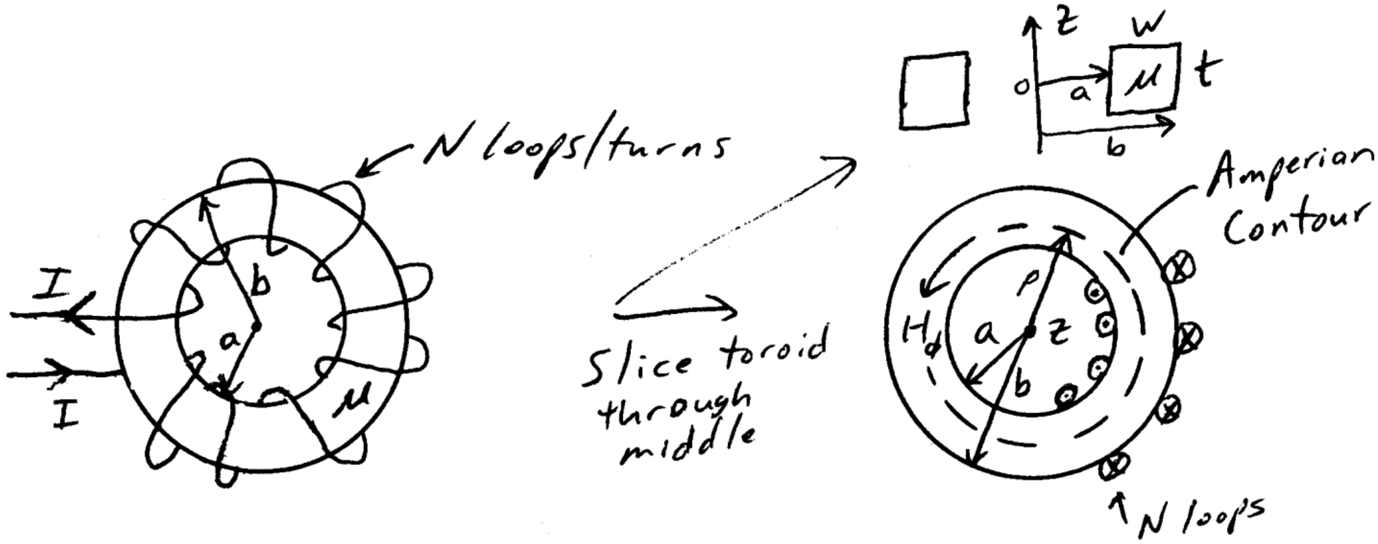
Example- For a toroidal inductor with N turns of wire tightly wound around a core (μ) with a rectangular cross-section ($w \times l$), find the magnetic field and magnetic flux density within the toroid. Then, find the inductance.



By symmetry, we expect $\bar{H} = \hat{a}_\phi H_\phi$ and choose an Amperian contour to be a circle of radius ρ where $a < \rho < b$ and at height $-\frac{t}{2} < z < \frac{t}{2}$.

Then, $\oint_C \bar{H} \cdot d\bar{l} = \int H_\phi \, dl = H_\phi (2\pi\rho)$

$I_{enc} = NI$ ← By RHR I is positive

$\hookrightarrow H_\phi (2\pi\rho) = NI \Rightarrow H_\phi = \dfrac{NI}{2\pi\rho}$

$$\bar{H} = \hat{a}_\phi \frac{NI}{2\pi\rho} \quad \left.\begin{matrix} a < \rho < b \\ -\frac{t}{2} < z < \frac{t}{2} \end{matrix}\right\} \text{inside toroid}$$

$$\bar{B} = \mu \bar{H} = \hat{a}_\phi \frac{\mu N I}{2\pi\rho} \quad \begin{matrix} a < \rho < b \\ -\frac{t}{2} < z < \frac{t}{2} \end{matrix}$$

Use $\bar{B}$ to find Ψ_m for toroid through a single loop

$$\Psi_m = \iint_{\substack{\text{loop}\\ \text{cross}\\ \text{section}}} \bar{B} \cdot d\bar{s} = \int_{z=-\frac{t}{2}}^{t/2} \int_{\rho=a}^{b} \hat{a}_\phi \frac{\mu N I}{2\pi\rho} \cdot \hat{a}_\phi \, d\rho \, dz$$

$$= \frac{\mu N I}{2\pi} \int_{-t/2}^{t/2} dz \int_a^b \frac{d\rho}{\rho} = \frac{\mu N I}{2\pi} (z)\Big|_{-t/2}^{t/2} (\ln \rho)\Big|_a^b$$

$$\underline{\Psi_m = \frac{\mu N I t}{2\pi} \ln(b/a)}$$

$$\Lambda = N\Psi_m = \frac{\mu N^2 I t}{2\pi} \ln(b/a)$$

$$\underline{\underline{L = \frac{\Lambda}{I} = \frac{\mu N^2 t}{2\pi} \ln(b/a)}}$$

If $w << a$, use $b = a + w$ and $\ln(1+x) = x - \frac{x^2}{2} + \ldots$

$$\underline{L \simeq \frac{\mu N^2 \overbrace{tw}^{\text{cross-sectional Area}}}{2\pi a} = \frac{\mu N^2 A}{2\pi a}}$$

useful for toroids w/ non-rectang. cross-sections