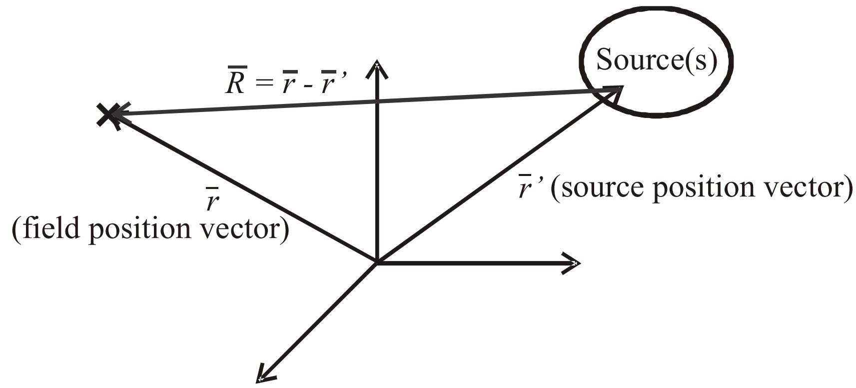
Chapter 1 Vector Algebra

Scalars- magnitude   Vectors- direction & magnitude

Position vectors- point from origin to a field point ($\overline{r}$) or to a source point ($\overline{r}'$)

Distance vectors- usually point from a source point to a field point $\overline{R} = \overline{r} - \overline{r}'$

The distance between the source and the field point is $R = |\overline{R}| = |\overline{r} - \overline{r}'|$



Scalar/dot product: $\overline{A}\bullet\overline{B} = |\overline{A}||\overline{B}|\cos(\theta_{AB}) = \overline{B}\bullet\overline{A}$

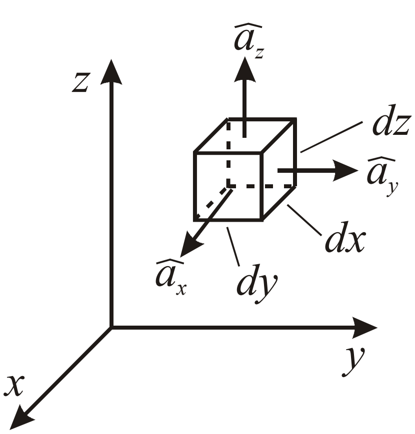
Vector/cross product: $\overline{A}\times\overline{B} = \hat{a}_n |\overline{A}||\overline{B}|\sin(\theta_{AB})$ where $\hat{a}_n$ determined by RHR. Also, $\overline{A}\times\overline{B} = -\overline{B}\times\overline{A}$.

Cross product hint: Remember order of (x, y, z), (ρ, ϕ, z), and (r, θ, ϕ)

Unit vector $\hat{a}_A = \dfrac{\overline{A}}{|\overline{A}|}$ where the vector magnitude is $|\overline{A}| = \sqrt{\overline{A}\bullet\overline{A}} = \sqrt{|\overline{A}|^2}$.

Scalar projection of vector $\overline{A}$ onto $\overline{B}$ is $A_B = \overline{A}\bullet\hat{a}_B$. The vector projection of vector $\overline{A}$ onto $\overline{B}$ is $\overline{A}_B = \hat{a}_B\left(\overline{A}\bullet\hat{a}_B\right)$.

Chapter 2 Coordinate Systems and Transformation & Chapter 3 Vector Calculus



Rectangular/Cartesian Coordinates (x, y, z) $\overline{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$

$d\overline{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$d\overline{s}_x = dy\,dz\,\hat{a}_x \quad d\overline{s}_y = dx\,dz\,\hat{a}_y \quad d\overline{s}_z = dx\,dy\,\hat{a}_z$

$dv = dx\,dy\,dz$

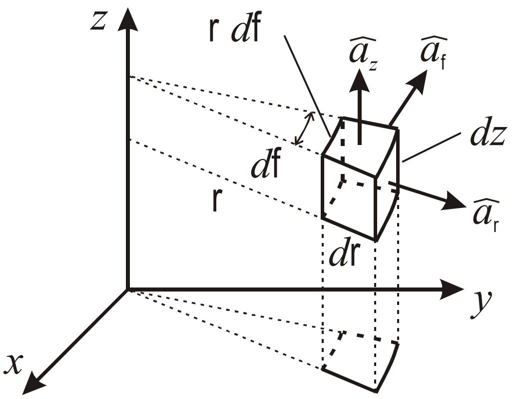
$\overline{r} = x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z$

$$\overline{\nabla}\Phi = \hat{a}_x\frac{\partial\Phi}{\partial x} + \hat{a}_y\frac{\partial\Phi}{\partial y} + \hat{a}_z\frac{\partial\Phi}{\partial z}$$

$$\overline{\nabla}\bullet\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\overline{\nabla}\times\overline{A} = \hat{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$



<u>Cylindrical Coordinates (ρ,ϕ,z)</u> $\quad \overline{A} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z$

$d\overline{l} = d\rho\,\hat{a}_\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$

$d\overline{s}_\rho = \rho\,d\phi\,dz\,\hat{a}_\rho, \quad d\overline{s}_\phi = d\rho\,dz\,\hat{a}_\phi, \quad d\overline{s}_z = \rho\,d\rho\,d\phi\,\hat{a}_z$

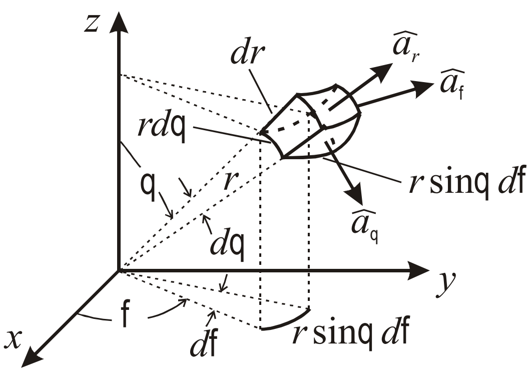
$dv = \rho\,d\rho\,d\phi\,dz, \quad \overline{r} = \rho\,\hat{a}_\rho + z\,\hat{a}_z$ (Note: $\hat{a}_\rho$ has ϕ dependence)

$$\overline{\nabla}\Phi = \hat{a}_\rho\frac{\partial\Phi}{\partial\rho} + \hat{a}_\phi\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{a}_z\frac{\partial\Phi}{\partial z}$$

$$\overline{\nabla}\bullet\overline{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\overline{\nabla}\times\overline{A} = \hat{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{a}_z\left(\frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$



<u>Spherical Coordinates (r,θ,ϕ)</u> $\quad \overline{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$

$d\overline{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi, \quad d\overline{s}_r = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r,$

$d\overline{s}_\theta = r\sin\theta\,dr\,d\phi\,\hat{a}_\theta, \quad d\overline{s}_\phi = r\,dr\,d\theta\,\hat{a}_\phi, \quad dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

$\overline{r} = r\,\hat{a}_r$ (Note: $\hat{a}_r$ has both θ & ϕ dependence)

$$\overline{\nabla}\Phi = \hat{a}_r\frac{\partial\Phi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\overline{\nabla}\bullet\overline{A} = \frac{1}{r^2}\frac{\partial\left(r^2A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\,A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\overline{\nabla}\times\overline{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial\left(\sin\theta A_\phi\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{a}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial\left(rA_\phi\right)}{\partial r}\right] + \hat{a}_\phi\frac{1}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Gradient $\overline{\nabla}\Phi$ - vector representing magnitude & direction of maximum spatial rate of change of a scalar field

Divergence $\overline{\nabla}\bullet\overline{A}$ - net outward flux per unit volume of a vector field (scalar)

Flux of $\overline{A}$ through surface S is $\iint_S \overline{A}\bullet d\overline{s}$ and Divergence Theorem- $\oint\!\!\oint_S \overline{A}\bullet d\overline{s} = \iiint_V \overline{\nabla}\bullet\overline{A}\,dv$

Curl $\overline{\nabla}\times\overline{A}$ - circulation of a vector field per unit area w/ area oriented for maximize (vector whose direction is normal to surface by RHR

Circulation of $\overline{A}$ around contour c is $\oint_c \overline{A}\bullet d\overline{l}$ and Stoke's Theorem- $\oint_c \overline{A}\bullet d\overline{l} = \iint_S(\overline{\nabla}\times\overline{A})\bullet d\overline{s}$

Vector field classification- $\overline{\nabla}\bullet\overline{A} = 0 \Rightarrow$ divergenceless/solenoidal; $\overline{\nabla}\times\overline{A} = 0 \Rightarrow$ irrotational & conservative since $\oint_c \overline{A}\bullet d\overline{l} = \iint_S(\overline{\nabla}\times\overline{A})\bullet d\overline{s} = 0$

Cartesian Coordinates (x, y, z) $\Leftrightarrow$ Cylindrical Coordinates (ρ, ϕ, z)

Point/variable conversions :

$$\rho = \sqrt{x^2 + y^2} \qquad x = \rho\cos\phi \qquad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad y = \rho\sin\phi \qquad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}$$

$$z = z \qquad z = z$$

Vector conversions : $\overline{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + A_z \hat{a}_z = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$$\hat{a}_x = \cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi \qquad \hat{a}_\rho = \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$$

$$\hat{a}_y = \sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_\phi \qquad \hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y \qquad \hat{a}_z = \hat{a}_z$$

$$A_\rho = A_x\cos\phi + A_y\sin\phi \qquad A_x = A_\rho\cos\phi - A_\phi\sin\phi$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi \qquad A_y = A_\rho\sin\phi + A_\phi\cos\phi \qquad A_z = A_z$$

Dot Products :

$$\hat{a}_x \bullet \hat{a}_\rho = \cos\phi \qquad \hat{a}_y \bullet \hat{a}_\rho = \sin\phi \qquad \hat{a}_z \bullet \hat{a}_\rho = 0$$

$$\hat{a}_x \bullet \hat{a}_\phi = -\sin\phi \qquad \hat{a}_y \bullet \hat{a}_\phi = \cos\phi \qquad \hat{a}_z \bullet \hat{a}_\phi = 0$$

$$\hat{a}_x \bullet \hat{a}_z = 0 \qquad \hat{a}_y \bullet \hat{a}_z = 0 \qquad \hat{a}_z \bullet \hat{a}_z = 1$$

Cartesian Coordinates (x, y, z) $\Leftrightarrow$ Spherical Coordinates (r, θ, ϕ)

Point/variable conversions:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\sin\theta\cos\phi \qquad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}} \qquad \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\phi = \tan^{-1}\frac{y}{x} \qquad y = r\sin\theta\sin\phi$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad z = r\cos\theta \qquad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}} \qquad \sin\theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

Vector conversions: $\overline{A} = \hat{a}_r A_r + A_\theta \hat{a}_\theta + \hat{a}_\phi A_\phi = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z \qquad \hat{a}_x = \sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi$$

$$\hat{a}_\theta = \cos\theta\cos\phi\,\hat{a}_x + \cos\theta\sin\phi\,\hat{a}_y - \sin\theta\,\hat{a}_z \qquad \hat{a}_y = \sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi$$

$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y \qquad \hat{a}_z = \cos\theta\,\hat{a}_r - \sin\theta\,\hat{a}_\theta$$

$$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \qquad A_x = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$

$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \qquad A_y = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi \qquad A_z = A_r\cos\theta - A_\theta\sin\theta$$

Dot Products:

$$\hat{a}_r \bullet \hat{a}_x = \sin\theta\cos\phi \qquad \hat{a}_r \bullet \hat{a}_y = \sin\theta\sin\phi \qquad \hat{a}_r \bullet \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \bullet \hat{a}_x = \cos\theta\cos\phi \qquad \hat{a}_\theta \bullet \hat{a}_y = \cos\theta\sin\phi \qquad \hat{a}_\theta \bullet \hat{a}_z = -\sin\theta$$

$$\hat{a}_\phi \bullet \hat{a}_x = -\sin\phi \qquad \hat{a}_\phi \bullet \hat{a}_y = \cos\phi \qquad \hat{a}_\phi \bullet \hat{a}_z = 0$$

Chapter 4 Electrostatic Fields

permittivity of free space: $\varepsilon_0 = 8.8541878 \times 10^{-12}$ F/m

Coulomb's Law:

point charges $\overline{F}_{12} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{R12} = \dfrac{Q_1 Q_2(\overline{r}_2 - \overline{r}_1)}{4\pi\varepsilon_0 |\overline{r}_2 - \overline{r}_1|^3}$ (N) force on Q_2 due to Q_1 (opposites attract, like repel)

Electric Field $\overline{E} = \overline{F} / q = -\overline{\nabla} V$ is conservative (i.e., $\overline{\nabla} \times \overline{E} = 0$ and $\oint_c \overline{E} \bullet d\overline{l} = 0$).

point charges: $\overline{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \dfrac{Q(\overline{r} - \overline{r}')}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|^3}$ (V/m) & $\overline{E} = \sum_{k=1}^{N} \dfrac{Q_k(\overline{r} - \overline{r}_k)}{4\pi\varepsilon_0 |\overline{r} - \overline{r}_k|^3}$ (V/m)

line charge density: $\overline{E} = \int_{c'} \dfrac{\rho_l (\overline{r} - \overline{r}')\, dl'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|^3}$ (V/m) where ρ_l can be a function of position (i.e. $\overline{r}'$).

surface charge density: $\overline{E} = \iint_{s'} \dfrac{\rho_s (\overline{r} - \overline{r}')\, ds'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|^3}$ (V/m) where ρ_s can be a function of position (i.e. $\overline{r}'$).

volume charge density: $\overline{E} = \iiint_{v'} \dfrac{\rho_v (\overline{r} - \overline{r}')\, dv'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|^3}$ (V/m) where ρ_v can be a function of position (i.e. $\overline{r}'$).

Electric Flux Density: $\overline{D} = \varepsilon\overline{E} = \varepsilon_r\varepsilon_0\overline{E}$ (C/m^2) and Electric Flux: $\psi = \iint_s \overline{D} \bullet d\overline{s}$ (C)

point charges: $\overline{D} = \dfrac{Q}{4\pi R^2}\hat{a}_R = \dfrac{Q(\overline{r} - \overline{r}')}{4\pi |\overline{r} - \overline{r}_1'|^3}$ (C/m^2) and $\overline{D} = \sum_{k=1}^{N} \dfrac{Q_k(\overline{r} - \overline{r}_k)}{4\pi |\overline{r} - \overline{r}_k|^3}$ (C/m^2)

line charge density: $\overline{D} = \int_{c'} \dfrac{\rho_l (\overline{r} - \overline{r}')\, dl'}{4\pi |\overline{r} - \overline{r}'|^3}$ (C/m^2) where ρ_l can be a function of position (i.e. $\overline{r}'$).

surface charge density: $\overline{D} = \iint_{s'} \dfrac{\rho_s (\overline{r} - \overline{r}')\, ds'}{4\pi |\overline{r} - \overline{r}'|^3}$ (C/m^2) where ρ_s can be a function of position (i.e. $\overline{r}'$).

volume charge density: $\overline{D} = \iiint_{v'} \dfrac{\rho_v (\overline{r} - \overline{r}')\, dv'}{4\pi |\overline{r} - \overline{r}'|^3}$ (C/m^2) where ρ_v can be a function of position (i.e. $\overline{r}'$).

Gauss' Law: Amount of electric flux through a closed surface is equal to amount of charge contained within surface. $\psi = \oiint_s \overline{D} \bullet d\overline{s} = Q_{\text{enc}}$ (C). In differential/point form $\overline{\nabla} \bullet \overline{D} = \rho_v$

Electrostatic potential V is work per unit charge to move from point A to B: $V_{AB} = V_B - V_A = \dfrac{W_{AB}}{q} = -\int_A^B \overline{E} \bullet d\overline{l}$

To get electric field from the electrostatic potential: $\overline{E} = -\overline{\nabla} V$

point charges: $V = \dfrac{Q}{4\pi\varepsilon_0 R} = \dfrac{Q}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|}$ (V) or $V = \sum_{k=1}^{N} \dfrac{Q_k}{4\pi\varepsilon_0 |\overline{r} - \overline{r}_k|}$ (V) (wrt to infinity)

line charge density: $V = \int_{c'} \dfrac{\rho_l\, dl'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|}$ (V) where ρ_l can be a function of position (i.e. $\overline{r}'$).

surface charge density: $V = \iint_{s'} \dfrac{\rho_s\, ds'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|}$ (V) where ρ_s can be a function of position (i.e. $\overline{r}'$).

volume charge density: $V = \iiint_{v'} \dfrac{\rho_v\, dv'}{4\pi\varepsilon_0 |\overline{r} - \overline{r}'|}$ (V) where ρ_v can be a function of position (i.e. $\overline{r}'$).

Energy stored in an electric field: $W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k = \frac{1}{2}\iiint_v \overline{D} \bullet \overline{E}\, dv = \frac{1}{2}\iiint_v \varepsilon_r \varepsilon_0 \left|\overline{E}\right|^2 dv = \frac{1}{2}CV^2$ (J)

Energy density of an electric field: $w_E = \frac{1}{2}\overline{D} \bullet \overline{E} = \frac{1}{2}\varepsilon_r \varepsilon_0 \left|\overline{E}\right|^2$ (J/m^3)

Chapter 5 Electric Fields in Material Space

Conduction current density: $\overline{J} = \sigma \overline{E}$ (A/m^2) and electrical current $I = \dfrac{dQ}{dt} = \iint_s \overline{J} \bullet d\overline{s}$ (A)

Resistance: $R = \dfrac{V}{I} = \dfrac{\ell}{\sigma S}$ (Ω) where ℓ is length, σ is conductivity (S/m), and S is surface area

Perfect electrical conductor (PEC)- $\overline{E}_{\text{inside}} = 0$, $\rho_{v,\text{ inside}} = 0$, and $V_{ab} = 0$ for any two points a and b inside.

Power dissipated: $P = \iiint_v \overline{E} \bullet \overline{J}\, dv = \iiint_v \sigma \left|\overline{E}\right|^2 dv$ (W) and power density $w_P = \overline{E} \bullet \overline{J} = \sigma \left|\overline{E}\right|^2$ (W/m^3)

Polarization vector: $\overline{P} = \chi_e \varepsilon_0 \overline{E}$ (C/m^2) relates electric field & flux density in materials per $\overline{D} = \varepsilon_0 \overline{E} + \overline{P}$.

Permittivity: $\varepsilon = (1+\chi_e)\varepsilon_0 = \varepsilon_r \varepsilon_0$ (F/m) where χ_e is electric susceptibility and ε_r is the relative permittivity.

Bound surface charge density $\rho_{ps} = \overline{P} \bullet \hat{a}_n$ (C/m^2) and bound volume charge density $\rho_{pv} = -\overline{\nabla} \bullet \overline{P}$ (C/m^3) can be used to represent polarized materials.

Equation of Continuity $I_{\text{out}} = \oiint_s \overline{J} \bullet d\overline{s} = \dfrac{-dQ_{\text{inside}}}{dt}$ or $\overline{\nabla} \bullet \overline{J} = \dfrac{-\partial \rho_v}{\partial t}$

Charge density in conductive materials $\rho_v(t) = \rho_v(0)\, e^{-t/T_r}$ (C/m^3) where $T_r = \varepsilon / \sigma$ is the relaxation time.

Capacitance: Relates stored charge to potential difference $C = \dfrac{Q}{V}$ (F) (e.g., parallel-plate $C = \dfrac{\varepsilon S}{d}$)

<u>Electrostatic Boundary conditions between dielectric regions</u>

Tangential- $E_{1t} = \dfrac{D_{1t}}{\varepsilon_1} = E_{2t} = \dfrac{D_{2t}}{\varepsilon_2}$, $\overline{E}_{1t} = \overline{E}_{2t}$, or $\hat{a}_{\text{n12}} \times (\overline{E}_2 - \overline{E}_1) = 0$ and

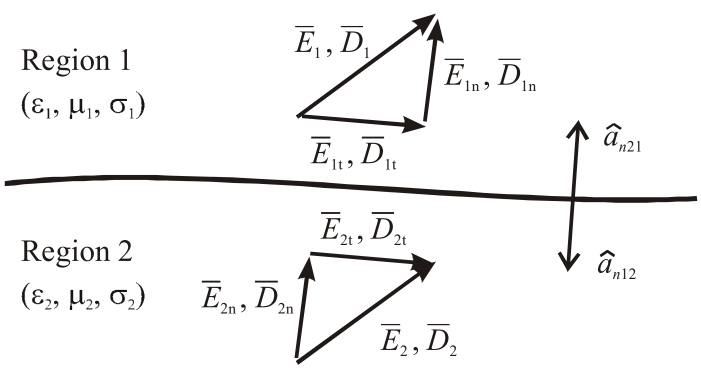
Normal- $D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$ or $\hat{a}_{\text{n12}} \cdot (\overline{D}_2 - \overline{D}_1) = \hat{a}_{\text{n21}} \cdot (\overline{D}_1 - \overline{D}_2) = \rho_s$.

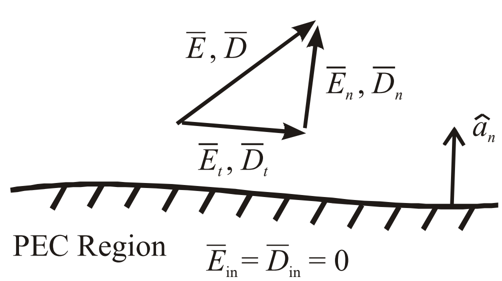
If $\rho_s = 0$, $D_{1n} = D_{2n}$ or $\hat{a}_{\text{n12}} \cdot (\overline{D}_2 - \overline{D}_1) = 0$. Surface normal $\hat{a}_{\text{n12}}$ points from region 1 into region 2, and $\mathcal{D}_{1n}$ points away from boundry while $\mathcal{D}_{2n}$ points toward the boundry

<u>Electrostatic Boundary conditions at dielectric-PEC interface</u>

Tangential- $E_t = 0$, $\overline{E}_t = 0$, or $\hat{a}_{\text{n}} \times \overline{E} = 0$ and Normal- $D_n = \rho_s$ or $\hat{a}_{\text{n}} \cdot \overline{D} = \rho_s$.

Surface normal $\hat{a}_{\text{n}}$ points from PEC region into dielectric region, i.e., $\mathcal{D}_n$ points away PEC.





Chapter 6 Electrostatic Boundary Value Problems

For problems where the electrostatic potential V at the boundaries as well as the materials inside the boundaries are known, Poisson's equation $\nabla^2 V = {-\rho_v}/{\varepsilon}$ or Laplace's equation $\nabla^2 V = 0$ can be solved for V. Then, use $\overline{E} = -\overline{\nabla} V$, $\overline{J} = \sigma \overline{E}$, $\overline{P} = \chi_e \varepsilon_0 \overline{E}$ and/or $\overline{D} = \varepsilon \overline{E} = \varepsilon_r \varepsilon_0 \overline{E}$ to find the electrostatic fields as well as:

Capacitance- $C = \dfrac{Q}{V} = \dfrac{\oiint_s \overline{D} \bullet d\overline{s}}{-\int_{\text{low}}^{\text{high}} \overline{E} \bullet d\overline{l}} = \dfrac{\varepsilon \oiint_s \overline{E} \bullet d\overline{s}}{-\int_{\text{low}}^{\text{high}} \overline{E} \bullet d\overline{l}}$ Resistance- $R = \dfrac{V}{I} = \dfrac{-\int_{\text{low}}^{\text{high}} \overline{E} \bullet d\overline{l}}{\oiint_s \overline{J} \bullet d\overline{s}} = \dfrac{-\int_{\text{low}}^{\text{high}} \overline{E} \bullet d\overline{l}}{\sigma \oiint_s \overline{E} \bullet d\overline{s}}$

Resistance and capacitance are related by $RC = \varepsilon/\sigma$ in a homogenous region.

Chapter 7 Magnetostatic Fields

permeability of free space: $\mu_0 = 4\pi \times 10^{-7}$ (H/m)

Biot-Savart Law:

Filamentary current $\overline{H} = \oint_{c'} \dfrac{I\, d\overline{l}' \times \hat{a}_R}{4\pi R^2} = \oint_{c'} \dfrac{I\, d\overline{l}' \times (\overline{r} - \overline{r}')}{4\pi |\overline{r} - \overline{r}'|^3}$ (A/m)

Surface current density $\overline{H} = \oiint_{s'} \dfrac{\overline{J}_s\, ds' \times \hat{a}_R}{4\pi R^2} = \oint_{c'} \int_{w'} \dfrac{J_s\, dw' d\overline{l}' \times (\overline{r} - \overline{r}')}{4\pi |\overline{r} - \overline{r}'|^3}$ (A/m)

Volume current density $\overline{H} = \oiiint_{v'} \dfrac{\overline{J}\, dv' \times \hat{a}_R}{4\pi R^2} = \oint_{c'} \iint_{s'} \dfrac{J\, ds' d\overline{l}' \times (\overline{r} - \overline{r}')}{4\pi |\overline{r} - \overline{r}'|^3}$ (A/m)

For a long straight line current: $\overline{H} = \dfrac{I}{2\pi \rho} \hat{a}_\phi$ (A/m)

Magnetic Flux Density: $\overline{B} = \mu_r \mu_0 \overline{H}$ (Wb/m^2 or T) and Magnetic Flux: $\psi_m = \iint_s \overline{B} \bullet d\overline{s}$ (Wb)

Ampere's Law: Line integral of the magnetic field around a closed path is equal to the amount of current passing through the surface enclosed. $\oint_c \overline{H} \bullet d\overline{l} = I_{\text{enclosed}}$ (A). Diff/point form $\overline{\nabla} \times \overline{H} = \overline{J}$.

Vector Magnetic Potential $\overline{A}$: related to magnetic flux density by $\overline{B} = \overline{\nabla} \times \overline{A}$ and to magnetic flux by $\psi_m = \oint_c \overline{A} \bullet d\overline{l}$ (Wb)

Filamentary current $\overline{A} = \oint_{c'} \dfrac{\mu I\, d\overline{l}'}{4\pi R} = \oint_{c'} \dfrac{\mu I\, d\overline{l}'}{4\pi |\overline{r} - \overline{r}'|}$ (Wb/m)

Surface current density $\overline{A} = \oiint_{s'} \dfrac{\mu \overline{J}_s\, ds'}{4\pi R} = \oint_{c'} \int_{w'} \dfrac{\mu J_s\, dw' d\overline{l}'}{4\pi |\overline{r} - \overline{r}'|}$ (Wb/m)

Volume current density $\overline{A} = \oint_{v'} \dfrac{\overline{J}\, dv'}{4\pi R} = \oint_{c'} \iint_{s'} \dfrac{J\, ds' d\overline{l}'}{4\pi |\overline{r} - \overline{r}'|}$ (Wb/m)

Maxwell's Equations for Static Fields

	Integral Form	Differential Form
Faraday's Law	$\oint_c \overline{E}\cdot d\overline{l} = 0$	$\overline{\nabla}\times\overline{E} = 0$
Ampere's Law	$\oint_c \overline{H}\cdot d\overline{l} = \int_s \overline{J}\cdot d\overline{s}$	$\overline{\nabla}\times\overline{H} = \overline{J}$
Gauss' Law	$\oint_s \overline{D}\cdot d\overline{s} = \int_V \rho_v\, dV$	$\overline{\nabla}\cdot\overline{D} = \rho_v$
	$\oint_s \overline{B}\cdot d\overline{s} = 0$	$\overline{\nabla}\cdot\overline{B} = 0$

Chapter 8 Magnetic Forces, Materials, and Devices

Force of a magnetic field on a point charge: $\overline{F}_m = Q\overline{u}\times\overline{B} = m\overline{a}$

Lorentz force equation: $\overline{F} = \overline{F}_e + \overline{F}_m = Q\left(\overline{E}+\overline{u}\times\overline{B}\right) = m\overline{a}$

Force of a magnetic field on current elements:

filamentary current: $\overline{F} = \oint_{c'} I\, d\overline{l}'\times\overline{B}$

surface current density- $\overline{F} = \oint_{c'}\int_{w'} J_s\, d\overline{l}' dw'\times\overline{B}$

volume current density- $\overline{F} = \oint_{c'}\iint_{s'} J\, d\overline{l}' ds'\times\overline{B}$

Force between two filamentary current elements (force exerted by element 2 on 1, $\overline{F}_{21} = -\overline{F}_{12}$):

filamentary currents- $\overline{F}_{21} = \dfrac{\mu_0 I_1 I_2}{4\pi}\oint_{c_1}\oint_{c_2}\dfrac{d\overline{l}_1\times\left[d\overline{l}_2\times(\overline{r}_1-\overline{r}_2)\right]}{\left|\overline{r}_1-\overline{r}_2\right|^3}$

Torque- $\overline{T} = \overline{r}\times\overline{F} = \overline{m}\times\overline{B}$ $(\mathrm{N}\cdot\mathrm{m})$ where $\overline{r}$ is the moment arm, $\overline{F}$ is force, $\overline{m} = IS\,\hat{a}_n$ or $\overline{m} = NIS\,\hat{a}_n$ (w/ multiple loops) is the magnetic dipole moment, and $\overline{B}$ is magnetic flux density.

Magnetization vector $\overline{M} = \chi_m\overline{H}$ (A/m) relates magnetic field and magnetic flux density in materials per $\overline{B} = \mu_0\left(\overline{H}+\overline{M}\right) = \mu_0\left(1+\chi_m\right)\overline{H} = \mu_0\mu_r\overline{H} = \mu\overline{H}$ where χ_m is the magnetic susceptibility and μ_r is the relative permeability.

Magnetostatic Boundary conditions between magnetic regions

Tangential- $\hat{a}_{n12}\times(\overline{H}_2-\overline{H}_1) = \hat{a}_{n21}\times(\overline{H}_1-\overline{H}_2) = \overline{J}_s$. If $\overline{J}_s = 0$, then $\overline{H}_{1t} = \overline{H}_{2t}$

Normal- $\overline{B}_{1n} = \overline{B}_{2n}$ or $\hat{a}_{n12}\cdot(\overline{B}_1-\overline{B}_2) = \hat{a}_{n21}\cdot(\overline{B}_2-\overline{B}_1) = 0$.

Surface normal $\hat{a}_{n12}$ points from region 1 into 2, and B_{1n} points away from boundry while B_{2n} points toward from boundry

Flux Linkage: $\lambda = N\Psi_m$

Self-inductance: $L = \lambda/I = N\Psi_m/I = 2W_m/I^2$ (H); Mutual-inductance: $M_{12} = \lambda_{12}/I_2 = N_1\Psi_{12}/I_2$ (H)

Energy stored in a magnetic field: $W_M = \frac{1}{2}\iiint_v \overline{B}\bullet\overline{H}\,dv = \frac{1}{2}\iiint_v \mu_r\mu_0 |\overline{H}|^2\,dv = \frac{1}{2}LI^2$ (J)

Energy density of a magnetic field: $w_M = \frac{1}{2}\overline{B}\bullet\overline{H} = \frac{1}{2}\mu_r\mu_0|\overline{H}|^2$ (J/m^3)

Magnetic Circuits (**not** covered 2023):

- Assume uniform magnetic flux density in core.
- Magnetomotive force $\underline{mmf \text{ (A•turns)} = \mathcal{F} = \int \overline{H}\bullet d\overline{l} = |\overline{H}|\ell}$ is analogous to emf or voltage.
- mmf source coils $\underline{\mathcal{F}_{\text{source}} = NI}$ (analogous to voltage sources) with coil inductance $\underline{L = \frac{\lambda}{I} = \frac{N\Psi}{I} = \frac{N^2}{\mathcal{R}}}$
- 'Ohm's Law' for mmf drops in magnetic circuits is $\underline{\mathcal{F} = \Psi\mathcal{R}}$ where Ψ (Wb) is magnetic flux (analogous to electrical current) and $\mathcal{R} = \frac{\ell}{\mu S}$ (A•turns/Wb) is reluctance (analogous to resistance).
- 'KCL' is $\underline{\sum \Psi_i = 0}$ and 'KVL' is $\underline{\sum \mathcal{F}_{drops} = 0}$. So, series & parallel reluctance, mmf (voltage) division, and flux (current) division rules work.
- Remember $\Psi = \iint_S \overline{B}\bullet d\overline{s} = |\overline{B}|S$ and $|\overline{B}| = \frac{\Psi}{S} = \mu|\overline{H}|$

Chapter 11 Transmission Lines

Table 11.1 Distributed Transmission Line Parameters at High Frequencies

Parameters	Coaxial Line	Two-wire Line	Planar Line
R (Ω/m)	$\frac{1}{2\pi\delta\sigma_c}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{1}{\pi a\delta\sigma_c}$	$\frac{2}{w\delta\sigma_c}$ where $\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}}$ is conductor skin depth
L (H/m)	$\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{d}{2a}\right)$	$\frac{\mu d}{w}$
G (S/m)	$2\pi\sigma / \ln\left(\frac{b}{a}\right)$	$\pi\sigma / \cosh^{-1}\left(\frac{d}{2a}\right)$	$\frac{\sigma w}{d}$
C (F/m)	$2\pi\varepsilon / \ln\left(\frac{b}{a}\right)$	$\pi\varepsilon / \cosh^{-1}\left(\frac{d}{2a}\right)$	$\frac{\varepsilon w}{d}$

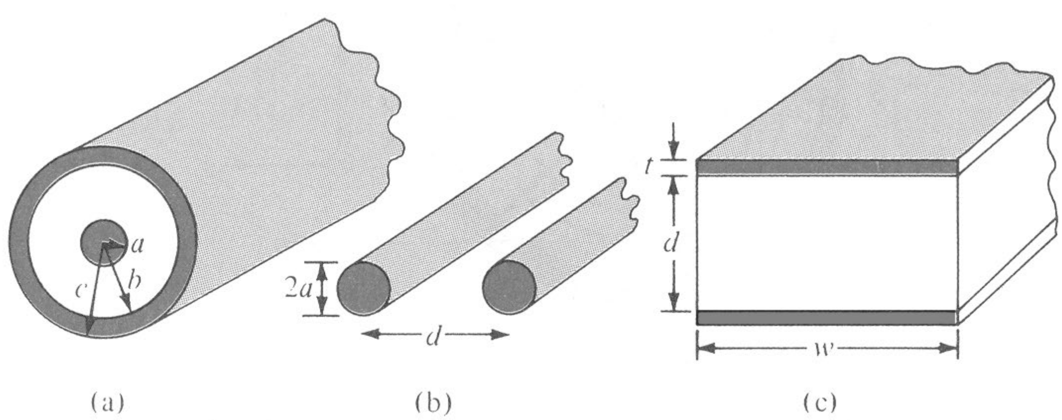
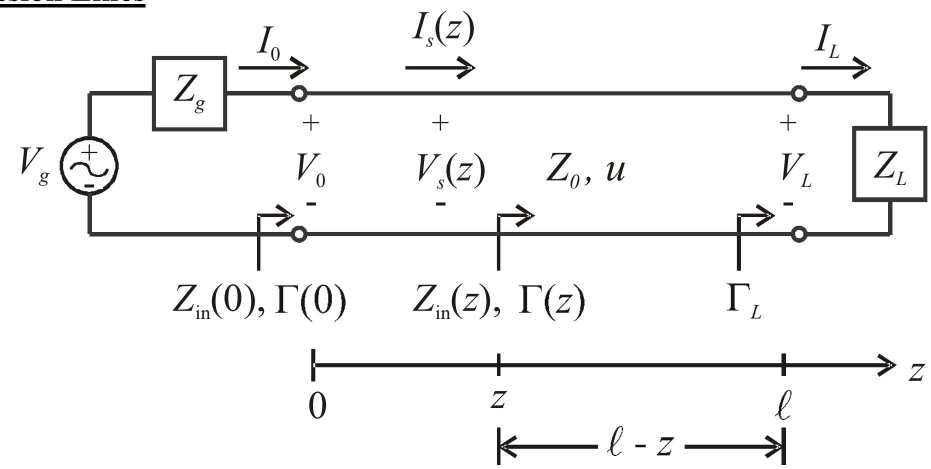


Figure 11.2 Common transmission lines: **(a)** coaxial line, **(b)** two-wire line, **(c)** planar line.

Lossless Transmission Lines



Phase velocity: $u = f\lambda = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$ (m/s)

Phase constant: $\beta = \dfrac{\omega}{u} = \omega\sqrt{LC} = \dfrac{2\pi}{\lambda}$ (rad/m)

Wavelength: $\lambda = \dfrac{u}{f} = \dfrac{2\pi}{\beta}$ (m)

Characteristic impedance: $Z_0 = \dfrac{V_0^+}{I_0^+} = -\dfrac{V_0^-}{I_0^-} = \sqrt{\dfrac{L}{C}}$ (Ω)

Phasor voltage:

$$V_S(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \quad \text{(V)}$$
$$= V_0^+ e^{-j\beta z}[1+\Gamma(z)] \quad \text{(V)}$$

Phasor current:

$$I_S(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} \quad \text{(A)}$$
$$= \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z} \text{(A)}$$
$$= \frac{V_0^+}{Z_0} e^{-j\beta z}[1-\Gamma(z)] \quad \text{(A)}$$

Forward and backward traveling voltages and currents:

Note that, at the input ($z = 0$), $V_S(0) = V_0 = V_0^+ + V_0^-$ and $I_S(0) = I_0 = I_0^+ + I_0^- = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}$

$$V_0^+ = 0.5(V_0 + I_0 Z_0) = \frac{V_0}{1+\Gamma(0)} = 0.5(V_L + I_L Z_0)\, e^{+j\beta\ell} \text{ and } I_0^+ = \frac{V_0^+}{Z_0}$$

$$V_0^- = 0.5(V_0 - I_0 Z_0) = V_0 - V_0^+ = 0.5(V_L - I_L Z_0)\, e^{-j\beta\ell} \text{ and } I_0^- = \frac{-V_0^-}{Z_0}.$$

Time-domain voltage: $V(z,t) = |V_0^+|\cos(\omega t - \beta z + \theta^+) + |V_0^-|\cos(\omega t + \beta z + \theta^-)$ (V) for $0 \le z \le \ell$

Time-domain current: $I(z,t) = \dfrac{|V_0^+|}{Z_0}\cos(\omega t - \beta z + \theta^+) - \dfrac{|V_0^-|}{Z_0}\cos(\omega t + \beta z + \theta^-)$ (A) for $0 \le z \le \ell$

Reflection Coefficient: $\Gamma(z) = \dfrac{V_{ref}(z)}{V_{inc}(z)} = -\dfrac{I_{ref}(z)}{I_{inc}(z)} = \dfrac{Z_{in}(z) - Z_0}{Z_{in}(z) + Z_0} = \Gamma_L\, e^{-j2\beta(\ell - z)}$ for $0 \le z \le \ell$,

of particular interest: load reflection coefficient $\Gamma_L = \dfrac{V_0^-}{V_0^+} e^{+j2\beta\ell} = \Gamma(0) e^{+j2\beta\ell} = \Gamma_{in} e^{+j2\beta\ell} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ and

input reflection coefficient $\Gamma_{in} = \Gamma(0) = \Gamma_L\, e^{-j2\beta\ell} = \dfrac{Z_{in}(0) - Z_0}{Z_{in}(0) + Z_0} = \dfrac{Z_{in} - Z_0}{Z_{in} + Z_0}$.

Notes: $|\Gamma(z)|$ is constant and $0 \le |\Gamma(z)| \le 1$ for passive loads.

Input impedance: $Z_{\text{in}}(z) = \dfrac{V(z)}{I(z)} = Z_0 \left[\dfrac{Z_L + jZ_0 \tan[\beta(\ell - z)]}{Z_0 + jZ_L \tan[\beta(\ell - z)]} \right] = Z_0 \left[\dfrac{1+\Gamma(z)}{1-\Gamma(z)} \right]$ for $0 \le z \le \ell$

at input $z = 0$: $Z_{\text{in}}(0) = Z_{\text{in}} = \dfrac{V_0}{I_0} = Z_0 \left[\dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right] = Z_0 \left[\dfrac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)} \right] = Z_0 \left[\dfrac{1+\Gamma(0)}{1-\Gamma(0)} \right] = Z_0 \left[\dfrac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}} \right]$

at load $z = \ell$: $Z_{\text{in}}(\ell) = Z_L = \dfrac{V_L}{I_L} = Z_0 \left[\dfrac{1+\Gamma_L}{1-\Gamma_L} \right]$

Standing wave ratio: $SWR = S = \dfrac{V_{\max}}{V_{\min}} = \dfrac{I_{\max}}{I_{\min}} = \dfrac{1+|\Gamma|}{1-|\Gamma|}$ and $1 \le S < \infty$ for passive loads.

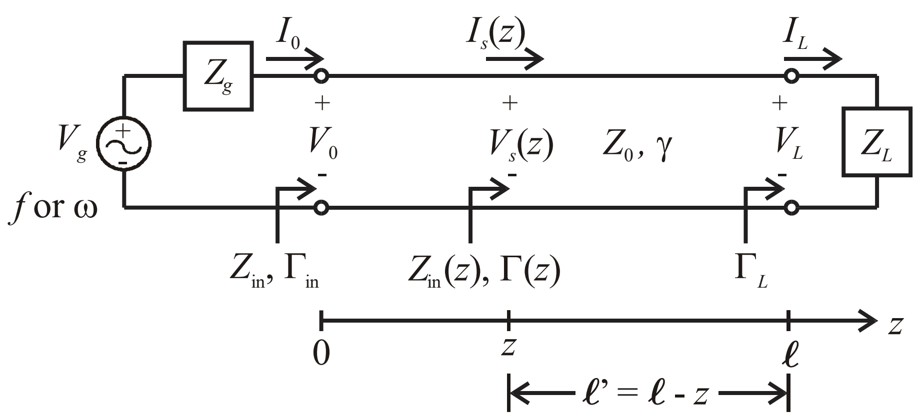
$$P_{\text{ave}}(z) = 0.5\,\text{Re}\{V(z) I^*(z)\} = 0.5 \frac{|V_0^+|^2}{Z_0} \left[1 - |\Gamma(z)|^2\right] \quad \text{(W)}$$

Input Power: $= P_{\text{ave}}(0) = 0.5\,\text{Re}\{V_0 I^*{}_0\} = 0.5 \dfrac{|V_0^+|^2}{Z_0} \left[1 - |\Gamma(0)|^2\right]$ (W) for $0 \le z \le \ell$

$= P_{\text{ave}}(\ell) = 0.5\,\text{Re}\{V_L I^*{}_L\}$ (W)

Note: Time-average real power is <u>constant</u> on a lossless transmission line.

<u>Lossy Transmission Lines (TLs)</u>



Propagation constant (1/m): $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$

Attenuation constant (np/m): $\alpha = \text{Re}(\gamma)$ & **Phase constant (rad/m):** $\beta = \text{Im}(\gamma) = \dfrac{\omega}{u} = \dfrac{2\pi}{\lambda}$

Phase velocity (m/s): $u = f\lambda = \dfrac{\omega}{\beta}$ where f is frequency (Hz) and $\omega = 2\pi f$ is radian frequency (rad/s).

Wavelength (m): $\lambda = \dfrac{u}{f} = \dfrac{2\pi}{\beta}$

Characteristic impedance (Ω): $Z_0 = \dfrac{V_0^+}{I_0^+} = \dfrac{-V_0^-}{I_0^-} = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}} = \dfrac{R + j\omega L}{\gamma} = \dfrac{\gamma}{G + j\omega C} = R_0 + jX_0 = |Z_0| \angle \theta_{Z0}$

Phasor voltage: $V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} = V_0^+ e^{-\gamma z}[1 + \Gamma(z)]$ for $0 \le z \le \ell$.

Phasor current: $I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \dfrac{V_0^+}{Z_0} e^{-\gamma z} - \dfrac{V_0^-}{Z_0} e^{+\gamma z} = \dfrac{V_0^+}{Z_0} e^{-\gamma z}[1-\Gamma(z)]$ for $0 \le z \le \ell$.

where $V_0^+ = |V_0^+| \angle\theta_V^+$, $V_0^- = |V_0^-| \angle\theta_V^-$, $I_0^+ = |I_0^+| \angle\theta_I^+$, and $I_0^- = |I_0^+| \angle\theta_I^+$ are complex coefficients representing the amplitudes of the forward and backward traveling voltages and currents.

Forward and backward traveling voltages (V) and currents (A):

Note that, at the input ($z = 0$), $V_S(0) = V_0 = V_0^+ + V_0^-$ and $I_S(0) = I_0 = I_0^+ + I_0^- = \dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}$

$$V_0^+ = 0.5(V_0 + I_0 Z_0) = \frac{V_0}{1+\Gamma(0)} = 0.5(V_L + I_L Z_0)\, e^{+\gamma\ell} \quad \text{and} \quad I_0^+ = \frac{V_0^+}{Z_0}$$

$$V_0^- = 0.5(V_0 - I_0 Z_0) = V_0 - V_0^+ = 0.5(V_L - I_L Z_0)\, e^{-\gamma\ell} = V_0^+\,\Gamma(0) \quad \text{and} \quad I_0^- = \frac{-V_0^-}{Z_0}.$$

Time-domain voltage: $V(z,t) = |V_0^+|\, e^{-\alpha z}\cos(\omega t - \beta z + \theta_V^+) + |V_0^-|\, e^{\alpha z}\cos(\omega t + \beta z + \theta_V^-)$ for $0 \le z \le \ell$.

Time-domain current:

$$\begin{aligned} I(z,t) &= |I_0^+|\, e^{-\alpha z}\cos(\omega t - \beta z + \theta_I^+) + |I_0^-|\, e^{\alpha z}\cos(\omega t + \beta z + \theta_I^-) \\ &= \frac{|V_0^+|}{|Z_0|} e^{-\alpha z}\cos(\omega t - \beta z + \theta_V^+ - \theta_{Z0}) - \frac{|V_0^-|}{|Z_0|} e^{\alpha z}\cos(\omega t + \beta z + \theta_V^- - \theta_{Z0}) \end{aligned} \quad 0 \le z \le \ell.$$

Reflection coefficient (unitless): $\Gamma(z) = \dfrac{V_{\text{ref}}(z)}{V_{\text{inc}}(z)} = \dfrac{-I_{\text{ref}}(z)}{I_{\text{inc}}(z)} = \dfrac{Z_{\text{in}}(z) - Z_0}{Z_{\text{in}}(z) + Z_0} = \Gamma_L\, e^{-2\gamma(l-z)} = \Gamma_{\text{in}}\, e^{2\gamma z}$ for $0 \le z \le \ell$.

of particular interest are: **load reflection coefficient** $\Gamma_L = \dfrac{V_0^-}{V_0^+} e^{+2\gamma\ell} = \Gamma(0)\, e^{+2\gamma\ell} = \Gamma_{\text{in}}\, e^{+2\gamma\ell} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

input reflection coefficient $\Gamma_{\text{in}} = \Gamma(0) = \dfrac{V_0^-}{V_0^+} = \dfrac{Z_{\text{in}}(0) - Z_0}{Z_{\text{in}}(0) + Z_0} = \dfrac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \Gamma_L\, e^{-2\gamma\ell}$.

Note: $0 \le |\Gamma(z)| \le 1$ for passive loads and transmission lines.

Input impedance (Ω): $Z_{\text{in}}(z) = \dfrac{V_s(z)}{I_s(z)} = Z_0\left[\dfrac{1+\Gamma(z)}{1-\Gamma(z)}\right] = Z_0\left[\dfrac{Z_L + Z_0\tanh[\gamma(\ell - z)]}{Z_0 + Z_L\tanh[\gamma(\ell - z)]}\right]$ for $0 \le z \le \ell$.

at input $z = 0$: $Z_{\text{in}}(0) = Z_{\text{in}} = \dfrac{V_0}{I_0} = Z_0\left[\dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right] = Z_0\left[\dfrac{Z_L + Z_0\tanh(\gamma\ell)}{Z_0 + Z_L\tanh(\gamma\ell)}\right] = Z_0\left[\dfrac{1+\Gamma(0)}{1-\Gamma(0)}\right] = Z_0\left[\dfrac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}\right]$

at load $z = \ell$: $Z_{\text{in}}(l) = Z_L = \dfrac{V_L}{I_L} = Z_0\left[\dfrac{1+\Gamma_L}{1-\Gamma_L}\right]$

Standing wave ratio* (unitless): $SWR = S = \dfrac{V_{\max}}{V_{\min}} = \dfrac{I_{\max}}{I_{\min}} = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|}$ and $1 \le S < \infty$ for passive loads.

[* Defined at **load** as a measure of mismatch. Really only meaningful for lossless/low loss &/or short TLs.]

Power (W):

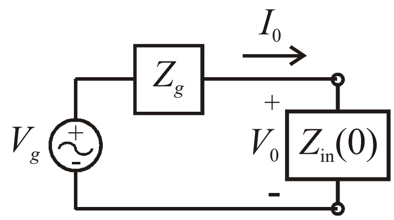
$$\begin{aligned} P_{\text{ave}}(z) &= 0.5\,\text{Re}\left\{V_s(z)\, I_s^*(z)\right\} \qquad 0 \le z \le \ell \\ &= \underbrace{0.5\frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha z}\cos(\theta_{Z0})}_{\text{forward traveling power}} - \underbrace{0.5\frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha z}\cos(\theta_{Z0})|\Gamma(z)|^2}_{\text{backward traveling power}} - \underbrace{\frac{|V_0^+|^2}{|Z_0|} e^{-2\alpha z}\sin(\theta_{Z0})\,\text{Im}(\Gamma(z))}_{\text{cross coupled power (usually very small)}} \end{aligned}$$

Input Power (W): $P_{\text{ave}}(0) = P_{\text{in}} = 0.5\,\text{Re}\left\{V_0\, I_0^*\right\} = 0.5\dfrac{|V_0^+|^2}{|Z_0|}\cos(\theta_{Z0})\left[1 - |\Gamma_{\text{in}}|^2\right] - \dfrac{|V_0^+|^2}{|Z_0|}\sin(\theta_{Z0})\,\text{Im}(\Gamma_{\text{in}})$

Load Power (W):

$$P_{\text{ave}}(\ell) = P_L = 0.5\,\text{Re}\left\{V_L\, I_L^*\right\} = 0.5\frac{|V_0^+|^2}{|Z_0|}e^{-2\alpha\ell}\cos(\theta_{Z0})\left[1 - |\Gamma_L|^2\right] - \frac{|V_0^+|^2}{|Z_0|}e^{-2\alpha\ell}\sin(\theta_{Z0})\,\text{Im}\left(\Gamma_L\right)$$

The input phasor voltage $V_S(0) = V_0$ and current $I_S(0) = I_0$ can be found using the equivalent circuit-



$$V_0 = V_g\left(\frac{Z_{\text{in}}(0)}{Z_g + Z_{\text{in}}(0)}\right) \quad \text{and} \quad I_0 = \frac{V_g}{Z_g + Z_{\text{in}}(0)}.$$

Smith Charts

- Graphical representation of **reflection coefficient** Γ. $\Gamma = |\Gamma|\angle\Gamma$ can be plotted/read. $|\Gamma|$ can be set/read using 'REFL. COEFF. V or I' scale at bottom of Smith chart. $\angle\Gamma$ can be set/read using 'ANGLE OF REFLECTION CEFFICIENT IN DEGREES' scale, an outside ring around Smith chart.
- **Normalized impedance** $z = Z/Z_0 = r + jx$ (Ω/Ω) can be plotted/read. Circles of constant *r* originate at RHS of Smith chart, centered on horizontal axis. Arcs of constant *x* originate at RHS of Smith chart, $x > 0$ (inductive reactance) arcs are above horizontal axis while $x < 0$ (capacitive reactance) arcs are below.
- **Normalized admittance** $y = Y/Y_0 = g + jb$ (S/S) can be plotted/read. Plot *z*, draw circle of constant $|\Gamma|$, centered on Smith chart through *z*, and go 180° around circle to location of *y*. Now, assume circles of constant *g* (dual use with *r*) originate at RHS of Smith chart, centered on horizontal axis while arcs of constant *b* (dual use with *x*) originate at RHS of Smith chart, $b > 0$ (capacitive susceptance) arcs are above horizontal axis while $b < 0$ (inductive susceptance) arcs are below.
- **Standing wave ratio (SWR/VSWR)** can be plotted/read using the 'SWR (VSWR)' scale at bottom of Smith chart by measuring distance from center of Smith chart to location of *y*, *z*, and/or Γ.
- Find $\boldsymbol{z_{\max} = r_{\max}}$ by drawing circle, centered on Smith chart, through the location of *z* and/or Γ. Where the circle intersects the horizontal axis to the <u>right</u> of the origin is $\boldsymbol{z_{\max} = r_{\max}}$. Note that the **SWR =** $\boldsymbol{r_{\max}}$. This location is where $V_{\max}$ and $I_{\min}$ for standing waves occur along the transmission line.
- Find $\boldsymbol{z_{\min} = r_{\min}}$ by drawing circle, centered on Smith chart, through the location of *z* and/or Γ. Where the circle intersects the horizontal axis to the <u>left</u> of the origin is $\boldsymbol{z_{\min} = r_{\min}}$. Note that the **SWR = 1/**$\boldsymbol{r_{\min}}$. This location is where $V_{\min}$ and $I_{\max}$ for standing waves occur along the transmission line.
- If we know one of z_L, Γ_L, z_{in}, or Γ_{in}, all the remaining quantities can be found by first plotting the known quantity on the Smith chart and reading the corresponding co-located quantity. Draw a circle, centered on Smith chart, through the location of the known quantity. Then, draw a radial line from the center to the outer rings through the location of the known quantity. Next, assuming the transmission line length ℓ and operation wavelength λ are known, move along the circle a distance ℓ/λ from the location of the known quantities to the location of those on the other end of the transmission line using either the 'WAVELENGTHS TOWARD GENERATOR' or 'WAVELENGTHS TOWARD LOAD' scales which are outside rings around Smith chart.

Simple Smith Chart