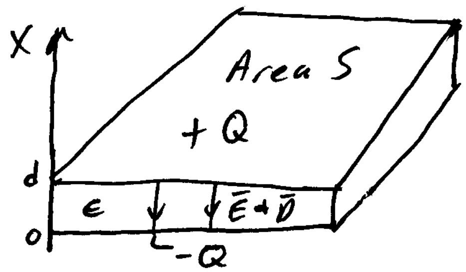
ex. Find the capacitance of a parallel-plate capacitor.

<u>Method 1</u> → Assume top plate has charge $+Q$ and bottom plate has charge $-Q$

→ Assume plate spacing is much smaller than plate dimensions (no fringing) $\Rightarrow \bar{D} = -\hat{a}_x D_x$ & $\bar{E} = -\hat{a}_x E_x$ between plates, zero elsewhere



Gauss' Law $\oiint_S \bar{D} \cdot d\bar{s} = Q_{enc}$

Choose a conformal box around the top plate as the Gaussian surface

$$\oiint_S \bar{D} \cdot d\bar{s} = \underbrace{\iint_{top} 0 \cdot +d\bar{s}_x}_{\to 0} + \underbrace{\iint_{sides} -\hat{a}_x D_x \cdot d\bar{s}_{sides}}_{\to 0} + \iint_{Bottom} -\hat{a}_x D_x \cdot -d\bar{s}_x = Q$$

$$D_x S = Q \Rightarrow D_x = Q/S \text{ between plates}$$

$$\epsilon E_x S = Q \Rightarrow E_x = \frac{Q}{\epsilon S} \quad \text{" "}$$

$$\bar{E} = \begin{cases} -\hat{a}_x \dfrac{Q}{\epsilon S} & \text{between plates} \\ 0 & \text{elsewhere} \end{cases}$$

$$V = -\int_{Bott}^{top} \bar{E} \cdot d\bar{\ell} = -\int_{x=0}^{d} -\hat{a}_x \frac{Q}{\epsilon S} \cdot (dx\,\hat{a}_x + \underbrace{dy}_{\to 0}\,\hat{a}_y + \underbrace{dz}_{\to 0}\,\hat{a}_z)$$

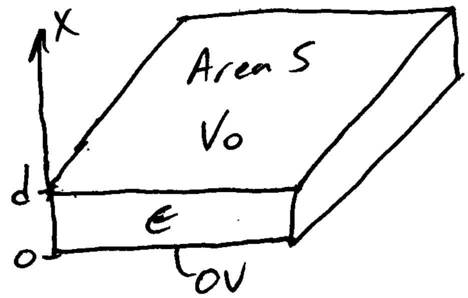
path straight up

$$= +\frac{Q}{\epsilon S} \int_{x=0}^{d} dx = \frac{Q}{\epsilon S}(d-0) \Rightarrow \underline{V = \frac{Qd}{\epsilon S}}$$

$$C = \frac{Q^{\leftarrow \text{assumed}}}{V} = \frac{Q}{\frac{Qd}{\epsilon S}} \Rightarrow \underline{\underline{C = \frac{\epsilon S}{d}}}$$

<u>Method 2</u> → Assume top plate at potential V_0 while bottom plate at 0V

→ Assume plate spacing is much smaller than plate dimensions ⇒ no changes in y + z directions



→ No free charges in dielectric

Use Laplace's Eq'n

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \underbrace{\frac{\partial^2 \cancel{V}}{\partial y^2}}_{\to 0} + \underbrace{\frac{\partial^2 \cancel{V}}{\partial z^2}}_{\to 0} = 0$$

$$\int \frac{d^2 V}{dx^2}\, dx = \int 0\, dx \;\rightarrow\; \frac{dV}{dx} = A$$

$$\int \frac{dV}{dx}\, dx = \int A\, dx \;\rightarrow\; \underline{V = Ax + B} \leftarrow \text{General Sol'n}$$

<u>Apply boundary conditions</u>

$$V(x=0) = 0 = A(0) + B \;\Rightarrow\; B = 0$$

$$V(x=d) = V_0 = A(d) \;\Rightarrow\; A = V_0/d$$

$$\underline{V(x) = \frac{V_0}{d} x \quad \text{for } 0 \le x \le d} \;\text{(between plates)}$$

$$\bar{E} = -\bar{\nabla} V = -\hat{a}_x \frac{\partial\left(\frac{V_0}{d}x\right)}{\partial x} - \hat{a}_y \underbrace{\frac{\partial \cancel{V(x)}}{\partial y}}_{\to 0} - \hat{a}_z \underbrace{\frac{\partial \cancel{V(x)}}{\partial z}}_{\to 0}$$

$$\bar{E} = -\hat{a}_x \frac{V_0}{d} \quad \text{(between plates)}$$

$$\underline{\bar{D} = \epsilon \bar{E} = -\hat{a}_x \frac{V_0 \epsilon}{d}} \quad \text{(between plates)}$$

<u>Find charge on top plate</u>

use $\rho_s = D_n$ boundary condition

$$\rho_s = \hat{a}_n \cdot \bar{D} = -\hat{a}_x \cdot -\hat{a}_x \frac{V_0 \epsilon}{d} = \frac{V_0 \epsilon}{d}$$

$$Q_{\substack{top \\ plate}} = \iint_S \rho_s \, ds = \frac{V_0 \epsilon}{d} \underbrace{\iint_S ds}_{S} = \frac{V_0 \epsilon S}{d}$$

$$C = \frac{Q}{\underbrace{V}_{\text{assumed}}} = \frac{\left(\frac{V_0 \epsilon S}{d}\right)}{V_0} \Rightarrow \underline{\underline{C = \frac{\epsilon S}{d}}} \text{ (SAME!)}$$

__Bonus__ What is the force between the capacitor plates?

→ Expect it always to be attractive as plates have opposite charge.

$$W_E = \frac{1}{2} \iiint_V \bar{D} \cdot \bar{E} \, dv = \int \bar{F} \cdot d\bar{\ell}$$

$$= \frac{1}{2} \iiint_V -\hat{a}_x \frac{Q}{S} \cdot -\hat{a}_x \frac{Q}{\epsilon S} \, dv = \frac{1}{2} \iiint_V -\hat{a}_x \frac{V_0}{d} \cdot -\hat{a}_x \frac{V_0 \epsilon}{d} \, dv$$

$$= \frac{Q^2}{2 \epsilon S^2} \overbrace{(S d)}^{\text{volume}} = \frac{V_0^2 \epsilon}{2 d^2} (S d)$$

$$= \frac{Q^2 d}{2 \epsilon S} = \frac{V_0^2 \epsilon S}{2 d}$$

From $W_E = \int \bar{F} \cdot d\bar{\ell} \rightarrow |F| = \left| \frac{d W_E}{d \ell} \right|$

let the spacing $d = x$

$$|\bar{F}| = \left| \frac{d \, \frac{Q^2 x}{2 \epsilon S}}{d x} \right| = \left| \frac{d \, \frac{V_0^2 \epsilon S}{2 x}}{d x} \right|$$

$$\underline{\underline{|\bar{F}| = \frac{Q^2}{2 \epsilon S}}} = \frac{V_0^2 \epsilon S}{2 x^2} \bigg|_{x=d} = \underline{\underline{\frac{V_0^2 \epsilon S}{2 d^2}}} \text{ (N)}$$