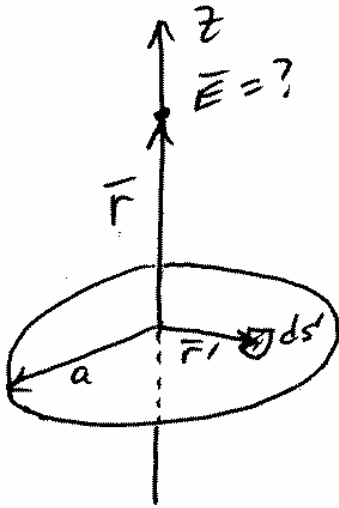


ex. Find the electric field @ some height  $z$  on the axis of symmetry due to a total charge of  $Q$  uniformly distributed on a thin circular disk of radius  $a$  located on the  $x$ - $y$  plane + centered on the origin.



$$\text{use } \vec{E} = \int_S \frac{\rho_s (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} ds'$$

Note: By symmetry,  $\vec{E}$  should only have a  $z$ -directed component.

$$\rho_s = \frac{Q}{A_{\text{area}}} = \frac{Q}{\pi a^2}$$

Field position vector  $\vec{r} = z \hat{a}_z$

Source position vector  $\vec{r}' = \rho' \hat{a}_{\rho'}$

$$ds' = |d\vec{s}'| = \rho' d\rho' d\phi'$$

So

$$\vec{E} = \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^a \frac{\frac{Q}{\pi a^2} (z \hat{a}_z - \rho' \hat{a}_{\rho'})}{4\pi\epsilon_0 |z \hat{a}_z - \rho' \hat{a}_{\rho'}|^3} \rho' d\rho' d\phi'$$

→ Before evaluating the integral, we know that  $\hat{a}_{\rho'}$  is a function of position (i.e. function of  $\phi'$ ). Convert to express  $\hat{a}_{\rho'}$  in terms of  $\hat{a}_x$ ,  $\hat{a}_y$ , +  $\hat{a}_z$

ex. cont.

$$\hat{a}_{\rho'} = \cos \phi' \hat{a}_x + \sin \phi' \hat{a}_y$$

$$\bar{E} = \frac{Q}{4\pi^2 a^2 \epsilon_0} \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^a \frac{(z \hat{a}_z - \rho' \cos \phi' \hat{a}_x - \rho' \sin \phi' \hat{a}_y)}{[(-\rho' \cos \phi')^2 + (-\rho' \sin \phi')^2 + z^2]^{3/2}} \rho' d\rho' d\phi'$$

but  $\cos^2 A + \sin^2 A = 1$ , so the denominator becomes  $[\rho'^2 + z^2]^{3/2}$

$$= \frac{Q}{4\pi^2 a^2 \epsilon_0} \left\{ \hat{a}_z z \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^a \frac{\rho' d\rho' d\phi'}{(\rho'^2 + z^2)^{3/2}} - \hat{a}_x \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^a \frac{\rho'^2 \cos \phi' d\rho' d\phi'}{(\rho'^2 + z^2)^{3/2}} - \hat{a}_y \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^a \frac{\rho'^2 \sin \phi' d\rho' d\phi'}{(\rho'^2 + z^2)^{3/2}} \right\}$$

Separate integrals and use integral tables

$$\bar{E} = \frac{Q}{4\pi^2 a^2 \epsilon_0} \left\{ \hat{a}_z z \int_{\phi'=0}^{2\pi} d\phi' \int_{\rho'=0}^a \frac{\rho' d\rho'}{(\rho'^2 + z^2)^{3/2}} - \left( \int_{\rho'=0}^a \frac{\rho'^2 d\rho'}{(\rho'^2 + z^2)^{3/2}} \right) \left[ \hat{a}_x \int_{\phi'=0}^{2\pi} \cos \phi' d\phi' + \hat{a}_y \int_{\phi'=0}^{2\pi} \sin \phi' d\phi' \right] \right\}$$

$\int_{\phi'=0}^{2\pi} \cos \phi' d\phi' \rightarrow 0$        $\int_{\phi'=0}^{2\pi} \sin \phi' d\phi' \rightarrow 0$   
 full cycle                                      full cycle

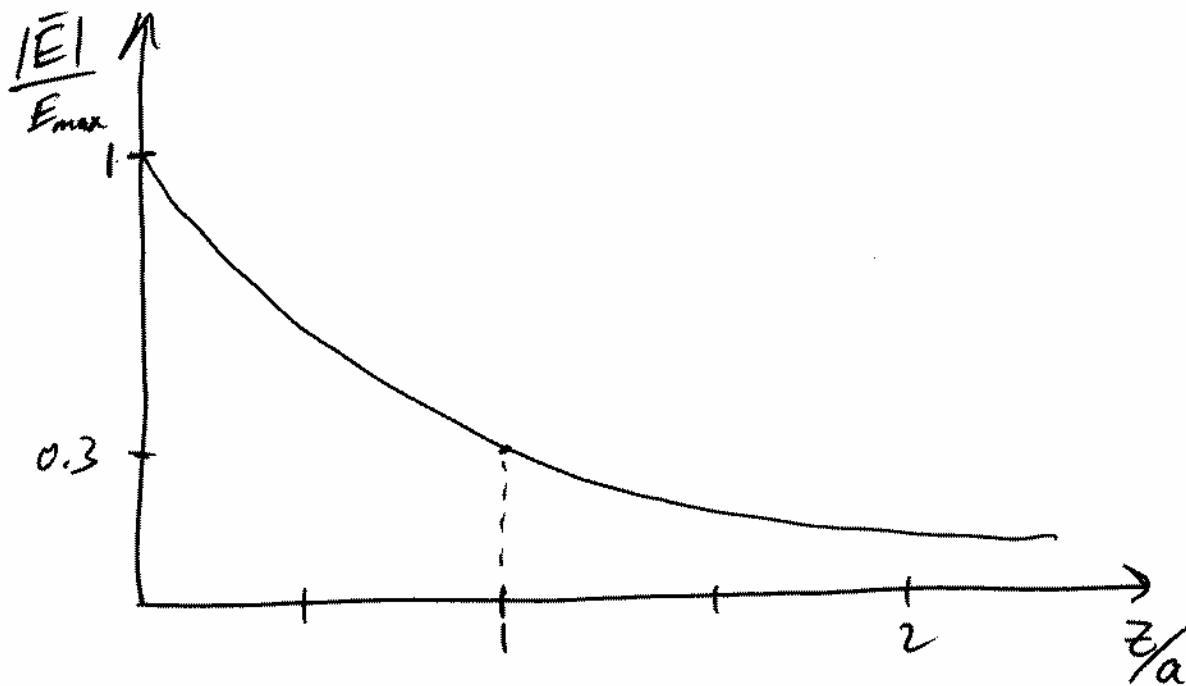
ex. cont.

$$\bar{E} = \frac{Q}{4\pi^2 a^2 \epsilon_0} \hat{a}_z z \left( \phi' \right) \Big|_{\phi'=0}^{2\pi} \left( \frac{-1}{\sqrt{\rho^2 + z^2}} \right) \Big|_{\rho=0}^a$$

$$= \hat{a}_z \frac{Q z (2\pi - 0)}{4\pi^2 a^2 \epsilon_0} \left[ \frac{-1}{\sqrt{a^2 + z^2}} + \frac{1}{z} \right]$$

$$\bar{E} = \hat{a}_z \frac{Q}{2\pi a^2 \epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + a^2}} \right] \quad (\text{V/m}) \quad z > 0$$


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ex. cont.

This result can be adapted and expanded on. For example, what if we knew the surface charge density  $\rho_s$  rather than the total charge on the disk?

$$\text{Note: } \rho_s = \frac{Q}{\pi a^2}$$

$$\text{So } \vec{E} = \hat{a}_z \frac{Q}{2\pi a^2 \epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + a^2}} \right] \text{ (V/m) } z > 0$$

$$\vec{E} = \hat{a}_z \frac{\rho_s}{2\epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + a^2}} \right] \text{ (V/m) } z > 0$$

what if we want to know the electric field for an infinite flat plane?

Use above result and let  $a \rightarrow \infty$

which yields

$$\vec{E} = \hat{a}_z \frac{\rho_s}{2\epsilon_0} \text{ (V/m)}$$

This in turn can be generalized to apply to any infinite flat plane w/ uniform  $\rho_s$

$$\vec{E} = \hat{a}_n \frac{\rho_s}{2\epsilon_0} \text{ (V/m) where } \hat{a}_n \text{ is the surface normal}$$