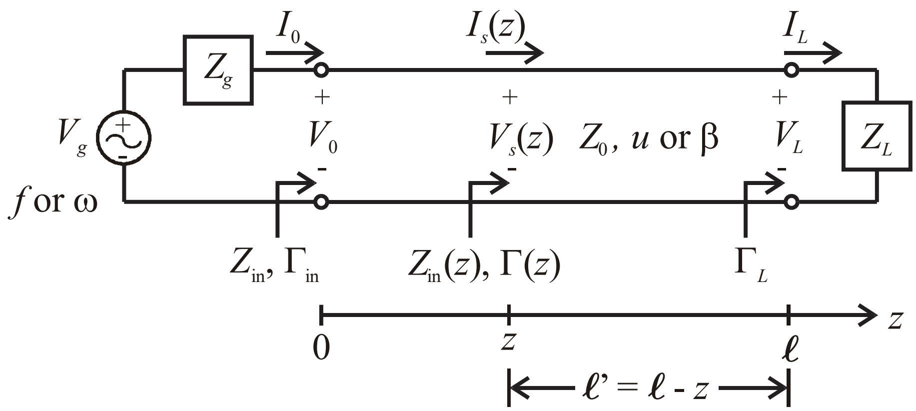
Lossless Transmission Lines (TLs)



Phase velocity (m/s): $u = f\lambda = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$

where f is frequency (Hz), ω is radian frequency (rad/s), and L & C are the per-unit-length inductance & capacitance.

Phase constant (rad/m): $\beta = \dfrac{\omega}{u} = \dfrac{2\pi}{\lambda} = \omega\sqrt{LC}$

Wavelength (m): $\lambda = \dfrac{u}{f} = \dfrac{2\pi}{\beta}$

Characteristic impedance (Ω): $Z_0 = \dfrac{V_0^+}{I_0^+} = \dfrac{-V_0^-}{I_0^-} = \sqrt{\dfrac{L}{C}} = R_0$ (real positive value)

Phasor voltage: $V_s(z) = V_0^+\, e^{-j\beta z} + V_0^-\, e^{+j\beta z} = V_0^+\, e^{-j\beta z}[1+\Gamma(z)]$ for $0 \le z \le \ell$.

Phasor current: $I_s(z) = I_0^+\, e^{-j\beta z} + I_0^-\, e^{+j\beta z} = \dfrac{V_0^+}{Z_0} e^{-j\beta z} - \dfrac{V_0^-}{Z_0}\, e^{+j\beta z} = \dfrac{V_0^+}{Z_0} e^{-j\beta z}[1-\Gamma(z)]$ for $0 \le z \le \ell$.

where $V_0^+ = |V_0^+|\angle\theta_V^+, V_0^- = |V_0^-|\angle\theta_V^-, I_0^+ = |I_0^+|\angle\theta_I^+$, and $I_0^- = |I_0^+|\angle\theta_I^+$ are complex coefficients representing the amplitudes of the forward and backward traveling voltages and currents.

Forward and backward traveling voltages (V) and currents (A):

Note that, at the input (z = 0), $V_S(0) = V_0 = V_0^+ + V_0^-$ and $I_S(0) = I_0 = I_0^+ + I_0^- = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}$

$$V_0^+ = 0.5(V_0 + I_0 Z_0) = \frac{V_0}{1+\Gamma(0)} = 0.5(V_L + I_L Z_0)\, e^{+j\beta l} \quad \text{and} \quad I_0^+ = \frac{V_0^+}{Z_0}$$

$$V_0^- = 0.5(V_0 - I_0 Z_0) = V_0 - V_0^+ = 0.5(V_L - I_L Z_0)\, e^{-j\beta l} = V_0^+\,\Gamma(0) \quad \text{and} \quad I_0^- = \frac{-V_0^-}{Z_0}.$$

Time-domain voltage: $V(z,t) = |V_0^+|\cos(\omega t - \beta z + \theta_V^+) + |V_0^-|\cos(\omega t + \beta z + \theta_V^-)$ for $0 \le z \le \ell$.

Time-domain current:

$$\begin{aligned} I(z,t) &= |I_0^+|\cos(\omega t - \beta z + \theta_I^+) + |I_0^-|\cos(\omega t + \beta z + \theta_I^-) \\ &= \frac{|V_0^+|}{Z_0}\cos(\omega t - \beta z + \theta_V^+) - \frac{|V_0^-|}{Z_0}\cos(\omega t + \beta z + \theta_V^-) \end{aligned}$$ for $0 \le z \le \ell$.

Reflection coefficient (unitless): $\Gamma(z) = \dfrac{V_{\text{ref}}(z)}{V_{\text{inc}}(z)} = \dfrac{-I_{\text{ref}}(z)}{I_{\text{inc}}(z)} = \dfrac{Z_{\text{in}}(z) - Z_0}{Z_{\text{in}}(z) + Z_0} = \Gamma_L\, e^{-j2\beta(l-z)} = \Gamma_{in}\, e^{j2\beta z}$ for $0 \le z \le \ell$.

of particular interest are: **load reflection coefficient** $\Gamma_L = \dfrac{V_0^-}{V_0^+} e^{+j2\beta l} = \Gamma(0)\, e^{+j2\beta l} = \Gamma_{\text{in}} e^{+j2\beta l} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

and

input reflection coefficient $\Gamma_{\text{in}} = \Gamma(0) = \dfrac{Z_{\text{in}}(0) - Z_0}{Z_{\text{in}}(0) + Z_0} = \dfrac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \Gamma_L\, e^{-j2\beta l}$.

Notes: $|\Gamma(z)|$ is constant and $0 \le |\Gamma(z)| \le 1$ for passive loads and transmission lines.

Input impedance (Ω): $Z_{\text{in}}(z) = \dfrac{V_s(z)}{I_s(z)} = Z_0\left[\dfrac{Z_L + jZ_0 \tan\left[\beta(\ell - z)\right]}{Z_0 + jZ_L \tan\left[\beta(\ell - z)\right]}\right] = Z_0\left[\dfrac{1+\Gamma(z)}{1-\Gamma(z)}\right]$ for $0 \le z \le \ell$.

at input $z = 0$: $Z_{\text{in}}(0) = Z_{\text{in}} = \dfrac{V_0}{I_0} = Z_0\left[\dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right] = Z_0\left[\dfrac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}\right] = Z_0\left[\dfrac{1+\Gamma(0)}{1-\Gamma(0)}\right] = Z_0\left[\dfrac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}\right]$

at load $z = l$: $Z_{\text{in}}(l) = Z_L = \dfrac{V_L}{I_L} = Z_0\left[\dfrac{1+\Gamma_L}{1-\Gamma_L}\right]$

Standing wave ratio (unitless): $SWR = S = \dfrac{V_{\text{max}}}{V_{\text{min}}} = \dfrac{I_{\text{max}}}{I_{\text{min}}} = \dfrac{1+|\Gamma|}{1-|\Gamma|}$ and $1 \le S < \infty$ for passive loads.

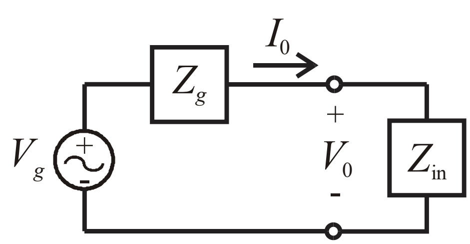
Power (W): $P_{\text{ave}}(z) = 0.5\,\text{Re}\left\{V(z)\, I^*(z)\right\} = 0.5\dfrac{|V_0^+|^2}{Z_0}\left[1 - |\Gamma(z)|^2\right]$ for $0 \le z \le \ell$.

Input Power (W): $P_{\text{ave}}(0) = P_{\text{in}} = 0.5\,\text{Re}\left\{V_0\, I_0^*\right\} = 0.5\dfrac{|V_0^+|^2}{Z_0}\left[1 - |\Gamma_{\text{in}}|^2\right]$

Load Power (W): $P_{\text{ave}}(\ell) = P_L = 0.5\,\text{Re}\left\{V_L\, I_L^*\right\} = 0.5\dfrac{|V_0^+|^2}{Z_0}\left[1 - |\Gamma_L|^2\right]$

Note: Time-average real power is <u>constant</u> on a lossless transmission line, i.e., $P_{\text{ave}}(z) = P_{\text{in}} = P_L$.

The input phasor voltage $V_s(0) = V_0$ and current $I_s(0) = I_0$ can be found using the equivalent circuit-



where, by circuit theory, $V_0 = V_g\left(\dfrac{Z_{\text{in}}}{Z_g + Z_{\text{in}}}\right)$ and $I_0 = \dfrac{V_g}{Z_g + Z_{\text{in}}}$.