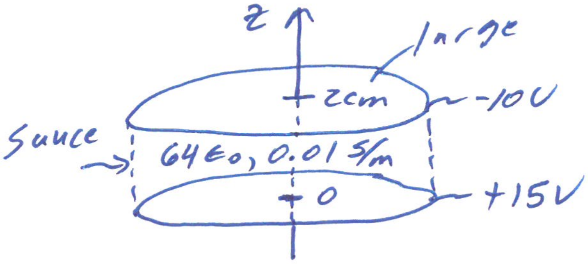
EE 381 Electric and Magnetic Fields Quiz #10 (Fall 2024)

Name Key A

Instructions: **Closed book & notes**. Place answers in indicated spaces and show all work for credit.

Eager for a snack, Emma Engineer places cranberry sauce (a tart mixture of cranberries, water, spices & sugar $\Rightarrow \varepsilon = 64\varepsilon_0$, $\sigma = 0.01$ S/m) between large circular metal plates at $z = 0$ and $z = 2$ cm. To warm the sauce, she connects a dual-channel DC power supply to the plates, lower plate at +15 V and upper plate at -10 V. After 2 seconds, does Laplace's or Poisson's equation apply? Justify answer by finding the relaxation time for any initial free charges in the sauce. Find the potential, electric field, and current density in the cranberry sauce at $z = 0.4$ cm. Extra credit- Determine the *power density* in the sauce.



$$T_r = \varepsilon/\sigma = \frac{64(8.8541878\times10^{-12})}{0.01} = \underline{5.66669\times10^{-8}\,s}$$

After 2s, any initial ρ_v would be long gone $\Rightarrow \rho_v = 0 \Rightarrow$ Laplace's Eqn

cyl. coordinates

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\underbrace{\frac{\partial V}{\partial\rho}}_{\to 0}\Big) + \frac{1}{\rho^2}\underbrace{\frac{\partial^2 V}{\partial\phi^2}}_{\to 0} + \frac{\partial^2 V}{\partial z^2} = 0$$

no change wrt ρ & ϕ

$$\int \frac{d^2V}{dz^2}dz = \int 0\,dz \rightarrow \frac{dV}{dz} = A \rightarrow \int \frac{dV}{dz}dz = \int A\,dz$$

$\rightarrow V = Az + B$ @ $z=0$ $V = 15 = A(0) + B \Rightarrow B = 15\,V$

@ $z = 0.02m$, $V = -10 = A(0.02) + 15 \rightarrow A = -1250$ V/m

$V(z) = -1250z + 15\,V$ $0 \le z \le 0.02m$ $\underline{V(0.004m) = 10\,V}$

$\bar{E} = -\bar{\nabla}V = -\hat{a}_z\frac{d}{dz}(-1250z + 15) = \underline{\hat{a}_z\,1250\text{ V/m}}$ $0 \le z \le 2cm$

$\bar{J} = \sigma\bar{E} = 0.01\,\hat{a}_z 1250 = \underline{\hat{a}_z\,12.5\text{ A/m}^2}$ $0 \le z \le 2cm$

Extra: $w_p = \sigma|\bar{E}|^2 = 0.01(1250)^2 = \underline{15{,}625\text{ W/m}^3}$

Laplace's or Poisson's equation? (Circle correct answer) Laplace's T_{relax} = 56.67 ns

$V(z = 0.4\text{ cm})$ = 10 V $\bar{E}(z = 0.4\text{ cm})$ = $\hat{a}_z$ 1250 V/m

$\bar{J}(z = 0.4\text{ cm})$ = $\hat{a}_z$ 12.5 A/m² Extra - *power density* = 15,625 W/m³

Cylindrical Coordinates (ρ,ϕ,z)

$$d\bar{l} = d\rho\,\hat{a}_\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$

$$d\bar{s}_\rho = \rho\, d\phi\, dz\,\hat{a}_\rho \qquad d\bar{s}_\phi = d\rho\, dz\,\hat{a}_\phi \qquad d\bar{s}_z = \rho\, d\rho\, d\phi\,\hat{a}_z$$

$$dv = \rho\, d\rho\, d\phi\, dz$$

$\bar{r} = \rho\,\hat{a}_\rho + z\,\hat{a}_z$ (Note: $\hat{a}_\rho$ has ϕ dependence)

$$\bar{\nabla}\Phi = \hat{a}_\rho\frac{\partial\Phi}{\partial\rho} + \hat{a}_\phi\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{a}_z\frac{\partial\Phi}{\partial z}$$

$$\bar{\nabla}\bullet\bar{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\bar{\nabla}\times\bar{A} = \hat{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{a}_z\left(\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

permeability of free space: $\mu_0 = 4\pi\bullet10^{-7}$ H/m permittivity of free space: $\varepsilon_0 = 8.8542\times10^{-12}$ F/m

Coulomb's Law: point charges $\bar{F}_{12} = \dfrac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{R12} = \dfrac{Q_1Q_2(\bar{r}_2-\bar{r}_1)}{4\pi\varepsilon_0|\bar{r}_2-\bar{r}_1|^3}$ (N) force on Q_2 due to Q_1

Electric Field $\bar{E} = \bar{F}/q = -\bar{\nabla}V$ is conservative (i.e., $\bar{\nabla}\times\bar{E}=0$ and $\oint_c \bar{E}\bullet d\bar{l} = 0$).

point charges: $\bar{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \dfrac{Q(\bar{r}-\bar{r}')}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|^3}$ (V/m) & $\bar{E} = \displaystyle\sum_{k=1}^{N}\frac{Q_k(\bar{r}-\bar{r}_k)}{4\pi\varepsilon_0|\bar{r}-\bar{r}_k|^3}$ (V/m)

volume charge density: $\bar{E} = \displaystyle\int_{v'}\frac{\rho_v(\bar{r}-\bar{r}')\,dv'}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|^3}$ (V/m) where ρ_v can be a function of position (i.e. $\bar{r}'$).

Electric Flux Density: $\bar{D} = \varepsilon_r\varepsilon_0\bar{E} = \varepsilon\bar{E}$ (C/m^2) and Electric Flux: $\psi = \int\bar{D}\bullet d\bar{s}$ (C)

Gauss' Law: Amount of electric flux through a closed surface is equal to amount of charge contained within surface. $\psi = \oint_s\bar{D}\bullet d\bar{s} = Q_{\text{enc}}$ (C). In differential/point form $\bar{\nabla}\bullet\bar{D} = \rho_v$

Charge density in conductive materials $\rho_v = \rho_v(0)\,e^{-t/T_r}$ where $T_r = \varepsilon/\sigma$ is the relaxation time.

Conduction current density: $\bar{J} = \sigma\bar{E}$ (A/m^2) and electrical current $I = \int_s\bar{J}\bullet d\bar{s} = \dfrac{dQ}{dt}$ (A)

Resistance: $R = \dfrac{V}{I} = \dfrac{l}{\sigma S}$ (Ω) where l is length, σ is conductivity (S/m), and S is surface area

Perfect electrical conductor (PEC)- $\bar{E}_{\text{inside}} = 0$, $\rho_{v,\text{ inside}} = 0$, and $V_{ab} = 0$ for any points a and b inside.

Power dissipation: $P = \int\bar{E}\bullet\bar{J}\,dv = \int\sigma|\bar{E}|^2\,dv$ (W) and power density $w_p = \bar{E}\bullet\bar{J} = \sigma|\bar{E}|^2$ (W/m^3)

Poisson's equation $\nabla^2V = {-\rho_v}/{\varepsilon}$ and Laplace's equation $\nabla^2V = 0$.

Capacitance- $C = \dfrac{Q}{V} = \dfrac{\oint_s\bar{D}\bullet d\bar{s}}{-\int_{\text{low}}^{\text{high}}\bar{E}\bullet d\bar{l}} = \dfrac{\varepsilon\oint_s\bar{E}\bullet d\bar{s}}{-\int_{\text{low}}^{\text{high}}\bar{E}\bullet d\bar{l}}$ Resistance- $R = \dfrac{V}{I} = \dfrac{-\int_{\text{low}}^{\text{high}}\bar{E}\bullet d\bar{l}}{\oint_s\bar{J}\bullet d\bar{s}} = \dfrac{-\int_{\text{low}}^{\text{high}}\bar{E}\bullet d\bar{l}}{\sigma\oint_s\bar{E}\bullet d\bar{s}}$

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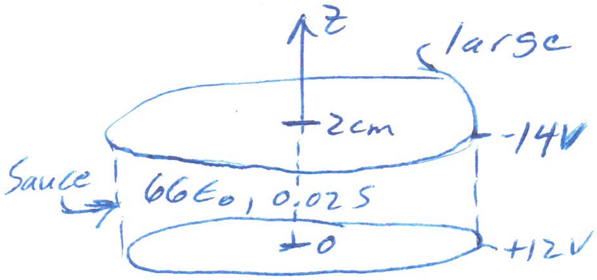
Name Key B

Instructions: **Closed book & notes**. Place answers in indicated spaces and show all work for credit.

Eager for a snack, Earl Engineer places cranberry sauce (a tart concoction of cranberries, water, spices & sugar $\Rightarrow \varepsilon = 66\varepsilon_0$, $\sigma = 0.02$ S/m) between large circular metal plates at $z = 0$ and $z = 2$ cm. To warm the sauce, he connects a dual-channel DC power supply to the plates, lower plate at +12 V and upper plate at -14 V. After 3 seconds, does Laplace's or Poisson's equation apply? Justify answer by finding the relaxation time for any initial free charges in the sauce. Find the potential, electric field, and current density in the cranberry sauce at $z = 0.4$ cm. Extra credit- Determine the *power density* in the sauce.



$T_r = \epsilon/\sigma = \dfrac{66(8.8541878 \times 10^{-12})}{0.02}$

$= \underline{2.92\,9 \times 10^{-8}\,s}$

↳ After 3 s, any initial free charges would be long gone $\Rightarrow \rho_v = 0$

$\Rightarrow$ Laplace's Eqn

cylindrical coord.

$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\underbrace{\left(\rho \frac{\partial V}{\partial \rho}\right)}_{\to 0} + \frac{1}{\rho^2}\underbrace{\frac{\partial^2 V}{\partial \phi^2}}_{\to 0} + \frac{\partial^2 V}{\partial z^2} = 0$

No change wrt ρ or ϕ

$\int \frac{d^2V}{dz^2}dz = \int 0\, dz \rightarrow \frac{dV}{dz} = A \rightarrow \int \frac{dV}{dz}dz = \int A\, dz$

$\rightarrow V = Az + B$ @ $z=0$ $V = 12V = A(0) + B \Rightarrow B = 12V$

@ $z = 0.02m$ $V = -14V = A(0.02) + 12 \Rightarrow A = -1300$ V/m

$V(z) = -1300z + 12$ V $0 \le z \le 2cm$ $V(0.004m) = \underline{6.8V}$

$\bar{E} = -\bar{\nabla}V = -\hat{a}_z \frac{d}{dz}(-1300z + 12) = \underline{\hat{a}_z 1300\ V/m}$ $0 \le z \le 2cm$

$\bar{J} = \sigma\bar{E} = 0.02\,\hat{a}_z 1300\ V/m = \underline{\hat{a}_z 26\ A/m^2}$ $0 \le z \le 2cm$

$w_p = \sigma|\bar{E}|^2 = 0.02(1300)^2 = 33{,}800\ W/m^3$

Laplace's or Poisson's equation? (Circle correct answer) T_{relax} = 29.92 ns

$V(z=0.4\text{ cm})$ = 6.8 V

$\bar{E}(z=0.4\text{ cm})$ = $\hat{a}_z$ 1300 V/m

$\bar{J}(z=0.4\text{ cm})$ = $\hat{a}_z$ 26 A/m²

Extra - *power density* = 33,800 W/m³

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$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

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