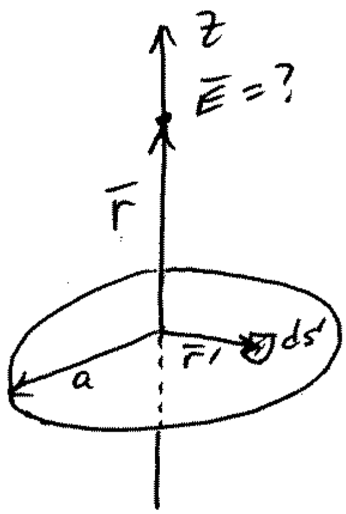
ex. Find the electric field @ some height z on the axis of symmetry due to a total charge of Q uniformly distributed on a thin circular disk of radius a located on the x-y plane + centered on the origin.



use $\bar{E} = \int_S \dfrac{\rho_s\,(\bar{r}-\bar{r}')}{4\pi\epsilon_0\,|\bar{r}-\bar{r}'|^3}\,ds'$

Note: By symmetry, $\bar{E}$ should only have a z-directed component.

Field position vector $\bar{r} = z\,\hat{a}_z$

$$\rho_s = \frac{Q}{Area} = \frac{Q}{\pi a^2}$$

Source position vector $\bar{r}' = \rho'\,\hat{a}_{\rho'}$

$$ds' = |d\bar{s}_z| = \rho'\,d\rho'\,d\phi'$$

So

$$\bar{E} = \int_{\phi'=0}^{2\pi}\int_{\rho'=0}^{a} \frac{\frac{Q}{\pi a^2}\left(z\,\hat{a}_z - \rho'\,\hat{a}_{\rho'}\right)}{4\pi\epsilon_0\,|z\,\hat{a}_z - \rho'\,\hat{a}_{\rho'}|^3}\,\rho'\,d\rho'\,d\phi'$$

→ Before evaluating the integral, we know that $\hat{a}_{\rho'}$ is a function of position (i.e. function of ϕ'). Convert to express $\hat{a}_{\rho'}$ in terms of $\hat{a}_x$, $\hat{a}_y$, + $\hat{a}_z$

ex. cont.

$$\hat{a}_{\rho}' = \cos\phi'\,\hat{a}_x + \sin\phi'\,\hat{a}_y$$

$$\bar{E} = \frac{Q}{4\pi^2 a^2 \epsilon_0} \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{(z\,\hat{a}_z - \rho'\cos\phi'\,\hat{a}_x - \rho'\sin\phi'\,\hat{a}_y)}{\left[(-\rho'\cos\phi')^2 + (-\rho'\sin\phi')^2 + z^2\right]^{3/2}}\,\rho'\,d\rho'\,d\phi'$$

but $\cos^2 A + \sin^2 A = 1$, so the denominator becomes $\left[\rho'^2 + z^2\right]^{3/2}$

$$= \frac{Q}{4\pi^2 a^2 \epsilon_0} \left\{ \hat{a}_z z \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{\rho'\,d\rho'\,d\phi'}{(\rho'^2 + z^2)^{3/2}} \right.$$

$$\left. - \hat{a}_x \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{\rho'^2 \cos\phi'\,d\rho'\,d\phi'}{(\rho'^2 + z^2)^{3/2}} - \hat{a}_y \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{\rho'^2 \sin\phi'\,d\rho'\,d\phi'}{(\rho' + z^2)^{3/2}} \right\}$$

Separate integrals and use integral tables

$$\bar{E} = \frac{Q}{4\pi^2 a^2 \epsilon_0} \left\{ \hat{a}_z z \int_{\phi'=0}^{2\pi} d\phi' \int_{\rho'=0}^{a} \frac{\rho'\,d\rho'}{(\rho'^2 + z^2)^{3/2}} \right.$$

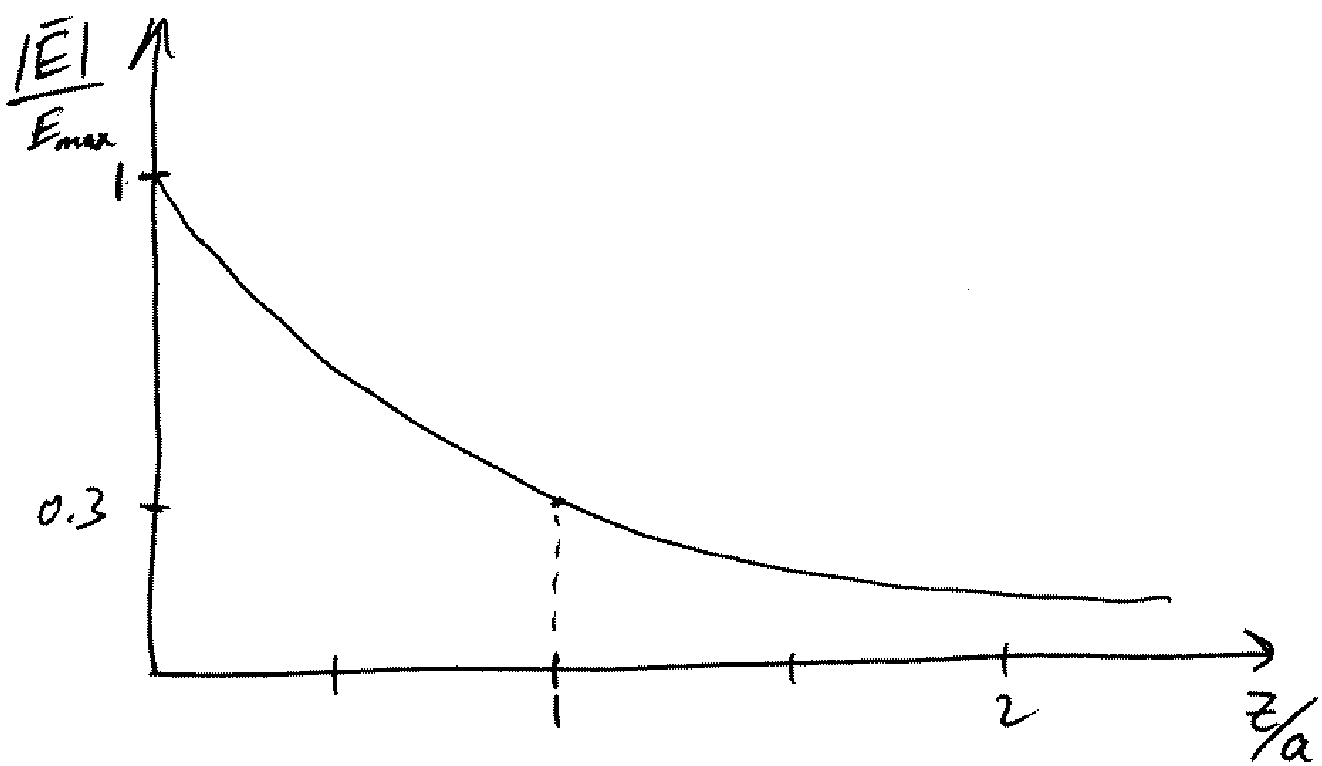
$$\left. - \left(\int_{\rho'=0}^{a} \frac{\rho'^2\,d\rho'}{(\rho'^2 + z^2)^{3/2}} \right) \left[\hat{a}_x \underbrace{\int_{\phi'=0}^{2\pi} \cos\phi'\,d\phi'}_{\to 0 \text{ full cycle}} + \hat{a}_y \underbrace{\int_{\phi'=0}^{2\pi} \sin\phi'\,d\phi'}_{\to 0 \text{ full cycle}} \right] \right\}$$

ex. cont.

$$\bar{E} = \frac{Q}{4\pi^2 a^2 \epsilon_0}\,\hat{a}_z\, z\,\left(\phi'\right)\Big|_{\phi'=0}^{2\pi}\left(\frac{-1}{\sqrt{\rho'^2+z^2}}\right)\Big|_{\rho'=0}^{a}$$

$$= \hat{a}_z\,\frac{Q\,z\,(2\pi-0)}{4\pi^2 a^2 \epsilon_0}\left[\frac{-1}{\sqrt{a^2+z^2}}+\frac{1}{z}\right]$$

$$\bar{E} = \hat{a}_z\,\frac{Q}{2\pi a^2 \epsilon_0}\left[1-\frac{z}{\sqrt{z^2+a^2}}\right]\ (V/m) \qquad z>0$$



ex. cont.

This result can be adapted and expanded on. For example, what if we knew the surface charge density ρ_s rather than the total charge on the disk?

Note: $\rho_s = \dfrac{Q}{\pi a^2}$

So $$\bar{E} = \hat{a}_z \frac{Q}{2\pi a^2 \epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right] \quad (V/m) \quad z>0$$

$$\underline{\bar{E} = \hat{a}_z \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right] \quad (V/m) \quad z>0}$$

what if we want to know the electric field for an infinite flat plane?

Use above result and let $a \to \infty$ which yields

$$\underline{\bar{E} = \hat{a}_z \frac{\rho_s}{2\epsilon_0} \quad (V/m)}$$

This in turn can be generalized to apply to any infinite flat plane w/ uniform ρ_s

$\underline{\bar{E} = \hat{a}_n \dfrac{\rho_s}{2\epsilon_0} \quad (V/m)}$ where $\hat{a}_n$ is the surface normal