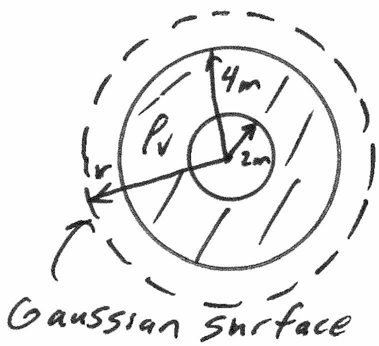
EE 381 Electric & Magnetic Fields Examination #2 (Fall 2021)

Name Key B

Instructions: Closed book. Put answers in indicated spaces, use notation as given in class for coordinates & vectors, & show all work for credit. Insert equation sheet in exam. $\varepsilon_0 = 8.8542\times10^{-12}$ F/m & $\mu_0 = 4\pi\times10^{-7}$ H/m

1) A spherically symmetric volume charge density is given by: $\rho_v = \begin{cases} \dfrac{-160}{r}\ \text{mC/m}^3 & 2 < r < 4\ \text{m} \\ 0 & \text{elsewhere} \end{cases}$.

Find the total charge Q contained in this charge distribution. Then, find the electric flux density vector $\bar{D}$ at spherical points $M(1.9, \pi/3, \pi/2)$ and $E(6, 0, 0)$ [distances in meters].



$$(4.13c)\ Q = \iiint_V \rho_v\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=2}^{4} \frac{-0.16}{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= -0.16\, (\phi)\Big|_{\phi=0}^{2\pi} (-\cos\theta)\Big|_{\theta=0}^{\pi} \left(\frac{r^2}{2}\right)\Big|_{r=2}^{4}$$

$$= -0.16(2\pi - 0)(+1+1)\left(\frac{16}{2} - \frac{4}{2}\right) = \underline{-12.063716\ C}$$

(4.41) Gauss' Law $\oiint_s \bar{D}\cdot d\bar{s} = Q_{enc} = \iiint \rho_v\, dv$. By symmetry, we can expect $\bar{D} = \hat{a}_r D_r$ and for D_r to be constant @ a fixed radius r. Choose a sphere of radius r as Gaussian surface.

$$\oiint \bar{D}\cdot d\bar{s} = D_r \iint_{sphere} ds_r = D_r(4\pi r^2) = Q_{enc} \Rightarrow D_r = \frac{Q_{enc}}{4\pi r^2}$$

At point M, $r = 1.9$m which is inside $\rho_v \Rightarrow Q_{enc} = 0$
$\hookrightarrow D_r = 0$

At point E, $r = 6$m which is outside $\rho_v \Rightarrow Q_{enc} = -12.064$ C

$$D_{r,E} = \frac{-12.063716}{4\pi(6)^2} = -0.02\overline{6}\ \text{C/m}^2 \Rightarrow \bar{D}_E = \underline{-0.02\overline{6}\ \hat{a}_r\ \text{C/m}^2}$$

$Q =$ -12.0637 (C)

$\bar{D}_M =$ 0

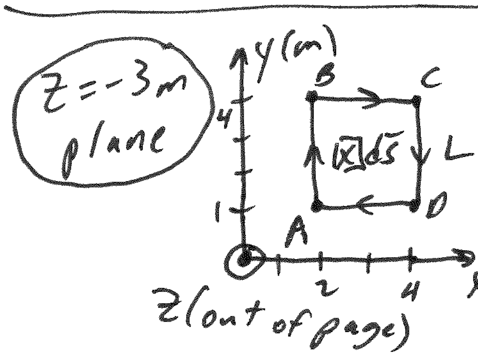
$\bar{D}_E =$ $-0.02\overline{6}\ \hat{a}_r$ (C/m^2)
$= -26.\overline{6}\ \hat{a}_r$ (mC/m^2)

$H_x \quad H_z \quad H_y = 0$

2) Find the curl and divergence the vector field $\bar{H} = 4y^2 e^{-0.25z}\,\hat{a}_x - 8xe^{-0.5y}\,\hat{a}_z$ (hobgoblins/m). Then, calculate the **circulation** of $\bar{H}$ (i.e., $\oint_C \bar{H}\cdot d\bar{l}$) around the contour defined going from Cartesian points $A(2, 1, -3) \to B(2, 4, -3) \to C(4, 4, -3) \to D(4, 1, -3) \to A$ [meters]. **Sketch contour**.

$$\text{div}\,\bar{H} = \bar{\nabla}\cdot\bar{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}^{\,0} + \frac{\partial H_z}{\partial z} = \frac{\partial}{\partial x}\left(4y^2 \underbrace{e^{-0.25z}}_{\to 0}\right) + \frac{\partial}{\partial z}\left(-8x \underbrace{e^{-0.5y}}_{\to 0}\right)$$
$$= \underline{0}$$

$$\text{Curl}\,\bar{H} = \bar{\nabla}\times\bar{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}^{\,0}\right]\hat{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial H_y}{\partial x}^{\,0} - \frac{\partial H_x}{\partial y}\right]\hat{a}_z$$
$$= -8xe^{-0.5y}(-0.5)\hat{a}_x + \left[4y^2 e^{-0.25z}(-0.25) + 8(1)e^{-0.5y}\right]\hat{a}_y - 4(2y)e^{-0.25z}\hat{a}_z$$
$$= \underline{4xe^{-0.5y}\hat{a}_x + \left[-y^2 e^{-0.25z} + 8e^{-0.5y}\right]\hat{a}_y - 8ye^{-0.25z}\hat{a}_z} \quad \frac{\text{hobgoblins}}{m^2}$$



By Stoke's Thm (3.57) $\oint_L \bar{A}\cdot d\bar{\ell} = \iint_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s}$

Here, by RHR, $d\bar{s} = -d\bar{s}_z = -\hat{a}_z\,dx\,dy$

$(\bar{\nabla}\times\bar{H})\cdot -\hat{a}_z\,dx\,dy = +8ye^{-0.25z}\,dx\,dy\,\Big|_{z=-3m}$

$$\text{Circulation} = \oint_L \bar{H}\cdot d\bar{\ell} = \int_{y=1}^{4}\int_{x=2}^{4} 8y\,e^{-0.25(-3)}\,dx\,dy$$
$$= 8e^{+0.75}\left(\frac{y^2}{2}\right)\Big|_{y=1}^{4}(x)\Big|_{x=2}^{4} = 8e^{0.75}\left(\frac{16}{2}-\frac{1}{2}\right)(4-2)$$
$$= \underline{254.04\ (\text{hobgoblins})}$$

curl $\bar{H}$ = $4xe^{-0.5y}\hat{a}_x + (8e^{-0.5y} - y^2 e^{-0.25z})\hat{a}_y - 8ye^{-0.25z}\hat{a}_z$ $\left(\frac{\text{hobgoblins}}{m^2}\right)$

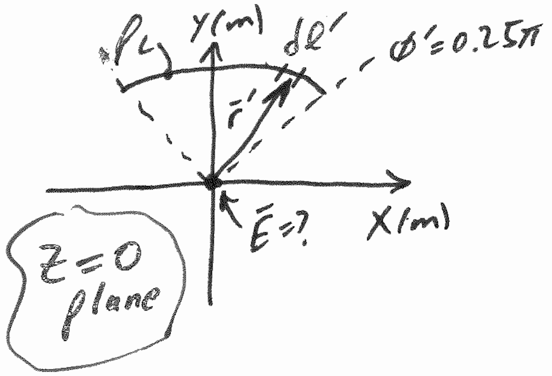
div $\bar{H}$ = 0

$\oint_C \bar{H}\cdot d\bar{l}$ = 254.04 (hobgoblins)

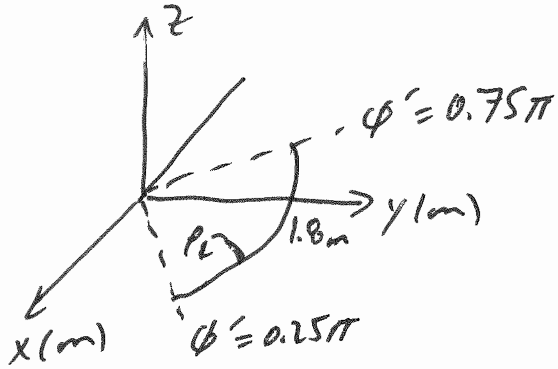
Extra credit: Is $\bar{G}$: Solenoidal? Yes / No Irrotational? Yes / No Conservative? Yes / No (Circle)

3) A total charge of -160 nC is uniformly distributed on an arc of radius 1.8 m subtending angles $0.25\pi \le \phi' \le 0.75\pi$ on the $z = 0$ (x-y) plane. We will calculate the electric field at the origin.

a) Sketch a **fully labeled** picture of the problem geometry.



OR



b) Write the appropriate general integral(s) in position vector form to find the vector **electric field** for this problem. Do NOT define any variables or simplify.

line charge distribution (4.14)

$$\bar{E} = \int_L \frac{\rho_L (\bar{r} - \bar{r}')\, dl'}{4\pi\epsilon_0 |\bar{r} - \bar{r}'|^3}$$

c) For the integral(s), what is the easiest coordinate system? Cartesian Cylindrical Spherical

following an arc (circle correct answer)

d) What is the specific **charge density** for this problem (e.g., ρ_l, ρ_s, or ρ_v)? Give value.

$$\rho_L = \frac{Q}{\text{length}} = \frac{-160 \times 10^{-9}}{\frac{1}{4}[2\pi 1.8]} = -56.588424 \times 10^{-9} \text{ C/m}$$

quarter of a circle

$$\rho_L = -56.588 \text{ (nC/m)}$$

e) From b), what is the appropriate **source differential**? $d\bar{l}'$, dl', $d\bar{s}'$, ds', $d\bar{v}'$, dv'
(circle correct answer)

f) Determine the specific **source differential** given the geometry of this problem

$$dl' = |d\bar{l}_{\phi}'| = \underbrace{\rho'}_{1.8} d\phi'$$

$$dl' = 1.8\, d\phi' \text{ (m)}$$

g) Determine the specific **source position vector** for the geometry of this problem

$$\bar{r}' = \underbrace{\rho'}_{1.8} \hat{a}_\rho' + \underbrace{z'}_{0} \hat{a}_z$$

Note: $\hat{a}_\rho' = \cos\phi' \hat{a}_x + \sin\phi' \hat{a}_y$

$$\bar{r}' = 1.8 \hat{a}_\rho' = 1.8(\cos\phi' \hat{a}_x + \sin\phi' \hat{a}_y) \text{ (m)}$$

h) Determine the specific **field position vector** for the geometry of this problem

want $\bar{E}$ @ origin $\Rightarrow \bar{r} = 0$

$$\bar{r} = 0$$

3) cont.

i) Determine the specific **distance vector** from the source to the field point in a form where all position dependence is shown (i.e., change unit vectors to Cartesian coordinates if necessary)

$$\bar{R} = \bar{r} - \bar{r}' = 0 - 1.8\hat{a}_\rho' = -1.8(\cos\phi'\hat{a}_x + \sin\phi'\hat{a}_y)$$

$$\underline{\bar{r} - \bar{r}' = -1.8(\cos\phi'\hat{a}_x + \sin\phi'\hat{a}_y)\ (m)}$$

j) Determine the **scalar distance** from the source to the field point

$$R = |\bar{r} - \bar{r}'| = |-1.8\hat{a}_\rho'| = \sqrt{(-1.8)^2} = 1.8\ m$$

$$\underline{|\bar{r} - \bar{r}'| = 1.8\ (m)}$$

k) Evaluate the integral to compute the **electric field vector** at point (0, 0, 0)

$$\bar{E} = \int_L \frac{\rho_L(\bar{r} - \bar{r}')\,d\ell'}{4\pi\epsilon_0 |\bar{r} - \bar{r}'|^3} = \int_{\phi'=0.25\pi}^{0.75\pi} \frac{-56.59\times10^{-9}(-1.8)(\cos\phi'\hat{a}_x + \sin\phi'\hat{a}_y)1.8\,d\phi'}{4\pi\epsilon_0 (1.8)^3}$$

$$= \frac{+56.588424\times10^{-9}(1.8)^2}{4\pi\, 8.8542\times10^{-12}(1.8)^3}\left\{\hat{a}_x\int_{\phi'=0.25\pi}^{0.75\pi}\cos\phi'\,d\phi' + \hat{a}_y\int_{\phi'=0.25\pi}^{0.75\pi}\sin\phi'\,d\phi'\right\}$$

$$= 282.550384\left\{\hat{a}_x \sin\phi'\Big|_{\phi'=0.25\pi}^{0.75\pi} + \hat{a}_y(-\cos\phi')\Big|_{\phi'=0.25\pi}^{0.75\pi}\right\}$$

$$= 282.55\{\hat{a}_x(\sin 0.75\pi - \sin 0.25\pi)$$
$$+ \hat{a}_y(-\cos 0.75\pi + \cos 0.25\pi)\}$$

$$= 282.55\{\hat{a}_x\underbrace{(0.707 - 0.707)}_{0} + \hat{a}_y(0.707 + 0.707)\}$$

$$= 399.586585\,\hat{a}_y\ V/m$$

$$\underline{\bar{E} = 399.5866\,\hat{a}_y\ (V/m)}$$