Z_g + Z_{\text{in}}}\right)$ and $I_0 = \dfrac{V_g}{Z_g + Z_{\text{in}}}$.