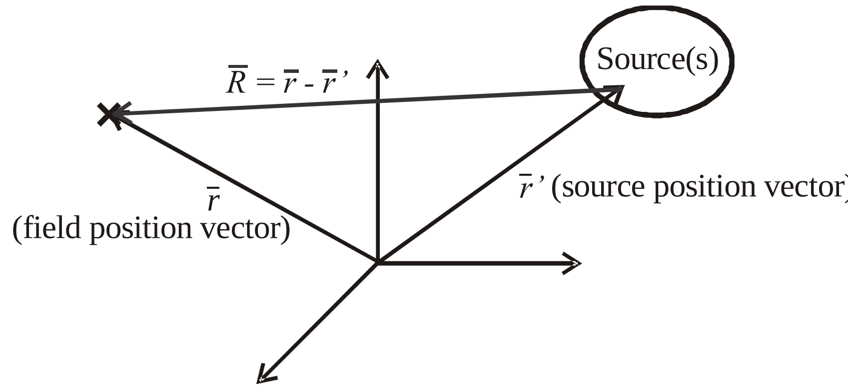
## Chapter 1 Vector Algebra

Scalars- magnitude  Vectors- direction & magnitude

Position vectors- point from origin to a field point ( $\bar{r}$ ) or to a source point ( $\bar{r}'$ )

Distance vectors- usually point from a source point to a field point $\bar{R} = \bar{r} - \bar{r}'$

The distance between the source and the field point is $R = |\bar{R}| = |\bar{r} - \bar{r}'|$




Scalar/dot product: $\bar{A} \bullet \bar{B} = |\bar{A}||\bar{B}|\cos(\theta_{AB}) = \bar{B} \bullet \bar{A}$

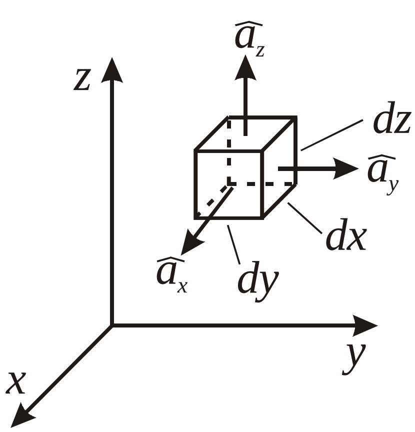
Vector/cross product: $\bar{A} \times \bar{B} = \hat{a}_n |\bar{A}||\bar{B}|\sin(\theta_{AB})$ where $\hat{a}_n$ determined by RHR. Also, $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$.

Cross product hint: Remember order of ($x$, $y$, $z$), ($\rho$, $\phi$, $z$), and ($r$, $\theta$, $\phi$)

Unit vector $\hat{a}_A = \dfrac{\bar{A}}{|\bar{A}|}$ where the vector magnitude is $|\bar{A}| = \sqrt{\bar{A} \bullet \bar{A}} = \sqrt{|\bar{A}|^2}$.

Scalar projection of vector $\bar{A}$ onto $\bar{B}$ is $A_B = \bar{A} \bullet \hat{a}_B$. The vector projection of vector $\bar{A}$ onto $\bar{B}$ is $\bar{A}_B = \hat{a}_B \left( \bar{A} \bullet \hat{a}_B \right)$.

## Chapter 2 Coordinate Systems and Transformation & Chapter 3 Vector Calculus




Rectangular/Cartesian Coordinates ($x$,$y$,$z$) $\bar{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$d\bar{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$d\bar{s}_x = dy\,dz\,\hat{a}_x \quad d\bar{s}_y = dx\,dz\,\hat{a}_y \quad d\bar{s}_z = dx\,dy\,\hat{a}_z$

$dv = dx\,dy\,dz$

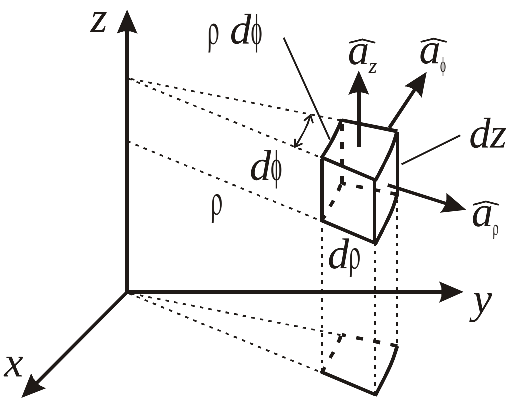
$\bar{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$

$$\bar{\nabla}\Phi = \hat{a}_x \frac{\partial \Phi}{\partial x} + \hat{a}_y \frac{\partial \Phi}{\partial y} + \hat{a}_z \frac{\partial \Phi}{\partial z}$$

$$\bar{\nabla} \bullet \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\bar{\nabla} \times \bar{A} = \hat{a}_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{a}_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{a}_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$




Cylindrical Coordinates $(\rho,\phi,z)$ $\bar{A} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z$

$d\bar{l} = d\rho\,\hat{a}_\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$

$d\bar{s}_\rho = \rho\, d\phi dz\,\hat{a}_\rho \qquad d\bar{s}_\phi = d\rho\, dz\,\hat{a}_\phi \qquad d\bar{s}_z = \rho\, d\rho\, d\phi\,\hat{a}_z$

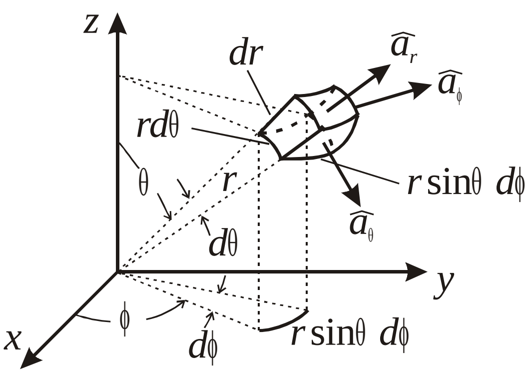
$dv = \rho\, d\rho\, d\phi\, dz$

$\bar{r} = \rho\,\hat{a}_\rho + z\,\hat{a}_z$ (Note: $\hat{a}_\rho$ has $\phi$ dependence)

$$\bar{\nabla}\Phi = \hat{a}_\rho \frac{\partial \Phi}{\partial \rho} + \hat{a}_\phi \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} + \hat{a}_z \frac{\partial \Phi}{\partial z}$$

$$\bar{\nabla}\bullet\bar{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\bar{\nabla}\times\bar{A} = \hat{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{a}_z\left(\frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$




Spherical Coordinates $(r,\theta,\phi)$ $\bar{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$

$d\bar{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi$

$d\bar{s}_r = r^2\sin\theta\, d\theta\, d\phi\,\hat{a}_r, \qquad d\bar{s}_\theta = r\sin\theta\, dr\, d\phi\,\hat{a}_\theta$

$d\bar{s}_\phi = r\, dr\, d\theta\,\hat{a}_\phi \qquad dv = r^2\sin\theta\, dr\, d\theta\, d\phi$

$\bar{r} = r\,\hat{a}_r$ (Note: $\hat{a}_r$ has both $\theta$ & $\phi$ dependence)

$$\bar{\nabla}\Phi = \hat{a}_r\frac{\partial\Phi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\bar{\nabla}\bullet\bar{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\bar{\nabla}\times\bar{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial\left(\sin\theta A_\phi\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{a}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial\left(rA_\phi\right)}{\partial r}\right] + \hat{a}_\phi\frac{1}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

Gradient $\bar{\nabla}\Phi$ - vector representing magnitude & direction of maximum spatial rate of change of a scalar field

Divergence $\bar{\nabla}\bullet\bar{A}$ - net outward flux per unit volume of a vector field (scalar)

Flux of $\bar{A}$ through surface $s$ is $\int_s \bar{A}\bullet d\bar{s}$

Divergence Theorem- $\oint_s \bar{A}\bullet d\bar{s} = \int_v \bar{\nabla}\bullet\bar{A}\, dv$

Curl $\bar{\nabla}\times\bar{A}$ - circulation of a vector field per unit area w/ area oriented for maximize (vector whose direction is normal to surface by RHR

Circulation of $\bar{A}$ around contour $c$ is $\oint_c \bar{A}\bullet d\bar{l}$

Stoke's Theorem- $\oint_c \bar{A}\bullet d\bar{l} = \int_s (\bar{\nabla}\times\bar{A})\bullet d\bar{s}$

Vector field classification- $\bar{\nabla}\bullet\bar{A} = 0 \Rightarrow$ divergenceless/solenoidal; $\bar{\nabla}\times\bar{A} = 0 \Rightarrow$ irrotational & conservative since $\oint_c \bar{A}\bullet d\bar{l} = \int_s (\bar{\nabla}\times\bar{A})\bullet d\bar{s} = 0$

## Cartesian Coordinates $(x,y,z)$ $\Leftrightarrow$ Cylindrical Coordinates $(\rho,\phi,z)$

**Point/variable conversions:**

$$\rho=\sqrt{x^2+y^2} \qquad x=\rho\cos\phi \qquad \cos\phi=\frac{x}{\sqrt{x^2+y^2}}$$

$$\phi=\tan^{-1}\left(\frac{y}{x}\right) \qquad y=\rho\sin\phi \qquad \sin\phi=\frac{y}{\sqrt{x^2+y^2}}$$

$$z=z \qquad z=z$$

**Vector conversions:** $\bar{A}=\hat{a}_\rho A_\rho+\hat{a}_\phi A_\phi+A_z\hat{a}_z=A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z$

$$\hat{a}_x=\cos\phi\,\hat{a}_\rho-\sin\phi\,\hat{a}_\phi \qquad \hat{a}_\rho=\cos\phi\,\hat{a}_x+\sin\phi\,\hat{a}_y$$

$$\hat{a}_y=\sin\phi\,\hat{a}_\rho+\cos\phi\,\hat{a}_\phi \qquad \hat{a}_\phi=-\sin\phi\,\hat{a}_x+\cos\phi\,\hat{a}_y \qquad \hat{a}_z=\hat{a}_z$$

$$A_\rho=A_x\cos\phi+A_y\sin\phi \qquad A_x=A_\rho\cos\phi-A_\phi\sin\phi$$

$$A_\phi=-A_x\sin\phi+A_y\cos\phi \qquad A_y=A_\rho\sin\phi+A_\phi\cos\phi \qquad A_z=A_z$$

**Dot Products:**

$$\hat{a}_x\bullet\hat{a}_\rho=\cos\phi \qquad \hat{a}_y\bullet\hat{a}_\rho=\sin\phi \qquad \hat{a}_z\bullet\hat{a}_\rho=0$$

$$\hat{a}_x\bullet\hat{a}_\phi=-\sin\phi \qquad \hat{a}_y\bullet\hat{a}_\phi=\cos\phi \qquad \hat{a}_z\bullet\hat{a}_\phi=0$$

$$\hat{a}_x\bullet\hat{a}_z=0 \qquad \hat{a}_y\bullet\hat{a}_z=0 \qquad \hat{a}_z\bullet\hat{a}_z=1$$

## Cartesian Coordinates $(x,y,z)$ $\Leftrightarrow$ Spherical Coordinates $(r,\ \theta,\ \phi)$

**Point/variable conversions:**

$$r=\sqrt{x^2+y^2+z^2} \qquad x=r\sin\theta\cos\phi \qquad \cos\phi=\frac{x}{\sqrt{x^2+y^2}} \qquad \cos\theta=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\phi=\tan^{-1}\frac{y}{x} \qquad y=r\sin\theta\sin\phi$$

$$\theta=\tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \qquad z=r\cos\theta \qquad \sin\phi=\frac{y}{\sqrt{x^2+y^2}} \qquad \sin\theta=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

**Vector conversions:** $\bar{A}=\hat{a}_r A_r+A_\theta\hat{a}_\theta+\hat{a}_\phi A_\phi=A_x\hat{a}_x+A_y\hat{a}_y+A_z\hat{a}_z$

$$\hat{a}_r=\sin\theta\cos\phi\,\hat{a}_x+\sin\theta\sin\phi\,\hat{a}_y+\cos\theta\,\hat{a}_z \qquad \hat{a}_x=\sin\theta\cos\phi\,\hat{a}_r+\cos\theta\cos\phi\,\hat{a}_\theta-\sin\phi\,\hat{a}_\phi$$

$$\hat{a}_\theta=\cos\theta\cos\phi\,\hat{a}_x+\cos\theta\sin\phi\,\hat{a}_y-\sin\theta\,\hat{a}_z \qquad \hat{a}_y=\sin\theta\sin\phi\,\hat{a}_r+\cos\theta\sin\phi\,\hat{a}_\theta+\cos\phi\,\hat{a}_\phi$$

$$\hat{a}_\phi=-\sin\phi\,\hat{a}_x+\cos\phi\,\hat{a}_y \qquad \hat{a}_z=\cos\theta\,\hat{a}_r-\sin\theta\,\hat{a}_\theta$$

$$A_r=A_x\sin\theta\cos\phi+A_y\sin\theta\sin\phi+A_z\cos\theta \qquad A_x=A_r\sin\theta\cos\phi+A_\theta\cos\theta\cos\phi-A_\phi\sin\phi$$

$$A_\theta=A_x\cos\theta\cos\phi+A_y\cos\theta\sin\phi-A_z\sin\theta \qquad A_y=A_r\sin\theta\sin\phi+A_\theta\cos\theta\sin\phi+A_\phi\cos\phi$$

$$A_\phi=-A_x\sin\phi+A_y\cos\phi \qquad A_z=A_r\cos\theta-A_\theta\sin\theta$$

**Dot Products:**

$$\hat{a}_r\bullet\hat{a}_x=\sin\theta\cos\phi \qquad \hat{a}_r\bullet\hat{a}_y=\sin\theta\sin\phi \qquad \hat{a}_r\bullet\hat{a}_z=\cos\theta$$

$$\hat{a}_\theta\bullet\hat{a}_x=\cos\theta\cos\phi \qquad \hat{a}_\theta\bullet\hat{a}_y=\cos\theta\sin\phi \qquad \hat{a}_\theta\bullet\hat{a}_z=-\sin\theta$$

$$\hat{a}_\phi\bullet\hat{a}_x=-\sin\phi \qquad \hat{a}_\phi\bullet\hat{a}_y=\cos\phi \qquad \hat{a}_\phi\bullet\hat{a}_z=0$$

## Chapter 4 Electrostatic Fields

permittivity of free space: $\varepsilon_0$ = 8.8541878 × $10^{-12}$ F/m

Coulomb's Law:

point charges $\bar{F}_{12} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{R12} = \dfrac{Q_1 Q_2(\bar{r}_2 - \bar{r}_1)}{4\pi\varepsilon_0 |\bar{r}_2 - \bar{r}_1|^3}$ (N) force on $Q_2$ due to $Q_1$ (opposites attract, like repel)

Electric Field $\bar{E} = \bar{F}/q = -\bar{\nabla}V$ is conservative (i.e., $\bar{\nabla}\times\bar{E} = 0$ and $\oint_c \bar{E}\bullet d\bar{l} = 0$).

point charges: $\bar{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \dfrac{Q(\bar{r}-\bar{r}')}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3}$ (V/m) & $\bar{E} = \sum_{k=1}^{N} \dfrac{Q_k(\bar{r}-\bar{r}_k)}{4\pi\varepsilon_0 |\bar{r}-\bar{r}_k|^3}$ (V/m)

line charge density: $\bar{E} = \int_{c'} \dfrac{\rho_l(\bar{r}-\bar{r}')\,dl'}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3}$ (V/m) where $\rho_l$ can be a function of position (i.e. $\bar{r}'$).

surface charge density: $\bar{E} = \iint_{s'} \dfrac{\rho_s(\bar{r}-\bar{r}')\,ds'}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3}$ (V/m) where $\rho_s$ can be a function of position (i.e. $\bar{r}'$).

volume charge density: $\bar{E} = \iiint_{v'} \dfrac{\rho_v(\bar{r}-\bar{r}')\,dv'}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|^3}$ (V/m) where $\rho_v$ can be a function of position (i.e. $\bar{r}'$).

Electric Flux Density: $\bar{D} = \varepsilon\bar{E} = \varepsilon_r\varepsilon_0\bar{E}$ (C/m$^2$) and Electric Flux: $\psi = \iint_s \bar{D}\bullet d\bar{s}$ (C)

point charges: $\bar{D} = \dfrac{Q}{4\pi R^2}\hat{a}_R = \dfrac{Q(\bar{r}-\bar{r}')}{4\pi |\bar{r}-\bar{r}_1'|^3}$ (C/m$^2$) and $\bar{D} = \sum_{k=1}^{N} \dfrac{Q_k(\bar{r}-\bar{r}_k)}{4\pi |\bar{r}-\bar{r}_k|^3}$ (C/m$^2$)

line charge density: $\bar{D} = \int_{c'} \dfrac{\rho_l(\bar{r}-\bar{r}')\,dl'}{4\pi |\bar{r}-\bar{r}'|^3}$ (C/m$^2$) where $\rho_l$ can be a function of position (i.e. $\bar{r}'$).

surface charge density: $\bar{D} = \iint_{s'} \dfrac{\rho_s(\bar{r}-\bar{r}')\,ds'}{4\pi |\bar{r}-\bar{r}'|^3}$ (C/m$^2$) where $\rho_s$ can be a function of position (i.e. $\bar{r}'$).

volume charge density: $\bar{D} = \iiint_{v'} \dfrac{\rho_v(\bar{r}-\bar{r}')\,dv'}{4\pi |\bar{r}-\bar{r}'|^3}$ (C/m$^2$) where $\rho_v$ can be a function of position (i.e. $\bar{r}'$).

Gauss' Law: Amount of electric flux through a closed surface is equal to amount of charge contained within surface. $\psi = \oiint_s \bar{D}\bullet d\bar{s} = Q_{\text{enc}}$ (C). In differential/point form $\bar{\nabla}\bullet\bar{D} = \rho_v$

Electrostatic potential $V$ is work per unit charge to move from point $A$ to $B$: $V_{AB} = V_B - V_A = \dfrac{W_{AB}}{q} = -\int_A^B \bar{E}\bullet d\bar{l}$

To get electric field from the electrostatic potential: $\bar{E} = -\bar{\nabla}V$

point charges: $V = \dfrac{Q}{4\pi\varepsilon_0 R} = \dfrac{Q}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$ (V) or $V = \sum_{k=1}^{N} \dfrac{Q_k}{4\pi\varepsilon_0 |\bar{r}-\bar{r}_k|}$ (V) (wrt to infinity)

line charge density: $V = \int_{c'} \dfrac{\rho_l\,dl'}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$ (V) where $\rho_l$ can be a function of position (i.e. $\bar{r}'$).

surface charge density: $V = \iint_{s'} \dfrac{\rho_s\,ds'}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$ (V) where $\rho_s$ can be a function of position (i.e. $\bar{r}'$).

volume charge density: $V = \iiint_{v'} \dfrac{\rho_v\,dv'}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$ (V) where $\rho_v$ can be a function of position (i.e. $\bar{r}'$).

Energy stored in an electric field: $W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k = \frac{1}{2}\iiint_v \bar{D}\bullet\bar{E}\,dv = \frac{1}{2}\iiint_v \varepsilon_r\varepsilon_0 |\bar{E}|^2\,dv = \frac{1}{2}CV^2$ (J)

Energy density of an electric field: $w_E = \frac{1}{2}\bar{D}\bullet\bar{E} = \frac{1}{2}\varepsilon_r\varepsilon_0|\bar{E}|^2$ (J/m$^3$)

## Chapter 5 Electric Fields in Material Space

Conduction current density: $\bar{J} = \sigma\bar{E}$ (A/m$^2$) and electrical current $I = \frac{dQ}{dt} = \iint_s \bar{J}\bullet d\bar{s}$ (A)

Resistance: $R = \frac{V}{I} = \frac{l}{\sigma S}$ ($\Omega$) where $l$ is length, $\sigma$ is conductivity (S/m), and $S$ is surface area

Perfect electrical conductor (PEC)- $\bar{E}_{\text{inside}} = 0$, $\rho_{v,\text{ inside}} = 0$, and $V_{ab} = 0$ for any two points $a$ and $b$ inside.

Power dissipated: $P = \iiint_v \bar{E}\bullet\bar{J}\,dv = \iiint_v \sigma|\bar{E}|^2\,dv$ (W) and power density $w_P = \bar{E}\bullet\bar{J} = \sigma|\bar{E}|^2$ (W/m$^3$)

Polarization vector: $\bar{P} = \chi_e\varepsilon_0\bar{E}$ relates electric field & electric flux density in materials per $\bar{D} = \varepsilon_0\bar{E} + \bar{P}$.

Permittivity: $\varepsilon = (1+\chi_e)\varepsilon_0 = \varepsilon_r\varepsilon_0$ where $\chi_e$ is electric susceptibility and $\varepsilon_r$ is the relative permittivity.

Bound surface charge density $\rho_{ps} = \bar{P}\bullet\hat{a}_n$ and bound volume charge density $\rho_{pv} = -\bar{\nabla}\bullet\bar{P}$ can be used to represent polarized materials.

Equation of Continuity $I_{\text{out}} = \oiint_s \bar{J}\bullet d\bar{s} = \frac{-dQ_{\text{inside}}}{dt}$ or $\bar{\nabla}\bullet\bar{J} = \frac{-\partial\rho_v}{\partial t}$

Charge density in conductive materials $\rho_v = \rho_v(0)\,e^{-t/T_r}$ where $T_r = \varepsilon/\sigma$ is the relaxation time.

Capacitance: Relates stored charge to potential difference $C = \frac{Q}{V}$ (e.g., parallel-plate $C = \frac{\varepsilon S}{d}$)

<u>Electrostatic Boundary conditions between dielectric regions</u>

Tangential- $E_{1t} = \frac{D_{1t}}{\varepsilon_1} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$, $\bar{E}_{1t} = \bar{E}_{2t}$, or $\hat{a}_{\text{n}12}\times(\bar{E}_2 - \bar{E}_1) = 0$ and

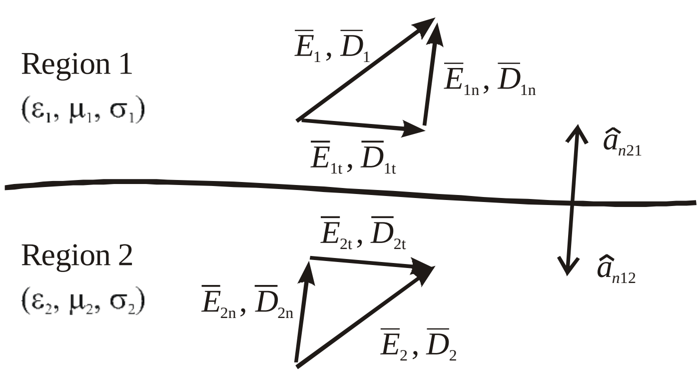
Normal- $D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$ or $\hat{a}_{\text{n}12}\cdot(\bar{D}_2 - \bar{D}_1) = \hat{a}_{\text{n}21}\cdot(\bar{D}_1 - \bar{D}_2) = \rho_s$.

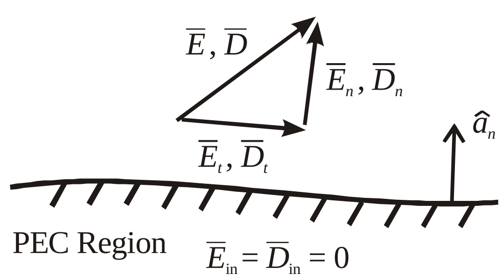
If $\rho_s = 0$, $D_{1n} = D_{2n}$ or $\hat{a}_{\text{n}12}\cdot(\bar{D}_2 - \bar{D}_1) = 0$. Surface normal $\hat{a}_{\text{n}12}$ points from region 1 into region 2, and $\mathcal{D}_{1n}$ points away from boundry while $\mathcal{D}_{2n}$ points toward the boundry

<u>Electrostatic Boundary conditions at dielectric-PEC interface</u>

Tangential- $E_t = 0$, $\bar{E}_t = 0$, or $\hat{a}_\text{n}\times\bar{E} = 0$ and Normal- $D_n = \rho_s$ or $\hat{a}_\text{n}\cdot\bar{D} = \rho_s$.

Surface normal $\hat{a}_\text{n}$ points from PEC region into dielectric region, i.e., $\mathcal{D}_n$ points away PEC.



## Chapter 6 Electrostatic Boundary Value Problems

Based on solving problems where the potential at the boundaries and materials inside the boundaries are known using Poisson's equation $\nabla^2 V = -\rho_v / \varepsilon$ or Laplace's equation $\nabla^2 V = 0$.

Capacitance- $C = \frac{Q}{V} = \frac{\oiint_s \bar{D} \bullet d\bar{s}}{-\int_{\text{low}}^{\text{high}} \bar{E} \bullet d\bar{l}} = \frac{\varepsilon \oiint_s \bar{E} \bullet d\bar{s}}{-\int_{\text{low}}^{\text{high}} \bar{E} \bullet d\bar{l}}$ Resistance- $R = \frac{V}{I} = \frac{-\int_{\text{low}}^{\text{high}} \bar{E} \bullet d\bar{l}}{\oiint_s \bar{J} \bullet d\bar{s}} = \frac{-\int_{\text{low}}^{\text{high}} \bar{E} \bullet d\bar{l}}{\sigma \oiint_s \bar{E} \bullet d\bar{s}}$

Resistance and capacitance are related by $RC = \varepsilon / \sigma$.

Method of Images- replace conducting planes with image charges so that boundary conditions still satisfied.

## Chapter 7 Magnetostatic Fields

permeability of free space: $\mu_0 = 4\pi \times 10^{-7}$ H/m

Biot-Savart Law:

Filamentary current $\bar{H} = \oint_{c'} \frac{I\, d\bar{l}' \times \hat{a}_R}{4\pi R^2} = \oint_{c'} \frac{I\, d\bar{l}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$ (A/m)

Surface current density $\bar{H} = \oiint_{s'} \frac{\bar{J}_s\, ds' \times \hat{a}_R}{4\pi R^2} = \oint_{c'} \int_{w'} \frac{J_s\, dw' d\bar{l}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$ (A/m)

Volume current density $\bar{H} = \oiiint_{v'} \frac{\bar{J}\, dv' \times \hat{a}_R}{4\pi R^2} = \oint_{c'} \iint_{s'} \frac{J\, ds' d\bar{l}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$ (A/m)

For a long straight line current: $\bar{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$ (A/m)

Magnetic Flux Density: $\bar{B} = \mu_r \mu_0 \bar{H}$ (Wb/m$^2$ or T) and Magnetic Flux: $\psi_m = \iint_s \bar{B} \bullet d\bar{s}$ (Wb)

Ampere's Law: Line integral of the magnetic field around a closed path is equal to the amount of current passing through the surface enclosed. $\oint_c \bar{H} \bullet d\bar{l} = I_{\text{enclosed}}$ (A). Diff/point form $\bar{\nabla} \times \bar{H} = \bar{J}$.

Vector Magnetic Potential $\bar{A}$: related to magnetic flux density by $\bar{B} = \bar{\nabla} \times \bar{A}$ and to magnetic flux by $\psi_m = \oint_c \bar{A} \bullet d\bar{l}$ (Wb)

Filamentary current $\bar{A} = \oint_{c'} \frac{\mu I\, d\bar{l}'}{4\pi R} = \oint_{c'} \frac{\mu I\, d\bar{l}'}{4\pi |\bar{r} - \bar{r}'|}$ (Wb/m)

Surface current density $\bar{A} = \oint_{s'} \frac{\mu \bar{J}_s\, ds'}{4\pi R} = \oint_{c'} \int_{w'} \frac{\mu J_s\, dw' d\bar{l}'}{4\pi |\bar{r} - \bar{r}'|}$ (Wb/m)

Volume current density $\bar{A} = \oint_{v'} \frac{\bar{J}\, dv'}{4\pi R} = \oint_{c'} \iint_{s'} \frac{J\, ds' d\bar{l}'}{4\pi |\bar{r} - \bar{r}'|}$ (Wb/m)

Maxwell's Equations for Static Fields

| | Integral Form | Differential Form |
|---|---|---|
| Faraday's Law | $\oint_c \bar{E}\cdot d\bar{l} = 0$ | $\bar{\nabla}\times\bar{E} = 0$ |
| Ampere's Law | $\oint_c \bar{H}\cdot d\bar{l} = \int_s \bar{J}\cdot d\bar{s}$ | $\bar{\nabla}\times\bar{H} = \bar{J}$ |
| Gauss' Law | $\oint_s \bar{D}\cdot d\bar{s} = \int_V \rho_v\, dV$ | $\bar{\nabla}\cdot\bar{D} = \rho_v$ |
| | $\oint_s \bar{B}\cdot d\bar{s} = 0$ | $\bar{\nabla}\cdot\bar{B} = 0$ |

## Chapter 8 Magnetic Forces, Materials, and Devices

Force of a magnetic field on a point charge: $\bar{F}_m = Q\bar{u}\times\bar{B} = m\bar{a}$

Lorentz force equation: $\bar{F} = \bar{F}_e + \bar{F}_m = Q\left(\bar{E}+\bar{u}\times\bar{B}\right) = m\bar{a}$

Force of a magnetic field on current elements:

filamentary current: $\bar{F} = \oint_{c'} I\, d\bar{l}'\times\bar{B}$

surface current density- $\bar{F} = \oint_{c'}\int_{w'} J_s\, d\bar{l}' dw'\times\bar{B}$

volume current density- $\bar{F} = \oint_{c'}\iint_{s'} J\, d\bar{l}' ds'\times\bar{B}$

Force between two filamentary current elements (force exerted by element 2 on 1, $\bar{F}_{21} = -\bar{F}_{12}$):

filamentary currents- $\bar{F}_{21} = \dfrac{\mu_0 I_1 I_2}{4\pi}\oint_{c_1}\oint_{c_2}\dfrac{d\bar{l}_1\times\left[d\bar{l}_2\times(\bar{r}_1-\bar{r}_2)\right]}{\left|\bar{r}_1-\bar{r}_2\right|^3}$

Torque- $\bar{T} = \bar{r}\times\bar{F} = \bar{m}\times\bar{B}$ $(\mathrm{N\cdot m})$ where $\bar{r}$ is the moment arm, $\bar{F}$ is force, and $\bar{m} = IS\,\hat{a}_n$ is the magnetic dipole moment.

Magnetization vector $\bar{M} = \chi_m\bar{H}$ relates magnetic field and magnetic flux density in materials per $\bar{B} = \mu_0\left(\bar{H}+\bar{M}\right) = \mu_0\left(1+\chi_m\right)\bar{H} = \mu_0\mu_r\bar{H} = \mu\bar{H}$.

Magnetostatic Boundary conditions between magnetic regions

Tangential- $\hat{a}_{\mathrm{n}12}\times(\bar{H}_2-\bar{H}_1) = \hat{a}_{\mathrm{n}21}\times(\bar{H}_1-\bar{H}_2) = \bar{J}_s$. If $\bar{J}_s = 0$, then $\bar{H}_{1t} = \bar{H}_{2t}$

Normal- $\bar{B}_{1n} = \bar{B}_{2n}$ or $\hat{a}_{\mathrm{n}12}\cdot(\bar{B}_1-\bar{B}_2) = \hat{a}_{\mathrm{n}21}\cdot(\bar{B}_2-\bar{B}_1) = 0$.

Surface normal $\hat{a}_{\mathrm{n}12}$ points from region 1 into 2, and $B_{1n}$ points away from boundry while $B_{2n}$ points toward from boundry

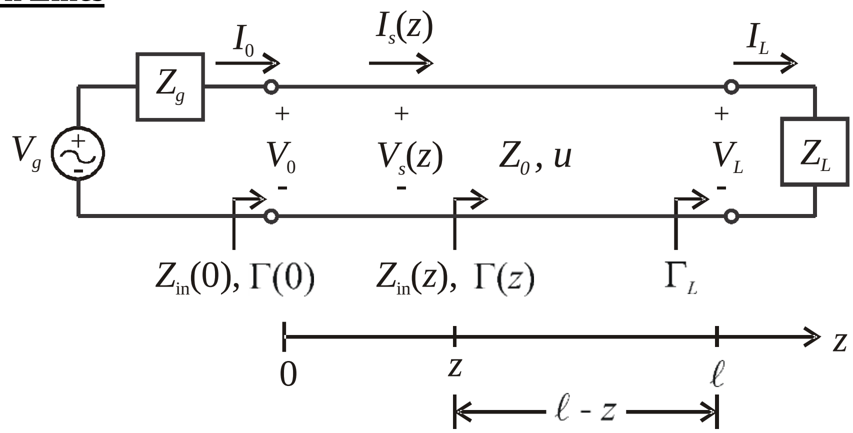
Energy stored in a magnetic field: $W_M = \tfrac{1}{2}\iiint_v \bar{B}\bullet\bar{H}\, dv = \tfrac{1}{2}\iiint_v \mu_r\mu_0\left|\bar{H}\right|^2 dv = \tfrac{1}{2}LI^2$ (J)

Energy density of a magnetic field: $w_M = \tfrac{1}{2}\bar{B}\bullet\bar{H} = \tfrac{1}{2}\mu_r\mu_0\left|\bar{H}\right|^2$ (J/m$^3$)

Magnetic Circuits:

- Assume uniform magnetic flux density in core:
- Magnetomotive force $\underline{mmf\ (\text{A}\bullet\text{turns}) = \mathcal{F} = \int \bar{H}\bullet d\bar{l} = |\bar{H}|l}$ is analogous to emf or voltage.
- Source coils (analogous to voltage sources) $\underline{\mathcal{F}_{\text{source}} = NI}$ with coil inductance $\underline{L = \frac{\lambda}{I} = \frac{N\Psi}{I} = \frac{N^2}{\mathcal{R}}}$
- 'Ohm's Law' for mmf drops in magnetic circuits is $\underline{\mathcal{F} = \Psi\mathcal{R}}$ where $\Psi$ (Wb) is magnetic flux (analogous to electrical current) and $\mathcal{R} = \frac{l}{\mu S}$ (A•turns/Wb) is reluctance (analogous to resistance).
- 'KCL' is $\underline{\sum \Psi_i = 0}$ and 'KVL' is $\underline{\sum \mathcal{F}_{drops} = 0}$. So, series & parallel reluctance, mmf (voltage) division, and flux (current) division rules work.
- Remember $\Psi = \int_S \bar{B}\bullet d\bar{s} = |\bar{B}|S$ and $|\bar{B}| = \frac{\Psi}{S} = \mu|\bar{H}|$

**Lossless Transmission Lines**




**Phase velocity:** $u = f\lambda = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$ (m/s) **Phase constant:** $\beta = \frac{\omega}{u} = \omega\sqrt{LC} = \frac{2\pi}{\lambda}$ (rad/m)

**Wavelength:** $\lambda = \frac{u}{f} = \frac{2\pi}{\beta}$ (m) **Characteristic impedance:** $Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \sqrt{\frac{L}{C}}$ ($\Omega$)

**Phasor voltage:**
$$V_S(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \quad \text{(V)}$$
$$= V_0^+ e^{-j\beta z}[1+\Gamma(z)] \quad \text{(V)}$$

**Phasor current:**
$$I_S(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} \quad \text{(A)}$$
$$= \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z} \text{(A)}$$
$$= \frac{V_0^+}{Z_0} e^{-j\beta z}[1-\Gamma(z)] \quad \text{(A)}$$

**Forward and backward traveling voltages and currents:**

Note that, at the input ($z = 0$), $V_S(0) = V_0 = V_0^+ + V_0^-$ and $I_S(0) = I_0 = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$

$$V_0^+ = 0.5(V_0 + I_0 Z_0) = \frac{V_0}{1+\Gamma(0)} = 0.5(V_L + I_L Z_0)\, e^{+j\beta l} \text{ and } I_0^+ = \frac{V_0^+}{Z_0}$$

$$V_0^- = 0.5(V_0 - I_0 Z_0) = V_0 - V_0^+ = 0.5(V_L - I_L Z_0)\, e^{-j\beta l} \text{ and } I_0^- = \frac{-V_0^-}{Z_0}.$$

**Time-domain voltage:** $V(z,t) = |V_0^+|\cos(\omega t - \beta z + \theta^+) + |V_0^-|\cos(\omega t + \beta z + \theta^-)$ (V)

**Time-domain current:** $I(z,t) = \dfrac{|V_0^+|}{Z_0}\cos(\omega t - \beta z + \theta^+) - \dfrac{|V_0^-|}{Z_0}\cos(\omega t + \beta z + \theta^-)$ (A)

**Reflection Coefficient:** $\Gamma(z) = \dfrac{V_{\text{ref}}(z)}{V_{\text{inc}}(z)} = -\dfrac{I_{\text{ref}}(z)}{I_{\text{inc}}(z)} = \dfrac{Z_{\text{in}}(z) - Z_0}{Z_{\text{in}}(z) + Z_0} = \Gamma_L\, e^{-j2\beta(l-z)}$,

of particular interest: load reflection coefficient $\Gamma_L = \dfrac{V_0^-}{V_0^+}e^{+j2\beta l} = \Gamma(0)e^{+j2\beta l} = \Gamma_{\text{in}}e^{+j2\beta l} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ and

input reflection coefficient $\Gamma_{\text{in}} = \Gamma(0) = \dfrac{Z_{\text{in}}(0) - Z_0}{Z_{\text{in}}(0) + Z_0} = \dfrac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \Gamma_L\, e^{-j2\beta l}$.

Notes: $|\Gamma(z)|$ is constant and $0 \le |\Gamma(z)| \le 1$ for passive loads.

**Input impedance:** $Z_{\text{in}}(z) = \dfrac{V(z)}{I(z)} = Z_0\left[\dfrac{Z_L + jZ_0\tan[\beta(\ell - z)]}{Z_0 + jZ_L\tan[\beta(\ell - z)]}\right] = Z_0\left[\dfrac{1+\Gamma(z)}{1-\Gamma(z)}\right]$ for $0 \le z \le l$

at input $z=0$: $Z_{\text{in}}(0) = Z_{\text{in}} = \dfrac{V_0}{I_0} = Z_0\left[\dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right] = Z_0\left[\dfrac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}\right] = Z_0\left[\dfrac{1+\Gamma(0)}{1-\Gamma(0)}\right] = Z_0\left[\dfrac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}\right]$

at load $z=l$: $Z_{\text{in}}(l) = Z_L = \dfrac{V_L}{I_L} = Z_0\left[\dfrac{1+\Gamma_L}{1-\Gamma_L}\right]$

**Standing wave ratio:** $SWR = S = \dfrac{V_{\max}}{V_{\min}} = \dfrac{I_{\max}}{I_{\min}} = \dfrac{1+|\Gamma|}{1-|\Gamma|}$ and $1 \le S < \infty$ for passive loads.

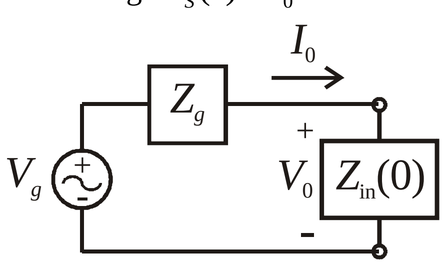
$$P_{\text{ave}}(z) = 0.5\,\text{Re}\{V(z)I^*(z)\} = 0.5\frac{|V_0^+|^2}{Z_0}\left[1-|\Gamma(z)|^2\right] \quad \text{(W)}$$

**Input Power:** $= P_{\text{ave}}(0) = 0.5\,\text{Re}\{V_0 I^*{}_0\} = 0.5\dfrac{|V_0^+|^2}{Z_0}\left[1-|\Gamma(0)|^2\right]$ (W)

$= P_{\text{ave}}(l) = 0.5\,\text{Re}\{V_L I^*{}_L\}$ (W)

Note: Time-average real power is <u>constant</u> on a lossless transmission line.

The input phasor voltage $V_S(0) = V_0$ and current $I_S(0) = I_0$ can be found using the equivalent circuit-




$V_0 = V_g\left(\dfrac{Z_{\text{in}}(0)}{Z_g + Z_{\text{in}}(0)}\right)$ and $I_0 = \dfrac{V_g}{Z_g + Z_{\text{in}}(0)}$.