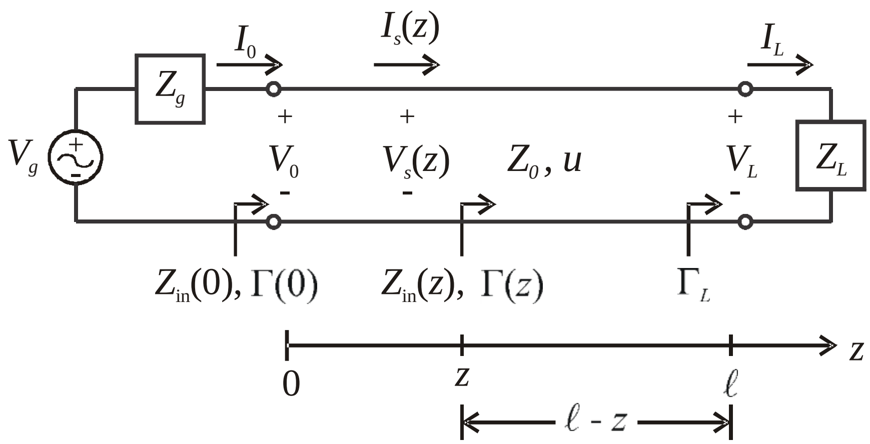
Lossless Transmission Lines



Phase velocity: $u = f\lambda = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$ (m/s)

where f is the frequency (Hz), ω is the radian frequency (rad/s), and L and C are the per-unit-length inductance and capacitance.

Phase constant: $\beta = \dfrac{\omega}{u} = \omega\sqrt{LC} = \dfrac{2\pi}{\lambda}$ (rad/m)

Wavelength: $\lambda = \dfrac{u}{f} = \dfrac{2\pi}{\beta}$ (m)

Characteristic impedance: $Z_0 = \dfrac{V_0^+}{I_0^+} = -\dfrac{V_0^-}{I_0^-} = \sqrt{\dfrac{L}{C}}$ (Ω)

Phasor voltage:
$$V_S(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \quad \text{(V)}$$
$$= V_0^+ e^{-j\beta z}[1+\Gamma(z)] \quad \text{(V)}$$

Phasor current:
$$I_S(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} \quad \text{(A)}$$
$$= \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z} \text{(A)}$$
$$= \frac{V_0^+}{Z_0} e^{-j\beta z}[1-\Gamma(z)] \quad \text{(A)}$$

where $V_0^+ = |V_0^+|\angle\theta^+, V_0^- = |V_0^-|\angle\theta^-, I_0^+$, and I_0^- are complex coefficients representing the forward and backward traveling voltages and currents.

Forward and backward traveling voltages and currents:

Note that, at the input ($z=0$), $V_S(0) = V_0 = V_0^+ + V_0^-$ and $I_S(0) = I_0 = I_0^+ + I_0^- = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}$

$$V_0^+ = 0.5(V_0 + I_0 Z_0) = \frac{V_0}{1+\Gamma(0)} = 0.5(V_L + I_L Z_0)\, e^{+j\beta l} \text{ and } I_0^+ = \frac{V_0^+}{Z_0}$$

$$V_0^- = 0.5(V_0 - I_0 Z_0) = V_0 - V_0^+ = 0.5(V_L - I_L Z_0)\, e^{-j\beta l} \text{ and } I_0^- = \frac{-V_0^-}{Z_0}.$$

Time-domain voltage: $V(z,t) = |V_0^+|\cos(\omega t - \beta z + \theta^+) + |V_0^-|\cos(\omega t + \beta z + \theta^-)$ (V)

Time-domain current: $I(z,t) = \dfrac{|V_0^+|}{Z_0}\cos(\omega t - \beta z + \theta^+) - \dfrac{|V_0^-|}{Z_0}\cos(\omega t + \beta z + \theta^-)$ (A)

Reflection Coefficient: $\Gamma(z) = \dfrac{V_{\text{ref}}(z)}{V_{\text{inc}}(z)} = -\dfrac{I_{\text{ref}}(z)}{I_{\text{inc}}(z)} = \dfrac{Z_{\text{in}}(z) - Z_0}{Z_{\text{in}}(z) + Z_0} = \Gamma_L\, e^{-j2\beta(l-z)}$,

of particular interest are: load reflection coefficient $\Gamma_L = \dfrac{V_0^-}{V_0^+} e^{+j2\beta l} = \Gamma(0)e^{+j2\beta l} = \Gamma_{\text{in}} e^{+j2\beta l} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

and

input reflection coefficient $\Gamma_{\text{in}} = \Gamma(0) = \dfrac{Z_{\text{in}}(0) - Z_0}{Z_{\text{in}}(0) + Z_0} = \dfrac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \Gamma_L\, e^{-j2\beta l}$.

Notes: $|\Gamma(z)|$ is constant and $0 \le |\Gamma(z)| \le 1$ for passive loads.

Input impedance: $Z_{\text{in}}(z) = \dfrac{V(z)}{I(z)} = Z_0\left[\dfrac{Z_L + jZ_0\tan[\beta(\ell - z)]}{Z_0 + jZ_L\tan[\beta(\ell - z)]}\right] = Z_0\left[\dfrac{1+\Gamma(z)}{1-\Gamma(z)}\right]$ for $0 \le z \le l$

at input $z = 0$: $Z_{\text{in}}(0) = Z_{\text{in}} = \dfrac{V(0)}{I(0)} = \dfrac{V_0}{I_0} = Z_0\left[\dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right] = Z_0\left[\dfrac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}\right] = Z_0\left[\dfrac{1+\Gamma(0)}{1-\Gamma(0)}\right] = Z_0\left[\dfrac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}\right]$

at load $z = l$: $Z_{\text{in}}(l) = Z_L = \dfrac{V_L}{I_L} = Z_0\left[\dfrac{1+\Gamma_L}{1-\Gamma_L}\right]$

Standing wave ratio: $SWR = S = \dfrac{V_{\max}}{V_{\min}} = \dfrac{I_{\max}}{I_{\min}} = \dfrac{1+|\Gamma|}{1-|\Gamma|}$ and $1 \le S < \infty$ for passive loads.

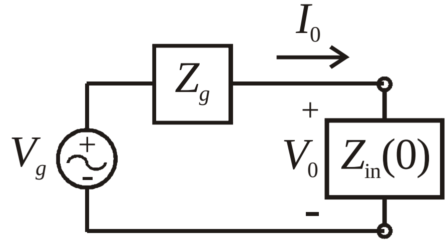
$$P_{\text{ave}}(z) = 0.5\,\text{Re}\{V(z)I^*(z)\} = 0.5\frac{|V_0^+|^2}{Z_0}\left[1 - |\Gamma(z)|^2\right] \quad \text{(W)}$$

Input Power: $= P_{\text{ave}}(0) = 0.5\,\text{Re}\{V_0 I^*{}_0\} = 0.5\dfrac{|V_0^+|^2}{Z_0}\left[1 - |\Gamma(0)|^2\right]$ (W)

$= P_{\text{ave}}(l) = 0.5\,\text{Re}\{V_L I^*{}_L\}$ (W)

Note: Time-average real power is <u>constant</u> on a lossless transmission line.

The input phasor voltage $V_S(0) = V_0$ and current $I_S(0) = I_0$ can be found using the equivalent circuit-



$V_0 = V_g\left(\dfrac{Z_{\text{in}}(0)}{Z_g + Z_{\text{in}}(0)}\right)$ and $I_0 = \dfrac{V_g}{Z_g + Z_{\text{in}}(0)}$.