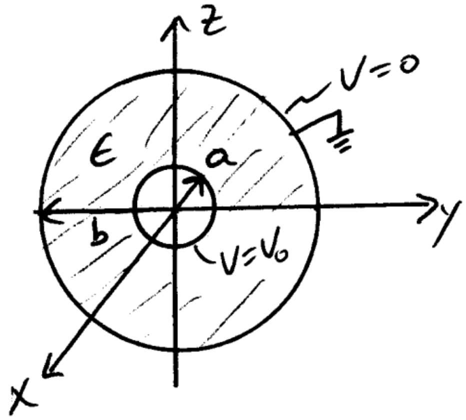
ex. Find the potential, electric field, electric flux density, and capacitance between two concentric spheres located at the origin and separated by a dielectric. Assume the inner sphere (radius a) is at potential V_0 and the outer sphere (radius b) is grounded.



* $\rho_v = 0 \rightarrow$ no free charge mentioned in dielectric
* $a < r < b$ is the bounded region
* Use Laplace's Eqn $\nabla^2 V = 0$ for spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \underbrace{\frac{\partial V}{\partial\theta}}_{\to 0}\right) + \frac{1}{r^2 \sin^2\theta}\underbrace{\frac{\partial^2 V}{\partial\phi^2}}_{\to 0}}_{\text{Nothing changes in } \theta \text{ or } \phi \text{ directions}} = 0$$

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$ ← Not partial derivatives anymore

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

$$\int d\left(r^2 \frac{dV}{dr}\right) = \int 0 \, dr$$

$$r^2 \frac{dV}{dr} = A$$ ← some constant

ex. cont.

$$\frac{dV}{dr} = \frac{A}{r^2}$$

unknown constants Rename constants

$$\int dV = \int \frac{A}{r^2}\, dr = -\frac{A}{r} + B = \frac{K_1}{r} + K_0$$

$$\underline{V = \frac{K_1}{r} + K_0} \quad \leftarrow \text{general sol'n}$$

Apply boundary conditions -

1) $V = 0$ at $r = b \Rightarrow 0 = \frac{K_1}{b} + K_0 \Rightarrow K_1 = -K_0 b$

$$\underline{V = \frac{-K_0 b}{r} + K_0}$$

2) $V = V_0$ at $r = a \Rightarrow V_0 = \frac{-K_0 b}{a} + K_0$

$$\rightarrow K_0 = V_0 \frac{a}{a-b} = \underline{-V_0 \frac{a}{b-a}} \quad \leftarrow \text{since } b > a$$

$$V = \left(V_0 \frac{ab}{b-a}\right)\frac{1}{r} - V_0 \frac{a}{b-a} \quad \leftarrow \text{unique solution}$$

$$\underline{\underline{V = V_0 \frac{ab}{b-a}\left[\frac{1}{r} - \frac{1}{b}\right] \quad a < r < b}} \quad \therefore$$

Check solution $V(r=b) = V_0 \frac{ab}{b-a}\left[\frac{1}{b} - \frac{1}{b}\right] = 0$ ok

$$V(r=a) = V_0 \frac{ab}{b-a}\left[\frac{1}{a} - \frac{1}{b}\right] = V_0 \frac{1}{b-a}\left[b-a\right]$$

$$V(r=a) = V_0 \quad \text{ok}$$

ex. cont.

$$\bar{E} = -\bar{\nabla}V = -\hat{a}_r \frac{\partial V}{\partial r} - \hat{a}_\theta \frac{1}{r}\underbrace{\cancel{\frac{\partial V}{\partial \theta}}}_{0} - \hat{a}_\phi \frac{1}{r\sin\theta}\underbrace{\cancel{\frac{\partial V}{\partial \phi}}}_{0}$$

$$= -\hat{a}_r V_0 \frac{ab}{b-a}\left[\frac{-1}{r^2} - 0\right]$$

$$\bar{E} = \hat{a}_r V_0 \frac{ab}{b-a}\frac{1}{r^2} \quad a<r<b$$

$$\bar{D} = \epsilon \bar{E} = \hat{a}_r V_0 \frac{ab\epsilon}{b-a}\frac{1}{r^2} \quad a<r<b$$

Capacitance – a potential difference of V_0 was assumed. Now, we need to find $+Q_f$ in terms of V_0.

By Gauss' Law, $\oint_S \bar{D}\cdot d\bar{s} = Q_{enc}$.

Choosing a spherical surface of $r=a^+$ (just encloses inner sphere), $Q_{enc} = +Q_f$ and $d\bar{s} = d\bar{s}_r = \hat{a}_r r^2 \sin\theta \, d\phi \, d\theta$

$$Q_f = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \hat{a}_r V_0 \frac{ab\epsilon}{b-a}\frac{1}{r^2} \cdot \hat{a}_r r^2 \sin\theta \, d\phi \, d\theta$$

$$= V_0 \frac{ab\epsilon}{b-a}\int_{\theta=0}^{\pi}\sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi = V_0 \frac{ab\epsilon}{b-a}\left(-\cos\theta\right)\Big|_0^{\pi}\left(\phi\right)\Big|_0^{2\pi}$$

$$Q_f = V_0 \frac{ab\epsilon}{b-a}(+1+1)(2\pi - 0) = V_0 \frac{4\pi ab\epsilon}{b-a}$$

$$C = Q_f/V_0 = \frac{4\pi\epsilon ab}{b-a} \quad \therefore$$