

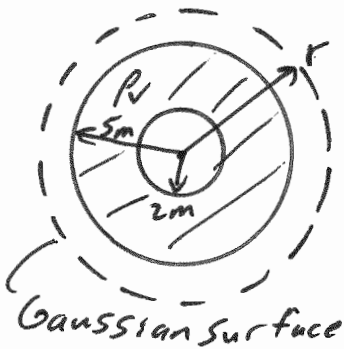
EE 381 Electric & Magnetic Fields Examination #2 (Fall 2021)

Name Key A

Instructions: Closed book. Put answers in indicated spaces, use notation as given in class for coordinates & vectors, & show all work for credit. Insert equation sheet in exam. $\epsilon_0 = 8.8542 \times 10^{-12}$ F/m & $\mu_0 = 4\pi \times 10^{-7}$ H/m

- 1) A spherically symmetric volume charge density is given by: $\rho_v = \begin{cases} \frac{-120}{r} \text{ mC/m}^3 & 2 < r < 5 \text{ m} \\ 0 & \text{elsewhere} \end{cases}$

Find the total charge Q contained in this charge distribution. Then, find the electric flux density vector \bar{D} at spherical points $M(1.9, \pi/3, \pi)$ and $E(6, 0, 0)$ [distances in meters].



$$\begin{aligned}
 (4.13c) \quad Q &= \iiint_V \rho_v \, dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=2}^5 \frac{-0.12}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
 &= -0.12 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=2}^5 r \, dr \\
 &= -0.12 (\phi) \Big|_{\phi=0}^{2\pi} (-\cos\theta) \Big|_{\theta=0}^{\pi} \left(\frac{r^2}{2}\right) \Big|_{r=2}^5 \\
 &= -0.12 (2\pi - 0) (+1 + 1) \left(\frac{25}{2} - \frac{4}{2}\right) = \underline{-15.83363 \text{ C}}
 \end{aligned}$$

(4.41) Gauss' Law $\oiint_S \bar{D} \cdot d\bar{S} = Q_{enc} = \iiint_V \rho_v \, dv$. By symmetry, we can expect $\bar{D} = \hat{a}_r D_r$ and for D_r to be constant @ a fixed value of r . Choose a sphere of radius r as Gaussian surface.

$$\oiint_S \bar{D} \cdot d\bar{S} = \iint_{\text{sphere}} D_r \, ds_r = D_r (4\pi r^2) = Q_{enc} \Rightarrow D_r = \frac{Q_{enc}}{4\pi r^2}$$

At point M , $r = 1.9 \text{ m}$ which is inside $\rho_v \Rightarrow Q_{enc} = 0$
 $\Rightarrow \underline{D_r = 0}$

At point E , $r = 6 \text{ m}$ which is outside $\rho_v \Rightarrow Q_{enc} = -15.83363 \text{ C}$

$$D_{r,E} = \frac{-15.83363}{4\pi (6)^2} = -0.035 \text{ C/m}^2 \Rightarrow \underline{\bar{D}_E = -\hat{a}_r 0.035 \text{ C/m}^2}$$

$\bar{D}_M = \underline{0}$

$Q = \underline{-15.83363 \text{ C}}$

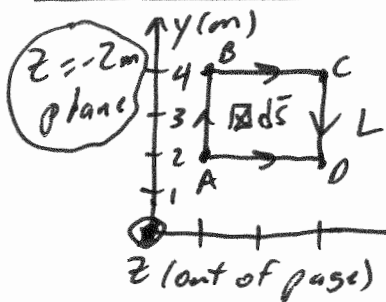
$\bar{D}_E = \underline{-\hat{a}_r 0.035 \text{ C/m}^2}$

$= \underline{-\hat{a}_r 35 \text{ mC/m}^2}$

2) Find the curl and divergence the vector field $\vec{G} = 5y^2e^{-0.2z}\hat{a}_x - 10xe^{-0.5y}\hat{a}_z$ (ghosts/m). Then, calculate the **circulation** of \vec{G} (i.e., $\oint_C \vec{G} \cdot d\vec{l}$) around the contour defined going from Cartesian points $A(1, 2, -2) \rightarrow B(1, 4, -2) \rightarrow C(3, 4, -2) \rightarrow D(3, 2, -2) \rightarrow A$ [meters]. **Sketch contour.**

$$\text{div } \vec{G} = \vec{\nabla} \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} = \frac{\partial}{\partial x} (5y^2e^{-0.2z}) + \frac{\partial}{\partial z} (-10xe^{-0.5y}) = 0$$

$$\begin{aligned} \text{curl } \vec{G} &= \vec{\nabla} \times \vec{G} = \left[\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \right] \hat{a}_x + \left[\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x} \right] \hat{a}_y + \left[\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \right] \hat{a}_z \\ &= -10xe^{-0.5y}(-0.5)\hat{a}_x + \left[5y^2e^{-0.2z}(-0.2) - (-10e^{-0.5y}) \right] \hat{a}_y - 5(2y)e^{-0.2z}\hat{a}_z \\ &= 5xe^{-0.5y}\hat{a}_x + (-y^2e^{-0.2z} + 10e^{-0.5y})\hat{a}_y - 10ye^{-0.2z}\hat{a}_z \quad \left(\frac{\text{ghost}}{\text{m}^2} \right) \end{aligned}$$



By Stokes's Thm (3.57) $\oint_L \vec{A} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$.

Here, by RHR, $d\vec{s} = -d\vec{s}_z = -\hat{a}_z dx dy$

$$(\vec{\nabla} \times \vec{G}) \cdot -\hat{a}_z dx dy = 0 + 0 + 10ye^{-0.2z} dx dy \Big|_{z=-2m}$$

$$\text{circulation} = \oint_L \vec{A} \cdot d\vec{l} = \int_{y=2}^4 \int_{x=1}^3 10ye^{-0.2(z)} dx dy$$

$$= 10e^{+0.4} \left(\frac{y^2}{2} \right) \Big|_{y=2}^4 (x) \Big|_{x=1}^3 = 10e^{+0.4} \left(\frac{16}{2} - \frac{4}{2} \right) (3-1)$$

$$= \underline{179.0189637 \text{ ghosts}}$$

$$\text{curl } \vec{G} = 5xe^{-0.5y}\hat{a}_x + (-y^2e^{-0.2z} + 10e^{-0.5y})\hat{a}_y - 10ye^{-0.2z}\hat{a}_z \quad \left(\frac{\text{ghosts}}{\text{m}^2} \right)$$

$$\text{div } \vec{G} = 0 \qquad \oint_C \vec{G} \cdot d\vec{l} = 179.02 \text{ (ghosts)}$$

Extra credit: Is \vec{G} : Solenoidal? Yes / No Irrotational? Yes / No Conservative? Yes / No (Circle)

$$\vec{\nabla} \cdot \vec{G} = 0$$

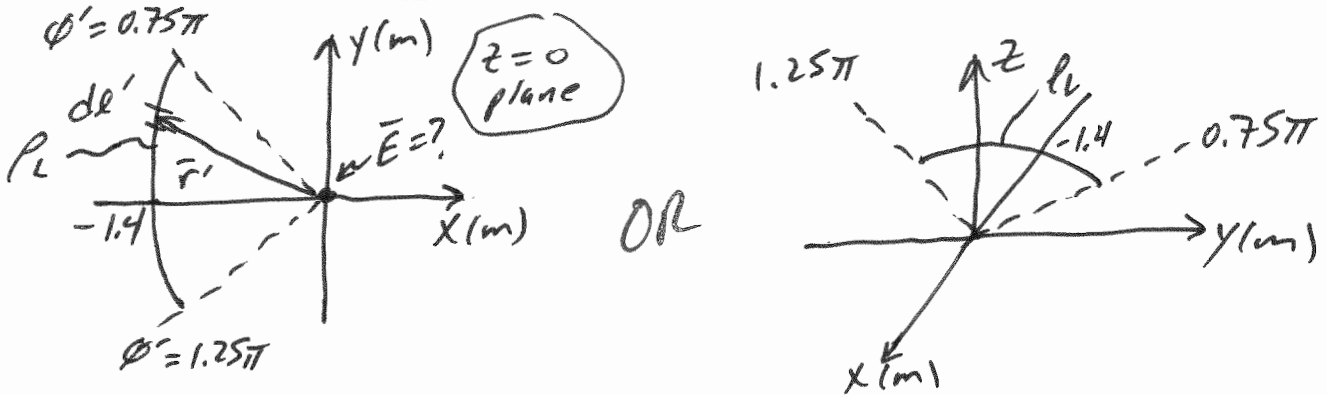
+2

+2

+1

3) A total charge of -120 nC is uniformly distributed on an arc of radius 1.4 m subtending angles $0.75\pi \leq \phi' \leq 1.25\pi$ on the $z = 0$ (x - y) plane. We will calculate the electric field at the origin.

a) Sketch a **fully labeled** picture of the problem geometry.



b) Write the appropriate general integral(s) in position vector form to find the vector electric field for this problem. Do NOT define any variables or simplify.

line charge distribution (4, 14)

$$\underline{\underline{\vec{E} = \int_L \frac{\rho_L (\vec{r} - \vec{r}') d\ell'}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}}}$$

c) For the integral(s), what is the easiest coordinate system? Cartesian Cylindrical Spherical (circle correct answer) following an arc

d) What is the specific charge density for this problem (e.g. $\rho_l, \rho_s,$ or ρ_v)? Give value.

$$\rho_L = \frac{Q}{\text{length}} = \frac{-120 \times 10^{-9}}{\frac{1}{4} [2\pi(1.4)]} = -54.567409 \times 10^{-9} \text{ C/m}$$

quarter of circle

$$\underline{\underline{\rho_L = -54.5674 \text{ (nC/m)}}}$$

e) From b), what is the appropriate source differential? $d\vec{l}', d\vec{s}', ds', d\vec{v}', dv'$ (circle correct answer)

f) Determine the specific source differential given the geometry of this problem

$$d\ell' = |d\vec{\ell}'| = \underset{1.4}{\rho'} d\phi' = 1.4 d\phi'$$

$$\underline{\underline{d\ell' = 1.4 d\phi' \text{ (m)}}}$$

g) Determine the specific source position vector for the geometry of this problem

$$\vec{r}' = \underset{1.4}{\rho'} \hat{a}_{\rho'} + \underset{0}{z'} \hat{a}_z \quad \text{Note: } \hat{a}_{\rho'} = \cos\phi' \hat{a}_x + \sin\phi' \hat{a}_y$$

$$\underline{\underline{\vec{r}' = 1.4 \hat{a}_{\rho'} = 1.4 (\cos\phi' \hat{a}_x + \sin\phi' \hat{a}_y) \text{ (m)}}}$$

h) Determine the specific field position vector for the geometry of this problem

$$\vec{E} \text{ @ origin } \Rightarrow \vec{r} = 0$$

$$\underline{\underline{\vec{r} = 0}}$$

3) cont.

- i) Determine the specific **distance vector** from the source to the field point in a form where all position dependence is shown (i.e., change unit vectors to Cartesian coordinates if necessary)

$$\vec{R} = \vec{r} - \vec{r}' = 0 - 1.4 \hat{a}_{\rho'} = -1.4 (\cos \phi' \hat{a}_x + \sin \phi' \hat{a}_y)$$

$$\vec{r} - \vec{r}' = -1.4 (\cos \phi' \hat{a}_x + \sin \phi' \hat{a}_y) \quad (\text{m})$$

- j) Determine the **scalar distance** from the source to the field point

$$R = |\vec{r} - \vec{r}'| = |-1.4 \hat{a}_{\rho'}| = \sqrt{(-1.4)^2} = 1.4 \text{ m}$$

$$|\vec{r} - \vec{r}'| = 1.4 \text{ (m)}$$

- k) Evaluate the integral to compute the **electric field vector** at point (0, 0, 0)

$$\vec{E} = \int_L \frac{\rho_L (\vec{r} - \vec{r}') d\ell'}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|^3} = \int_{\phi'=0.75\pi}^{1.25\pi} \frac{-54.57 \times 10^{-9} (-1.4) (\cos \phi' \hat{a}_x + \sin \phi' \hat{a}_y) 1.4 d\phi'}{4\pi \epsilon_0 (1.4)^3}$$

$$= \frac{+54.5679 \times 10^{-9} (1.4)^2}{4\pi (8.8542 \times 10^{-12}) 1.4^3} \left\{ \hat{a}_x \int_{\phi'=0.75\pi}^{1.25\pi} \cos \phi' d\phi' + \hat{a}_y \int_{\phi'=0.75\pi}^{1.25\pi} \sin \phi' d\phi' \right\}$$

$$= 350.3048 \left\{ \hat{a}_x \sin \phi' \Big|_{\phi'=0.75\pi}^{1.25\pi} + \hat{a}_y (-\cos \phi') \Big|_{\phi'=0.75\pi}^{1.25\pi} \right\}$$

$$= 350.3048 \left\{ \hat{a}_x (\sin 1.25\pi - \sin 0.75\pi) - \hat{a}_y (\cos 1.25\pi - \cos 0.75\pi) \right\}$$

$$= 350.3048 \left\{ \hat{a}_x (-0.7071 - 0.7071) - \hat{a}_y (0.7071 - 0.7071) \right\}$$

$\rightarrow 0$

$$\vec{E}_{\text{origin}} = -\hat{a}_x 495.4 \text{ V/m}$$

$$\vec{E}(0,0,0) = -495.4 \hat{a}_x \text{ (V/m)}$$